INTRODUCTION TO MODERN ALGEBRA I, GU4041, SPRING 2020

Practice Final, May 2020

1. True or False? If false, give a counterexample; if true, provide an explanation. The explanation can be brief but it is not enough to say that the statement was explained in the course.

(a) A group of order 392 has either 1 or 8 Sylow 7-subgroups.

(b) For any n let $A(n)$ denote the number of distinct non-isomorphic abelian groups of order n. Then $A(65) > A(64)$.

(c) Let G be a group of even order. Then it has at least one conjugacy class, not including the identity element, with an odd number of elements.

(d) Let H be a subgroup of the alternating group A_5. Suppose H contains every 3-cycle. Then $H = A_5$.

2. (a) Determine the centralizer of the product $(12)(34)(56)(78)$ of four 2-cycles in S_8. Use this to determine the number of all elements of S_8 that can be written as products of four 2-cycles.

(b) Determine the centralizer of the product $(12)(34)(56)(78)$ of four 2-cycles in S_{12}. Use this to determine the number of all elements of S_{12} that can be written as products of four 2-cycles.

3. How many elements of order 5 are there in $S_5 \times \mathbb{Z}_{25}$?

4. (a) What is the number of conjugacy classes of the dihedral group D_{2n}? Prove your answer, and note that it depends on whether n is odd or even.

(b) Write down the class equation for D_{2n} and identify the centralizer of each element.

5. Let G be a finite group, $N \subseteq G$ a normal subgroup. Let $H = G/N$ be the quotient group, and let $\pi : G \to H$ denote the quotient map.

Let X denote the set of conjugacy classes in the group N. In other words, two elements $n_1, n_2 \in N$ are in the same conjugacy class if there is an element $n \in N$ such that $n \cdot n_1 \cdot n^{-1} = n_2$. The conjugacy class of an element $n \in N$ is denoted $[n]$.

(a) Show that G acts on the set X by conjugation: if $n \in N$, and $g \in G$, then $g([n])$ is the conjugacy class $[gng^{-1}]$. Show that this action is well-defined: in other words, if $[n_1] = [n_2]$ then $g([n_1]) = g([n_2])$. (Warning: do not confuse conjugacy in G with conjugacy in N.)

(b) Write down the class equation for the action of G on X.

(c) Suppose N is abelian. Show that there is an action of H on X such that, for all $n \in N$, $g \in G$,

$$g([n]) = \pi(g)([n]).$$

6. (15 points) Construct two non isomorphic non-abelian groups of order 168, each of which contains a normal abelian subgroup of order 8. (Hint: try to use direct products of smaller groups.)

7. Show that there are no simple groups of order 38 and 40.

8. Let p be a prime number and let G be a finite p-group. Write down the steps of the proof that G is solvable.

9. Write down the class equation for the groups K_4, Q_8, and S_4.

10. Let G be a group, $H \subseteq G$, $K \trianglelefteq G$ two subgroups, with K normal. Suppose the derived subgroup $D(H) \subseteq H$ is strictly smaller than H and $H \cap K = \{e\}$.

Prove that $H \cdot K$ has a normal subgroup J such that $H \cdot K/J$ is abelian and $|H \cdot K/J| > 1$.