INTRODUCTION TO MODERN ALGEBRA I, GU4041, SPRING 2020

Midterm I, February 27, 2020

For any positive integer m, we denote by $\mathbb{Z}_m$ a cyclic group with m elements.

1. True or False? (Each question is worth 3 points.) If false, give a counterexample; if true, provide an explanation. The explanation can be brief but it is not enough to say that the statement was explained in the course.

(a) For any three sets A, B, C,

$$A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C).$$

(b) If H and J are subgroups of a group G, then so is $H \cup J$.

(c) $108 \equiv -3 \pmod{37}$.

(d) Let A, B, C be sets, and let $f : A \to B$ be injective and $g : B \to C$ be surjective. Then $g \circ f : A \to C$ is bijective.

(e) If n is an integer write $[n] = [n]_5$ for its residue class modulo 5 (i.e., its image in $\mathbb{Z}_5$). Let $f : \mathbb{Z}_5 \to \mathbb{Z}_5$ be the map that takes $[n]$ to $[3n]$. Then f is a bijection.

2. (15 points) (a) Carry out the following operations in modular arithmetic.

(i) In arithmetic modulo 35, find the number a between 1 and 35 such that

$$41 + 76 \equiv a \pmod{35}.$$

(ii) In arithmetic modulo 10 find the number a between 1 and 10 such that

$$1000000000001^2 \equiv a \pmod{10}.$$

(b) List the elements of the group $\mathbb{Z}_6$ that are *not* generators.

3. (15 points) Which of the following is an equivalence relation? Justify your answer.

(a) On the set X of residents of New York City, we say $a \sim b$ if a and b live on the same street.

(b) Let N be an integer. On the set $\mathbb{N}$ of natural numbers, we say $a \sim b$ if $gcd(a, N) = gcd(b, N)$.

(c) On the set $\mathbb{C}$ of complex numbers, we say $a \sim b$ if $a - b$ is the square of an integer.

4. (20 points) Let G be a group and let g, h, and j be elements of G. Prove carefully that if $jghj = jhgj$ then g and h commute.

5. (a) (15 points) Let $\mathbb{R}^{\times}$ be the group of non-zero real numbers under multiplication. Find a finite subgroup of $\mathbb{R}^{\times}$ that contains more than one element.

(b) Extra credit: show that the subgroup you found in (a) and the subgroup with one element are the only finite subgroups of $\mathbb{R}^{\times}$.

6. (20 points) List the sets of cyclic subgroups of $\mathbb{Z}_3 \times \mathbb{Z}_3$ and of $\mathbb{Z}_3 \times \mathbb{Z}_2$.