INTRODUCTION TO MODERN ALGEBRA I, GU4041, SPRING 2020

Final, May 2020

HONOR CODE AFFIRMATION.

I affirm that I will not plagiarize, use unauthorized materials, or give or receive illegitimate help on assignments, papers, or examinations. I will also uphold equity and honesty in the evaluation of my work and the work of others. I do so to sustain a community built around this Code of Honor.

Please sign and scan and submit this signed page:

Your name:

Your UNI:

This is an **open book test**. The exam will be available at 10:00 AM Eastern Daylight Time on May 13.

The exam is due at 10 PM (New York time) May 15 on Courseworks. It is strongly recommended that you submit this signed page as soon as you receive the exam; this will let you know well in advance whether or not the system is working properly. Please contact CUIT immediately, at askcuit@columbia.edu, or call 212-854-1919, if you encounter any obstacle.

The professor will be available intermittently on Zoom to answer questions; please write to mh2836@columbia.edu to request an appointment
Questions by email will be impossible to manage; please *do not* send any.

Each question is worth 20 points.

1. True or False? Provide an explanation in either case. The explanation can be brief but it is not enough to say that the statement was explained in the course.

(a) Let p be a prime. There are exactly three non-isomorphic abelian groups of order p^3.

(b) Let $p \neq q$ be distinct primes. There are exactly two non-isomorphic abelian groups of order pq.

(c) Let $p \neq q$ be distinct primes. There is exactly one simple group of order pq.

(d) A group of order 1000 has a unique 5-Sylow subgroup.

2. (a) List the conjugacy classes of the symmetric group S_6. Choose an element s in each conjugacy class and determine its centralizer in S_6. Use this information to determine the number of elements in each conjugacy class, and verify the class equation.

(b) Which of the conjugacy classes in (a) belong to the alternating group A_6?

3. (a) Prove that the absolute value map $z \mapsto |z|$ is a homomorphism from the multiplicative group $\mathbb{C}^\times$ of complex numbers to the multiplicative group $\mathbb{R}^\times$ of real numbers. What is its image and what is its kernel?

(b) Prove that any finite subgroup of $\mathbb{C}^\times$ is contained in the circle consisting of elements of absolute value 1.

(c) Prove that any finite subgroup of $\mathbb{C}^\times$ is cyclic.

4. (a) Let $G = Q_8$ be the quaternion group. Write down the derived subgroup $D(G)$ and its derived subgroup $D^2(G) = D(D(G))$.

(b) Same as (a), with $G = S_4$.

5. Let G be any group and $S \subseteq G$ any conjugacy class. Let $N = \langle S \rangle \subseteq G$ be the subgroup generated by S. Prove that N is a normal subgroup.

6. Construct non-abelian groups of order 21 and 55.

7. Show that there are no simple groups of order 56.

8. Let p be a prime number.

(a) Prove that any group of order p^2 is abelian.

(b) Give an example of a group of order p^3 that is not abelian (for every p).

9. Let G be a finite group with center Z. Suppose $[G : Z] = n$. Prove that any conjugacy class of G has at most n elements. Can you be more precise about the size of conjugacy classes?

10. Let K_4 be the Klein group. How many elements does $Aut(K_4)$ contain?