Modern Algebra I   HW 1 Solutions

Problem 1 First we show $X \setminus (Y \cup Z) \subseteq (X \setminus Y) \cap (X \setminus Z)$. Suppose $x \in X$, but x is not in the union of Y and Z. Then x is neither in Y nor in Z.

$$\left.\begin{array}{l} x \in X,\ x \notin Y \Rightarrow x \in X \setminus Y \\ x \in X,\ x \notin Z \Rightarrow x \in X \setminus Z \end{array}\right\} \Rightarrow x \in (X \setminus Y) \cap (X \setminus Z).$$

Now we show $(X \setminus Y) \cap (X \setminus Z) \subseteq X \setminus (Y \cup Z)$. Suppose $x \in (X \setminus Y) \cap (X \setminus Z)$. Then $x \in X \setminus Y$ and $x \in X \setminus Z$. This means that $x \in X$, but neither in Y nor in Z. Then $x \in X$ but $x \notin Y \cup Z$. So, $x \in X \setminus (Y \cup Z)$. □

Problem 2 First we show $A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)$. Suppose $a \in A$, but a is not in the intersection of B and C. Then either $a \notin B$ or $a \notin C$.

If $a \notin B$, then $a \in A \setminus B$. So, $a \in (A \setminus B) \cup (A \setminus C)$ by the definition of union.

Similarly, if $a \notin C$, then $a \in A \setminus C$ and then $a \in (A \setminus B) \cup (A \setminus C)$.

Now we show $(A \setminus B) \cup (A \setminus C) \subseteq A \setminus (B \cap C)$. Suppose $a \in (A \setminus B) \cup (A \setminus C)$. Then $a \in A \setminus B$ or $a \in A \setminus C$. If $a \in A \setminus B$, then $a \in A$ but not in B. If a is not in B, it also can't be in the intersection of B and C. So, $a \in A \setminus (B \cap C)$. Similarly, $a \in A \setminus C$ implies $A \setminus (B \cap C)$. □

Problem 3 (a) A has 8 subsets $\rightarrow$ $\phi = \{\}$, $\{22\}$, $\{100\}$, $\{5\}$, $\{22, 100\}$, $\{100, 5\}$, $\{22, 5\}$, $\{22, 100, 5\} = A$.

6 of these subsets (all except ϕ and $\{5\}$) contain an even number.

(b) • If "between" includes endpoints: $B = \{1, 2, 3, \ldots, 100\}$.

$\#$ 0-element subsets of $B = 1$ (empty set)

$\#$ 1-element subsets of $B = |B| = 100$

$\#$ 2-element subsets of $B = \binom{|B|}{2} = \binom{100}{2} = \frac{100 \times 99}{2} = 4950$

Total 5051 subsets of at most 2 elements.

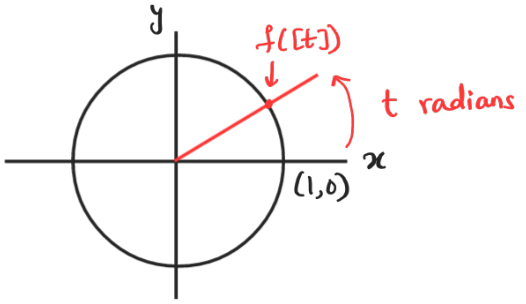
• If "between" doesn't include endpoints: $B = \{2, 3, \ldots, 99\}$.

Then the answer is $1 + 98 + \frac{98 \times 97}{2} = 4852$ □

Problem 4 (a) We need to show that $\sim$ is reflexive, symmetric and transitive.

- Reflexive $\rightarrow$ t radians is the same angle as itself $\Rightarrow t \sim t$.
- Symmetric $\rightarrow$ $t \sim t' \Leftrightarrow$ t and t' radians are the same angle $\Leftrightarrow$ t' and t radians are the same angle $\Leftrightarrow t' \sim t$.
- Transitive $\rightarrow$ If $t \sim t'$ and $t' \sim t''$, then t, t', t'' radians are all the same angle. So, $t \sim t''$.

(b) Given $t \in \mathbb{R}$, let $[t]$ be its equivalence class in $\mathbb{R}/\sim$. Define $f([t])$ to be the point on the unit circle corresponding to an angle of t radians as shown in the picture below. Note that if $[t] = [t']$,



then t and t' radians are the same angle. So, $f([t]) = f([t'])$ and f is a well-defined function on $\mathbb{R}/\sim$.

If $f([t]) = f([t'])$, then t and t' radians must be the same angle, and $[t] = [t']$. So, f is injective.

For any given point P on the unit circle, let t be the length of the arc from $(1,0)$ to P. Then the angle formed by P, the origin, the positive x-axis is t radians and $P = f([t])$. So, f is surjective.

(c) $f([t]) = (\cos t, \sin t)$. $\square$

Problem 5 A function from ϕ to S is a relation on ϕ and S, i.e., a subset R of $\phi \times S$ such that for every $x \in \phi$, there is a unique $y \in S$ with $(x,y) \in R$.

At most one function $\rightarrow$ Now $\phi \times S = \phi$ and the only subset of ϕ is ϕ itself. So, if there is a function from ϕ to S, it must be the empty function ϕ.

At least one function $\rightarrow$ The statement "for every $x \in \phi$, there is a unique $y \in S$ with $(x,y) \in \phi$" is vacuously true. So, the relation ϕ on ϕ and S certainly defines a function.

Thus, there is a unique function $f : \phi \to S$ and its graph is ϕ.

f fails to be injective if there are $x_1 \neq x_2$ with $f(x_1) = f(x_2)$. Since ϕ doesn't have any elements, this never happens. f is injective for all S.

For f to be bijective, the following must hold:

For every $y \in S$, there is $x \in \phi$ such that $f(x) = y$.

This is vacuously true for $S = \phi$ and false if S is non-empty (there is no $x \in \phi$). So, f is bijective if and only if $S = \phi$.

$\square$

Problem 6 (a) ϕ is surjective.

$\Leftrightarrow$ For every $(e,f) \in \mathbb{R}^2$, there is $(x,y) \in \mathbb{R}^2$ such that $\phi(x,y) = (ax+by, cx+dy) = (e,f)$.

$\Leftrightarrow$ For every $e \in \mathbb{R}$ and $f \in \mathbb{R}$, there are $x \in \mathbb{R}$ and $y \in \mathbb{R}$ such that $ax+by = e$ and $cx+dy = f$.

$\Leftrightarrow$ For all $e \in \mathbb{R}$ and $f \in \mathbb{R}$, the system of linear equations

$$L_1: ax+by = e \quad , \quad L_2: cx+dy = f$$

has a solution.

(b) ϕ is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ and given by the matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Alternatively, show surjective $\Leftrightarrow ad-bc \neq 0$ $\Leftrightarrow$ injective by solving linear equations L_1, L_2

By the rank-nullity theorem, $\dim(\ker(\phi)) + \dim(\mathrm{im}(\phi)) = 2$.

ϕ is surjective $\Leftrightarrow$ The image of ϕ is $\mathbb{R}^2$

$\Leftrightarrow \dim(\mathrm{im}(\phi)) = 2$

$\Leftrightarrow \dim(\ker(\phi)) = 0$

$\Leftrightarrow \phi$ is injective.

(c) Consider $a=b=c=d=1$. Then $ax+by = cx+dy = x+y$.

$$\phi(x,y) = (x+y, x+y).$$

Then the image of ϕ is the set of points of the form $(x,x) \in \mathbb{R}^2$.

This is the line $x=y$ in the xy-plane.

$\square$

Problem 7 (a) Functions from B to A → For each element of B, there are $|A|$ outputs / images to choose from. So, there are

$$\underbrace{|A| \times |A| \times \cdots \times |A|}_{|B| \text{ times}} = |A|^{|B|} = 4^2 = 16 \text{ functions.}$$

By the same argument, there are $2^4 = 16$ functions $A \to B$.

(b) $c(1) = 2$ and $c(2) = 1 \Rightarrow c(c(1)) = 1$ and $c(c(2)) = 2$.

So, $c \circ c$ is the identity function on B.

Then for any $f \in F$, $C(C(f)) = c \circ c \circ f = f$.

This shows that for every $f \in F$, there is a function $C(f)$ in F which gets mapped to f by C. So, C is bijective.

Also, $C(f_1) = C(f_2) \Rightarrow C(C(f_1)) = C(C(f_2)) \Rightarrow f_1 = f_2$, so C is injective as well.

Finally, suppose $C(f) = f$. Then $c \circ f = f$. In particular, $c(f(u)) = f(u)$.

This is impossible since c changes all elements of B including $f(u)$.

There is no function $f \in F$ such that $C(f) = f$.

$\square$