Review of set theory

GU4041, fall 2023

Columbia University

September 7, 2023

Outline

Set theory is about its notation

One says about sets that *a set is a collection of objects*, but one doesn't stop to define "collection" or "object."

We treat "set" as an undefined term that we can nevertheless put to good use. In this course we study *groups*, and a group is in the first place a set. But it is a set that has specific properties and operations, and in order to explain what these are, we need to recall the notation of set theory.

Examples of sets

So X is a set, or a collection, its objects are called *elements*. If x is an element of X we write $x \in X$.

Some examples of sets: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$; or the set of lines of code in a program; or a data set.

The elements are, respectively, integers, rational numbers, real numbers, the lines of code, or the data.

A surprisingly handy set is the *emptyset*, denoted $\emptyset$, which is the set that contains *no elements*.

A finite set contains finitely many elements; if X is finite then its *cardinality* is the number of elements, written $|X|$.

Basic notation and terminology

If X and Y are sets, then we write $X = Y$ if they have the same elements.
We write $X \subset Y$ if every element of X is an element of Y; then X is a *subset* of Y.
So for example, if $Y = \mathbb{Z}$ is the set of integers, $X = \mathbb{N}$ is the set of positive integers, then $X \subset Y$. We also have $\{0, 1\} \subset \mathbb{Z}$.
The collection of subsets of Y is itself a set, called the *power set* of Y and written $\mathcal{P}(Y)$.

Fun fact: convince yourselves that the definition of $\subset$ implies that $\emptyset \subset Y$ for *any set* Y. So $\emptyset \in \mathcal{P}(Y)$ for any Y. This means that $\mathcal{P}(Y)$ is never the empty set!!

Basic notation and terminology

The *union* $X \cup Y$ is the set containing all the elements of X and of Y (and no others); the *intersection* $X \cap Y$ is the set containing all elements that are both in X and in Y.
The *complement* of Y in X, written $X \setminus Y$, or $X - Y$, is the set of elements of X not in Y. In symbols

$$x \in X \setminus Y \Leftrightarrow x \in X \text{ but } x \notin Y.$$

If $X \cap Y = \emptyset$, the *disjoint union* $X \coprod Y$ is $X \cup Y$.

First non-trivial fact: De Morgan's Laws

Let X be a set, with subsets Y and Z.

Theorem (De Morgan's Laws)

(a) $X \setminus (Y \cap Z) = (X \setminus Y) \cup (X \setminus Z)$.

(b) $X \setminus (Y \cup Z) = (X \setminus Y) \cap (X \setminus Z)$.

Point (b) is in the homework. The proof of (a) consists in showing that each side is contained in the other.
(i) To show $X \setminus (Y \cap Z) \subset (X \setminus Y) \cup (X \setminus Z)$: Suppose $x \in X$ but not in the intersection of Y and Z. So either $x \notin Y$ or $x \notin Z$. If $x \notin Y$ then $x \in X \setminus Y$; of $x \notin Z$ then $x \in X \setminus Z$.

(ii) To show $(X \setminus Y) \cup (X \setminus Z) \subset X \setminus (Y \cap Z)$: Suppose $x \in X \setminus Y$. Then $x \notin Y$ so $x \notin Y \cap Z$; hence $x \in X \setminus (Y \cap Z)$. The case of $x \in X \setminus Z$ is identical.

Complements and products

If all sets are taken to be subsets of a fixed X, then to $A \subset X$ we write ${}^cA$, the complement of A, for $X \setminus A$.

Fact

${}^{cc}A = A$.

"The elements of X that are not not in A are in A."

If X and Y are sets, then $X \times Y$ is the set of *ordered pairs* (x, y), where $x \in X$ and $y \in Y$. The most familiar example is when $X = Y = \mathbb{R}$, then $X \times Y = \mathbb{R}^2$; hence $X \times Y$ is called the *cartesian product*.

Examples of products

$$\mathbb{R} \times \mathbb{R}^2 = \mathbb{R}^3; \mathbb{R}^a \times \mathbb{R}^b = \mathbb{R}^{a+b}.$$

If X and Y are finite sets, then $|X \times Y| = |X||Y|$.

Challenge

Show that if A is any set, then $\emptyset \times A = \emptyset$.

Relations

Let X and Y be sets. A *relation* on X and Y is a subset $R \subset X \times Y$. If $(x, y) \in R$ we write xRy.

Example

If $X = Y = \mathbb{R}$, then the relation $x < y$ can be drawn as a subset; likewise $x \leq y$, or $x = y$.

The set of relations is exactly the power set $\mathcal{P}(X \times Y)$.
A *function* from X to Y is a special kind of relation: if for every $x \in X$ there is exactly one $y \in Y$ such that $(x, y) \in R$. We write $y = f(x)$ and then R is the *graph* of $f : X \to Y$.
More fun with the emptyset: Let X be a non-empty set. Show that there are no functions from X to $\emptyset$ but that there is exactly one function from $\emptyset$ to X.

Equivalence relations

An *equivalence relation* on X is a relation in $X \times X$ that is

- **Reflexive:** For all $x \in X$, xRx;
- **Symmetric:** If xRy then yRx;
- **Transitive:** If xRy and yRz then xRz.

So $=$ is an equivalence relation. On the other hand $x < y$ is transitive but not reflexive or symmetric; $x \leq y$ is reflexive and transitive but not symmetric.

If $X = Y$ is the set of people in New York, then the relation x *loves* y, is neither reflexive, symmetric, nor transitive.

More on equivalence relations

If $\sim$ is an equivalence relation on X an *equivalence class* is a subset $S \subset X$ such that $a, b \in S \Leftrightarrow a \sim b$.
Every element of X belongs to exactly one equivalence class, and the set of equivalence classes is a new set, written $X/\sim$.
The most familiar example of a non-trivial equivalence relation is the construction of the *rational numbers* $\mathbb{Q}$. Consider $X = Y = \mathbb{Z} \times \mathbb{Z} \setminus \{0\}$. So X is the set of ordered pairs (a, b) with $a, b \in \mathbb{Z}$, $b \neq 0$.
Say $(a, b) \sim (c, d)$ if and only if $ad = bc$. We construct a bijective function $f : X/\sim \; \rightarrow \mathbb{Q}$: $f((a, b)) = \frac{a}{b}$.
Of course $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$.

More examples

Define a relation on $\mathbb{Z}$ by saying $a \sim b$ if and only if $a + b$ is even.

Then $a \sim b$ if and only if either a and b are both odd or a and b are both even.
Thus there are two equivalence classes: the class containing 0 and the class containing 1.

A *partition* of the set X is a collection of subsets $X_\alpha \subset X$ such that

- Each $x \in X$ belongs to some X_α, and
- If $\alpha \neq \beta$ then $X_\alpha \cap X_\beta = \emptyset$.

Theorem

Let R be an equivalence relation on the set X. Then the set of equivalence classes for R is a partition of X. Moreover, every partition of X arises in this way.

Every function defines an equivalence relation on its domain

Let $f : X \to Y$ be a function. If $x, x' \in X$, say $x \sim_f x'$ if $f(x) = f(x')$.

Theorem

Every equivalence relation on X is of the form $\sim_f$ for some $f : X \to Y$.

Proof: Let $\sim$ be an equivalence relation on X. Define $Y = X/\sim$ and let $f : X \to Y$ be the function that takes x to its equivalence class. Then $\sim$ coincides with $\sim_f$.

Properties of functions

Let $f : X \to Y$ be a function. The *image* of f, denoted $f(X)$ is the set of $y \in Y$ such that $y = f(x)$ for some $x \in X$. We say f is *surjective* if $f(X) = Y$.

If $y \in Y$, the *inverse image* $f^{-1}(y)$ is the set of $x \in X$ such that $f(x) = y$. We say f is *injective* if, for any $y \in Y$, $|f^{-1}(y)| \leq 1$. The function f is *bijective* if it is both injective and surjective.

Lemma (Exercise)

Let $f : X \to Y$ be a function. Then y is in the image of f if and only if $f^{-1}(y)$ is NOT the empty set.

If $f : X \to Y$ is bijective, then for each $y \in Y$ the set $f^{-1}(y)$ contains exactly one element. This defines the *inverse function* $f^{-1} : Y \to X$.

Examples

Let $X = Y = \mathbb{R}, f(x) = x^n$ for $n \in \mathbb{N}$. Then f is bijective if and only if n is *odd*.

Let $X = Y = [0, \infty) \subset \mathbb{R}$. Then $f(x) = x^n$ is bijective for any $n = 1, 2, \cdots \in \mathbb{N}$. The inverse function is the n-th root.

Let $X = \mathbb{R}$, $Y = (0, \infty)$. Then $f(x) = e^x$ is bijective. The inverse function is $log(x)$ (natural logarithm).

Subsets of finite sets

Let $X = \{1, 2, \ldots, n\}$ for some positive integer n.

Theorem

The power set $\mathcal{P}(X)$ has 2^n elements.

Proof: Let $S \subset X$ be a subset. Define a function $f_S : X \to \{0, 1\}$ by setting $f_S(x) = 1$ if $x \in S$, $f_S(x) = 0$ if $x \notin S$. Thus $S = f_S^{-1}(1)$. Thus f_S completely determines S, and vice versa. In other words, the map $S \mapsto f_S$ from $\mathcal{P}(X)$ to functions from X to $\{0, 1\}$ is a bijection. But the set of such f_S is exactly the set of binary numbers with n digits, which has cardinality 2^n.

What is a number

Let $X = \{1, 2, \ldots, n\}$ as above.

There is a map $\mathcal{P}(X) \to \{0, 1, \ldots, n\}$ given by $S \mapsto |S|$.

Is a number an equivalence class of finite sets?

Binary operations

The most important functions in (modern) algebra are the *binary operations*:

Definition

Let X be a set. A binary operation on X is a function

$$m : X \times X \to X.$$

Examples: addition and multiplication for $X = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$.

$$m(x, y) = x + y \text{ if } x, y \in \mathbb{Z}.$$

These operations have familiar *properties*, which serve as the basis for the axioms of algebra. We return to these properties at the end of the second week.