MODERN ALGEBRA I GU4041

Homework 8, due November 2: Isomorphism theorems

1. Let n and m be two positive integers. Denote by $f : \mathbb{Z} \to \mathbb{Z}_n$ and $g : \mathbb{Z} \to \mathbb{Z}_m$ the two natural maps, and define

$$f \times g : \mathbb{Z} \to \mathbb{Z}_n \times \mathbb{Z}_m$$

by $(f \times g)(x) = (f(x), g(x))$.

(a) Suppose $(n, m) = 1$. Show that the kernel of $f \times g$ is the subgroup of multiples of nm in $\mathbb{Z}$.

(b) Still supposing $(n, m) = 1$, use the First Isomorphism Theorem to reprove the Chinese remainder theorem:

$$\mathbb{Z}_{nm} \xrightarrow{\sim} \mathbb{Z}_n \times \mathbb{Z}_m.$$

(c) If $n = m$, determine the image and kernel of $f \times g$, and show that this is consistent with the First Isomorphism Theorem.

2. We illustrate the Second Isomorphism Theorem by continuing question 1, but now not assuming n and m are relatively prime. In what follows $< a >$ denotes the subgroup of multiples of a in $\mathbb{Z}$, for any integer a.

Let $G = \mathbb{Z}$, $H =< m >= \ker(g) \subset \mathbb{Z}$, $N =< n >= \ker(f) \subset \mathbb{Z}$. Let $d = GCD(m, n)$, c the least common multiple $LCM(m, n)$.

(a) Use Bezout's theorem to show that $H \cdot N =< d > \subset \mathbb{Z}$. (The Second Isomorphism Theorem uses multiplicative notation for the group operation, but since H and N are subgroups of $\mathbb{Z}$ you may prefer to use additive notation.)

(b) Show that $H \cap N =< c > \subset \mathbb{Z}$.

(c) Show that the Second Isomorphism Theorem can be rewritten

$$\mathbb{Z}_{c/m} \xrightarrow{\sim} \mathbb{Z}_{n/d}$$

and that this comes down to the formula $m \cdot n = c \cdot d$.

(Hint: Multiplication by d is an isomorphism $\mathbb{Z} \xrightarrow{\sim} < d > \subset \mathbb{Z}$. What is the image of the subgroup $< n/d > \subset \mathbb{Z}$?)

3. Let G be a group and $N \triangleleft G$ be a normal subgroup. Suppose N is of prime index p in G. Let $H \subset G$ be any subgroup. Prove that exactly one of the following is true:

(i) $H \subset N$; or

(ii) $G = HN$ and $[H : H \cap N] = p$.

4. Let G be a group and $N \triangleleft G$ and $M \triangleleft G$ be normal subgroups. Suppose also that $G = NM$.

(a) Prove that there is an isomorphism

$$G/N \cap M \xrightarrow{\sim} G/N \times G/M.$$

(Hint: Use the First Isomorphism Theorem.)

(b) Use (a) and the Second Isomorphism Theorem to deduce that, if G is the product of two normal subgroups N and M such that $N \cap M = (\mathbf{e})$ then

$$G \xrightarrow{\sim} M \times N.$$

5. Judson book, section 11.4, exercises 14, 17.

Recommended reading

Howie notes, sections 6.4, 6.5; Judson, Chapter 11.