INTRODUCTION TO MODERN ALGEBRA I, GU4041, SPRING 2020

Practice Midterm 2

1. True or False? If false, give a counterexample; if true, provide an explanation. The explanation can be brief but it is not enough to say that the statement was explained in the course.

(a) No group of order 88 has a subgroup of order 16.

(b) There are exactly three non-isomorphic groups of order 8.

2. (a) Let S_8 denote the symmetric group on 8 letters, and let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 1 & 4 & 7 & 8 & 5 & 2 \end{pmatrix}$$

Write the cycle decomposition of σ.

(b) Let G be a finite group with 8 elements, and consider the homomorphism from the proof of Cayley's theorem

$$\alpha : G \to S_8,$$

where we view S_8 as the symmetric group of permutations of G, given by

$$\alpha(g)(h) = g \cdot h.$$

Show that there is no $g \in G$ such that $\alpha(g) = \sigma$ where σ is as in part (a).

3. (a) List the normal subgroups of the dihedral group D_{34}. For which integers m is there a surjective homomorphism $\alpha : D_{34} \to \mathbb{Z}_m$? Suppose we don't require α to be surjective?

(b) Quote a theorem that asserts that there is an isomorphism

$$\beta : \mathbb{Z}_9 \times \mathbb{Z}_2 \xrightarrow{\sim} \mathbb{Z}_{18}.$$

(c) Let D_{36} be the dihedral group of symmetries of the regular 18-gon. We view $\mathbb{Z}_{18}$ as the subgroup of rotations in D_{36} and let $f \in D_{36}$ denote the reflection in the vertical axis. Let $K = \beta(\mathbb{Z}_9) \subseteq \mathbb{Z}_{18}$, with β as in (b). Show that the subgroup $H \subseteq D_{36}$ generated by f and K is normal, and determine the group D_{36}/H.

4. List all the non-isomorphic abelian groups of order 75, 76, 77, and 72.

5. Let N, N' be subgroups of a group G, such that $N \triangleleft N'$. Let H be any subgroup of G. Let $K = N' \cap H$.

(a) Show that $N \cap H \triangleleft K$.

(b) Show that KN is a subgroup of G.

(c) Show that $K/(N \cap H)$ is isomorphic to a subgroup of N'/N. (Hint: Use the appropriate isomorphism theorem.)

6. (a) How many elements of each order are there in the alternating group A_5?

(b) Show by using (a) that A_5 is not isomorphic to the direct product $D_{10} \times S_3$.