SYMMETRIES OF REGULAR POLYHEDRA

1. The cube (regular hexahedron)

Any polyhedron with six faces is called a hexahedron; the cube is the (only) one with the property that all of its faces are polygons with the same angles and side lengths. The cube also has eight vertices (corners) and twelve 1-dimensional edges. You can check for yourself that *Euler's formula* holds for the cube:

$$V - E + F = 2$$

where V is the number of vertices, E the number of edges, and F is the number of faces. This is a purely *topological* property of any closed convex polyhedron; it has nothing to do with lengths or angles or even the number of faces.

We call the cube C and we let R denote the group of rotations of C. The elements of R are compositions of rotations around one of the axes between the midpoints of opposite faces. If we assume the vertices of C are the points

$$(1,1,1), (1,1,-1), (1,-1,1), (-1,1,1),$$
$$(1,-1,-1), (-1,1,-1), (-1,-1,1), (-1,-1,-1)$$

in Euclidean space, then the center is the origin $(0,0,0)$. The group R then has three generators: the rotations of 90° (clockwise, to be precise) around the x, y, and z-axes. Each of these generators is of order 4 (rotate 90° four times and you are back where you started.)

We prove in a moment that R is a group of order 24. By way of contrast, let *Rb* be the group of operations on the Rubik's cube. It has six generators – the 90° rotations around each of the six faces and its order is

$$43252003274489856000 = 2^{27}3^{14}5^{3}7^{2}11$$

which seems ridiculously large until you realize that you can construct a very large non-abelian group with just two generators.

The group R permutes the faces *transitively*: given any two faces A and B, there is a rotation r such that $r(A) = B$. This shows that R has at least 6 elements, which is the number of faces; but it also permutes the vertices and the edges transitively (it's easier to convince yourself with your own cube), so $|R| \geq 12$. And it permutes the diagonals of all six faces (of which there are 12) and the lines between midpoints of opposing edges (see Figure 5.29 on p. 72 of Judson's book) as well as the long diagonals, such as the one between $(-1,-1,-1)$ and $(1,1,1)$ (see Figure 5.28 on p. 72 of Judson's book).

Proposition 1.1. *The group* $|R|$ *has order* 24.

Proof. We use the action on the six faces. Let's call them F, B, U, D, L, and r. For every face X, let R_X denote the subset of $\sigma \in R$ such that $\sigma(U) = X$. Then R_U is the subgroup of rotations around the z axis (U stands for "up"), and we will write $H = R_U$ to avoid confusion.

For each X, choose $r_X \in R_X$; so $r_X(U) = X$. Suppose r' is another element of R_X. Thus

$$r'(U) = X = r_X(U)$$

which implies

$$(r_X^{-1} \cdot r')(U) = U;$$

in other words $r' \in r_X \cdot H$. Thus for any X, R_X is the left coset $r_X \cdot H$.

It follows that $|R_X| = |H| = 4$ for each X. On the other hand,

$$R = \coprod_X R_X = R_F \coprod R_B \coprod R_U \coprod R_D \coprod R_L \coprod R_r$$

and thus

$$|R| = \sum_X |R_X| = 4 + 4 + 4 + 4 + 4 + 4 = 24,$$

which concludes the proof. □

Now we think of the symmetric group S_4 as the group of permutations of the long diagonals. The group R permutes the long diagonals, and this defines a homomorphism

$$h : R \to S_4.$$

Proposition 1.2. *The homomorphism h is an isomorphism.*

Proof. Since both groups have order 24, it suffices to show that h is surjective. (One could also try to prove that it's injective, but this requires more visual imagination.) The proof in Judson's book shows that the image of h contains a transposition, which is obtained by rotating around the line between opposite midpoints. But this is also tricky to visualize. Here is an easier argument. Let $J = h(R)$ be the image of R.

Step 1. Show that J contains all the 4-cycles.
Step 2. Show that J contains all the products of two disjoint 2-cycles.
Step 3. Show that J contains at least two disjoint 3-cycles.
Step 4. Count.

Step 1 is easy. Any 4-cycle can be written to start with 1, so in the form $\begin{pmatrix}1 & a & b & c\end{pmatrix}$, where (a, b, c) is some permutation of $(2, 3, 4)$. There are six such permutations, and thus six 4-cycles. On the other hand, rotation clockwise 90° around each of the axes defines three 4-cycles, and rotation counterclockwise defines the other 3. (Here you should check with a cube, with the long diagonals numbered as in Figure 5.28 of Judson's book, that these are all distinct.)

Step 2 is just as easy: the square of any 90° rotation is a 180° rotation, which exchanges opposite corners. Again, checking on the cube, you see that all three such elements of S_4 are contained in J.

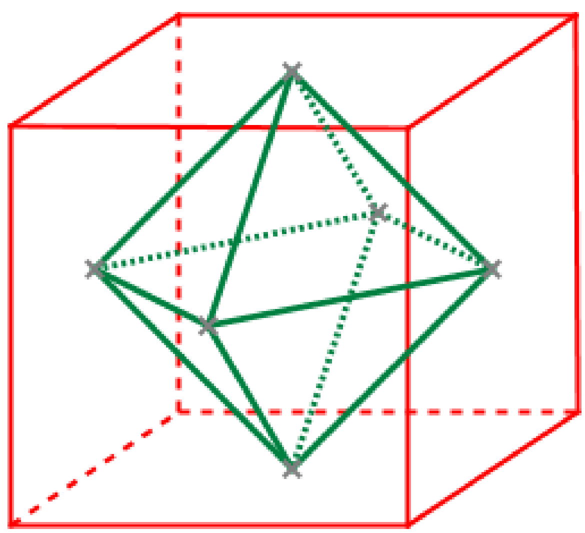
Step 3 can be computed: if $\sigma = \begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix}$ and $\tau = \begin{pmatrix}1 & 4 & 2 & 3\end{pmatrix}$ are two 4-cycles, then $\sigma\tau = \begin{pmatrix}2 & 4 & 3\end{pmatrix}$; thus R contains both $\begin{pmatrix}2 & 4 & 3\end{pmatrix}$ and its inverse $\begin{pmatrix}2 & 3 & 4\end{pmatrix}$. Similarly,

$$\begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix} \cdot \begin{pmatrix}1 & 3 & 2 & 4\end{pmatrix} = \begin{pmatrix}1 & 4 & 2\end{pmatrix}$$

Now we see that J contains six 4-cycles, three elements of the form $(ab)(cd)$ with a, b, c, and d all distinct; and at least four 3-cycles, as well as the identity. Thus $|J| \geq 6 + 3 + 4 + 1 = 14$ and the only such number that divides 24 is 24.

□

The octahedron. The regular octahedron (eight-faced polyhedron) has six vertices and eight faces, so by Euler's formula it must have twelve edges. You can construct a regular octahedron starting with a cube as follows. The vertices are the centers of the six faces of the cube. There are fifteen ways to choose two out of six vertices. Draw the corresponding fifteen segments connecting pairs of vertices, then remove the three that join opposite faces of the cube. Twelve edges remain, forming eight triangles. Fill in the triangles: this is the octahedron. Since every rotation of the cube preserves the vertices of the octahedron, we see that S_4 is a subgroup of the group of rotations of the octahedron; but it is easy to see that every rotation of the octahedron also preserves the faces of the cube, so the two figures have the same rotation group.



The symmetries of the remaining regular polyhedra are discussed in the notes `http://www.math.columbia.edu/~harris/website/content/2-courses1/1-mathematics-gu4041-spring-2020/platonic-solids.pdf`.