MODERN ALGEBRA I GU4041

Homework 6, due October 24: Lagrange's theorem, homomorphisms and normal subgroups

1. Howie notes, section 3.5, exercise 6.

2. Howie notes, section 6.6, exercises 1 and 2.

3. Choose a subgroup H of order 2 in S_3.

(a) Find $g \in S_3$ such that $gHg^{-1} \neq H$, thus demonstrating that H is not a normal subgroup.

(b) Write down representatives of the sets of left cosets S_3/H and right cosets $H\backslash S_3$.

4. (from the Judson book, section 6.5, exercise 11): Let H be a subgroup of a group G and let $g_1, g_2 \in G$. Show that the following are equivalent:

- $g_1 H = g_2 H$.
- $Hg_1^{-1} = Hg_2^{-1}$
- $g_1 H \subset g_2 H$
- $g_1 \in g_2 H$
- $g_1^{-1} g_2 \in H$.

5. Let G denote the set of 3×3 matrices with entries in $\mathbb{R}$, of the form

$$\begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & \lambda \end{pmatrix}$$

that satisfy the relation

$$(ad - bc)\lambda = 1.$$

(a) Show that G is a group.

(b) Show that the subset $H \subset G$ for which $a = d = 1$ and $b = c = 0$ is a subgroup.

(c) Show that H is a *normal* subgroup of G.

(d) Let $\phi : G \to GL(2, \mathbb{R})$ be the map

$$\phi(\begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & \lambda \end{pmatrix}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Show that ϕ is a homomorphism and that $\phi(g)$ is the identity matrix if and only if $g \in H$.

6. Let $n > 2$ be an integer. Show that the group of rotations of the regular n-gon is a normal subgroup of the dihedral group D_{2n}, and identify the quotient group.

Recommended reading

Howie notes, section 3.4, sections 6.1-6.3