GU4041: Intro to Modern Algebra I

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Midterm 2

1) True or False? If false, provide a counterexample; if true, provide an explanation. The explanation can be brief, but it is not enough to say that the statement was explained in the course.

a) Any finite group of order pq, where $p < q$ are each prime, is abelian.

This is false; as a counterexample, take S_3, which has order $6 = (2)(3)$. This is a nonabelian group; $(12)(23) = (123) \neq (132) = (23)(12)$. With a particular constraint on the values of p and q, this does hold; in particular, q cannot be equivalent to 1 mod p.

b) Suppose H and J are two subgroups of a group G. Then $HJ \subseteq G$ is a subgroup and

$$H/H \cap J \cong HJ/J$$

This is false; again let $G = S_3$. Let $H =< (12) >= \{e, (12)\}, K =< (23) >= \{e, (23)\}$. Then $HK = \{(e)(e), (e)(23), (12)(e), (12)(23)\} = \{e, (23), (12), (123)\}$. This is not a subgroup, by Lagrange's theorem, since it has order 4. With an appropriate normality condition, this is the second isomorphism theorem.

2)

a) Write down the cycle decompositions of the following products of permutations:

i) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 4 & 5 \end{pmatrix}^2$

$$(1243)(12)^2 = (1243)(12)(12) = (1243)$$

ii) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 4 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 3 & 1 & 4 \end{pmatrix}^2$

$$(123)(1254)^2 = (123)(1254)(1254) = (15243)$$

b) Write each of the following permutations

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 4 & 5 \end{pmatrix}$$

As a product of transpositions in S_5, and determine which belong to the alternating group A_5.

$$\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 2 & 4 & 1 & 3 & 5\end{pmatrix} = (1243) = (13)(14)(12); \begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 2 & 3 & 1 & 4 & 5\end{pmatrix} = (123) = (13)(12)$$

So the latter one belongs to A_5, but the former does not.

3) Find a non-abelian group G with two distinct normal subgroups H, K of index 2. Show that $H \cap K$ is a normal subgroup of G and that $G/H \cap K$ is an abelian group that is not cyclic.

The fact that $H \cap K$ is normal in G is true in general for normal subgroups H, K of G: We know $H \cap K$ is a subgroup, and if $x \in H \cap K, g \in G$ then $gxg^{-1} \in H$ since H is normal, and likewise for K. Thus $gxg^{-1} \in H \cap K$ and $H \cap K$ is normal.

Now we have $H \notin K$, since otherwise we'd have $[G : H] = [G : K][K : H] > [G : K] = 2$. Choose $h \in H \setminus K$. Then we have $G = K \coprod hK$ since $[G : K] = 2$. This implies in particular that $G = HK$. Then by the second isomorphism theorem,

$$G/K = HK/K \cong H/H \cap K,$$

so $[H : H \cap K] = 2$. Thus $[G : H \cap K] = [G : H][H : H \cap K] = 4$. But any group of order 4 is abelian, so we see that $G/H \cap K$ is abelian. To see that it is not cyclic, we can produce two distinct subgroups of order 2. We've already seen that $H/H \cap K$ is one, and by symmetry, $K/H \cap K$ is another. These are distinct since, for instance, we have

$$(H/H \cap K) \cdot (K/H \cap K) = G/H \cap K,$$

since $HK = G$, whereas if they were equal their product would be a group of order 2, not 4.

4) For any integer $n > 0$ let $A(n)$ denote the number of non-isomorphic abelian groups of order n. Consider the numbers $A(28), A(5), A(33), A(9), A(32)$. Which is largest? Which is smallest? Are any two equal? Explain.

By the fundamental theorem on finite abelian groups, the rules for computing $A(n)$ for an integer n with prime factorization $p_1^{e_1} \cdots p_k^{e_k}$ are: (a) $A(n) = A(p_1^{e_1}) \times \cdots \times A(p_k^{e_k})$, and (b) for a prime p, $A(p^e)$ is the number of ways of writing e as a sum of positive integers. We have $28 = 2^2 \cdot 7, 5 = 5, 33 = 3 \cdot 11, 9 = 3^2, 32 = 2^5$, thus $A(5) = 1 = A(33) < A(9) = 2 = A(28) < A(32) = 7$.

5) Let G be a finite group and $H \trianglelefteq G$ and $K \trianglelefteq G$. Suppose that $H \cap K = \{1\}$. Show that $|G| \leq [G : H][G : K]$ by constructing an injective homomorphism from G to a group of order $[G : H][G : K]$. When do we have equality?

Let $\alpha : G \to G/H$ and $\beta : G \to G/K$ be the natural maps. Note that $|G/H| = [G : H]$ and $|G/K| = [G : K]$. We know that α is a homomorphism since $\alpha(g_1)\alpha(g_2) = (g_1H)(g_2H) = g_1g_2H = \alpha(g_1g_2)$, and similarly for β. Then $(\alpha \times \beta) : G \to (G/H) \times (G/K)$ is a homomorphism, and $(G/H) \times (G/K)$ is a group of order $[G : H][G : K]$. To see that $\ker(\alpha \times \beta) = \{1\}$, we observe that α maps g to the identity coset in G/H iff $g \in H$, and β maps g to the identity coset in G/K iff $g \in K$; thus, $(\alpha \times \beta)(g) = (H, K)$ iff $g \in (H \cap K) = \{1\}$. Hence $(\alpha \times \beta) : G \to (G/H) \times (G/K)$ is injective, so $|G| \leq |(G/H) \times (G/K)| = [G : H][G : K]$.

Equality holds iff $(\alpha \times \beta)$ is bijective. Since the given conditions imply that $(\alpha \times \beta)$ is injective, we need to find an additional condition which implies that $(\alpha \times \beta)$ is surjective.
We claim that $(\alpha \times \beta)$ is surjective if $G = HK$. For any element $(aH, bK) \in (G/H) \times (G/K)$, we want to find $s \in G$ with $(\alpha \times \beta)(s) = (aH, bK)$. We can write $a^{-1}b = hk$ for $h \in H$ and $k \in K$ (since $G = HK$); letting $s := ah = bk^{-1} \in G$, we find $\alpha(s) = sH = ahH = aH$ and $\beta(s) = sK = bk^{-1}K = bK$, so $(\alpha \times \beta)(s) = (aH, bK)$. Hence $(\alpha \times \beta)$ is surjective. [Conversely, suppose $(\alpha \times \beta)$ is surjective; is it then true that $G = HK$? Yes. For any element $g \in G$, we can find

$s \in G$ such that $(\alpha \times \beta)(s) = (gH, K)$ (since $(\alpha \times \beta)$ is surjective). Then $gs^{-1} \in H$ and $s \in K$, with $(gs^{-1})s = g$.] Hence under the given conditions, $|G| = [G : H][G : K]$ iff $(\alpha \times \beta)$ is surjective, which occurs precisely when $G = HK$.

6) Construct two non-isomorphic non-abelian groups of order 192, each of which contains a normal abelian subgroup of order 8. [Hint: at least one of them can be a direct product of smaller groups.]

Consider $G_1 = \mathbb{Z}_8 \times S_4$ and $G_2 = D_{2\cdot 96} := \langle r, s \mid r^{96} = s^2 = 1, rs = sr^{-1} \rangle$.

For G_1, we have:

- The order of G_1 is $|G_1| = |\mathbb{Z}_8 \times S_4| = |\mathbb{Z}_8|\,|S_4| = 8 \cdot 24 = 192$.
- G_1 is non-abelian: e.g., $([0], (12)) \cdot ([0], (23)) = ([0], (123)) \neq ([0], (132)) = ([0], (23)) \cdot ([0], (12))$.
- G_1 contains a normal abelian subgroup of order 8: Define the subgroup $H_1 := \mathbb{Z}_8 \times \{\mathrm{id}\} \subseteq G_1$; clearly $|H_1| = 8$. For any $(a, \sigma) \in G_1$ and $(b, \mathrm{id}) \in H_1$, we have $(a, \sigma) \cdot (b, \mathrm{id}) = (a + b, \sigma) = (b + a, \sigma) = (b, \mathrm{id}) \cdot (a, \sigma)$; that is, the elements of H_1 commute with every element in G_1. This certainly implies that H_1 is a normal subgroup of G_1.

For G_2 (the group of symmetries of a 96-gon), we have:

- The order of G_2 is $|G_2| = |D_{2\cdot 96}| = 2 \cdot 96 = 192$.
- G_2 is non-abelian: e.g., $rs = sr^{-1} \neq sr$ by definition.
- G_2 contains a normal abelian subgroup of order 8: Define the cyclic subgroup $H_2 := \langle r^{12} \rangle \subseteq G_2$; clearly $|H_2| = 8$. For any $r^{12i} \in H_2$, we have $r^j\, r^{12i}\, (r^j)^{-1} = r^{12i} \in H_2$ and $(sr^j)\, r^{12i}\, (sr^j)^{-1} = sr^j\, r^{12i}\, (r^j)^{-1} s = s\, r^{12i} s = r^{-12i} \in H_2$. Hence $ghg^{-1} \in H_2$ for all $g \in G_2$ and $h \in H_2$, which means H_2 is a normal subgroup of G_2.

Finally, we observe that $r \in G_2$ has order 96, while $([1], (123)) \in G_1$ has order $\mathrm{lcm}(8, 3) = 24$ and no element in G_1 has greater order. This shows that G_1 is not isomorphic to G_2.

[*Note*: There are many other possible answers: $\mathbb{Z}_8 \times D_{2\cdot 12}$, $\mathbb{Z}_8 \times \mathbb{Z}_4 \times S_3$, $\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \times Q_8$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times A_4$, ...]

Midterm 2 Solutions

nolander

April 2020

1 Introduction

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since $HK = G$, whereas if they were equal their product would be a group of order 2, not 4.

An example where this occurs is the quaternion group Q_8. The subgroups $\langle i \rangle, \langle j \rangle, \langle k \rangle$ are all distinct subgroups of index 2.

4. For any integer $n > 0$ let $A(n)$ denote the number of non-isomorphic abelian groups of order n. Consider the numbers $A(28), A(5), A(33), A(9), A(32)$. Which is largest? Which is smallest? Are any two equal? Explain.

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