Modern Algebra I HW 10 Solutions

Problem 1 Exercise 2: $X_g = \{x \in X \mid gx = x\}$, $G_x = \{g \in G \mid gx = x\}$.

$\uparrow$ Elements fixed by g $\uparrow$ Elements which fix x.

(a) $X = \{1,2,3\}$, $G = S_3 = \{(1),(12),(13),(23),(123),(132)\}$

$X_{(1)} = X$, $X_{(12)} = \{3\}$, $X_{(13)} = \{2\}$, $X_{(23)} = \{1\}$

$X_{(123)} = X_{(132)} = \phi$.

$G_1 = \{(1),(23)\}$, $G_2 = \{(1),(13)\}$, $G_3 = \{(1),(12)\}$.

Each G_i is isomorphic to $\mathbb{Z}_2$.

(b) $X = \{1,2,3,4,5,6\}$, $G = \{(1),(12),(345),(354),(12)(345),(12)(354)\}$.

$X_{(1)} = X$, $X_{(12)} = \{3,4,5,6\}$, $X_{(345)} = X_{(354)} = \{1,2\}$,

$X_{(12)(345)} = X_{(12)(354)} = \{6\}$.

$G_1 = G_2 = \{(1),(345),(354)\} \cong \mathbb{Z}_3$

$G_3 = G_4 = G_5 = \{(1),(12)\} \cong \mathbb{Z}_2$

$G_6 = G$. $\square$

Exercise 3: $x \sim y$ if there is $g \in G$ with $gx = y$.

(a) $X = \{1,2,3\}$, $G = S_3 = \{(1),(12),(13),(23),(123),(132)\}$

$O_1 = O_2 = O_3 = X = \{1,2,3\}$ and $|G| = 6 = 3 \times 2 = |O_x| \cdot |G_x|$ for any $x \in X$.

(b) $X = \{1,2,3,4,5,6\}$, $G = \{(1),(12),(345),(354),(12)(345),(12)(354)\}$.

$O_1 = O_2 = \{1,2\}$ and $|G| = 6 = 2 \times 3 = |O_x| \cdot |G_x|$ for $x = 1,2$.

$O_3 = O_4 = O_5 = \{3,4,5\}$ and $|G| = 6 = 3 \times 2 = |O_x| \cdot |G_x|$ for $x = 3,4,5$.

$O_6 = \{6\}$ and $|G| = 6 = 6 \times 1 = |O_6| \cdot |G_6|$. $\square$

Problem 2 • $\mathbb{Z}_{12}$ is abelian. So, $gag^{-1} = a$ for every $a, g \in \mathbb{Z}_{12}$. Then there are $|\mathbb{Z}_{12}| = 12$ conjugacy classes of $\mathbb{Z}_{12}$, each containing a single element. The center of $\mathbb{Z}_{12}$ is $\mathbb{Z}_{12}$ and the class equation is satisfied.

- $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ under multiplication given by
  $i^2 = j^2 = k^2 = ijk = -1$, $(-1)^2 = 1$.

  $jij^{-1} = -jij = -jk = -i$

  $Z(Q_8) = \{\pm 1\}$

  The conjugacy classes are: $\{1\}$, $\{-1\}$,

  $\{i, -i\}$ $\quad jij^{-1} = j^{-1}ij = -jij = -jk = -i$,
  $kik^{-1} = k^{-1}ik = -kik = -jk = -i$.

  $\{j, -j\}$ $\quad$ Similar computations
  $\{k, -k\}$

  The centralizers are $C_x = C_{-x} = \{\pm 1, \pm x\}$ for $x = i, j, k$, and
  $|Q_8 : C_x| = |Q_8 : C_{-x}| = 2$ = size of the conjugacy class $\{\pm x\}$.

  The class equation is satisfied:
  $$|Q_8| = 8 = |Z(Q_8)| + \sum_{x \in \{i,j,k\}} |G : C_x|$$

- D_{14} is generated by rotation s, reflection f with
  $s^7 = f^2 = (fs)^2 =$ identity e.

  The elements of D_{14} are of the form $e, s, \ldots, s^6, f, fs, \ldots, fs^6$.

  $Z(D_{14}) = \{e\}$.

  $$\left.\begin{aligned} & s^a f s^{-a} = fs^{-2a}. \\ & (fs^a) f (fs^a)^{-1} = fs^a f s^{-a} f = s^{-2a} f = fs^{2a}. \end{aligned}\right\} \Rightarrow \text{All reflections form a conjugacy class.}$$

  $a = 1 \Rightarrow fs^{-2} = fs^5$ and fs^2 are in the conjugacy class of f.

  $a = 2 \Rightarrow fs^{-4} = fs^3$ and fs^4 are in the conjugacy class of f.

  $a = 3 \Rightarrow fs^{-6} = fs$ and fs^6 are in the conjugacy class of f.

  The centralizer of f is $\{e, f\}$. In general, $C_{fs^a} = \{e, fs^a\}$.

  Also,

  $$\left.\begin{aligned} & s^b s^a s^{-b} = s^a \\ & (fs^b) s^a (fs^b)^{-1} = fs^{b+a-b} f = fs^a f = s^{-a} \end{aligned}\right\} \Rightarrow \text{Pairs of rotations } s^{\pm a} \text{ form conjugacy classes.}$$

  The centralizer of any rotation is $C_{s^a} = \{e, s, s^2, \ldots, s^6\}$, the subgroup of rotations.

Summary:

Conjugacy classes are $\{e\}$, $\{fs^a \mid 0 \le a \le 6\}$, $\{s, s^6\}$, $\{s^2, s^5\}$, $\{s^3, s^4\}$ with cardinalities $1, 7, 2, 2, 2$.

$$|D_{14}| = 14 = 1 + 7 + 2 + 2 + 2$$
$$= |Z(D_{14})| + |D_{14} : C_f| + |D_{14} : C_s| + |D_{14} : C_{s^2}| + |D_{14} : C_{s^3}|. \qquad \square$$

Problem 3 (i) When we create a subset of A, we have two choices for each element of A $\rightarrow$ whether to include the element in the subset or exclude it. So, there are 2^n subsets of A.

$$|P(A)| = 2^n. \qquad \square$$

(ii) The identity permutation takes 1 to 1. For any $a \neq 1$, the transposition $(1\,a)$ takes 1 to a. Then the orbit containing 1 is the entire set A. The given group action has a single orbit and that orbit is A. $\square$

(iii) Let $S \in P(A)$ and $\sigma \in \Sigma_n$. Define

$$\sigma(S) = \{\sigma(a) \mid a \in S\}.$$

Clearly, $\sigma(S)$ is a subset of A and lies in $P(A)$. Thus, the above definition gives a map $\Sigma_n \times P(A) \rightarrow P(A)$.

If σ is the identity permutation, then for any $S \in P(A)$,

$$\sigma(S) = \{a \mid a \in S\} = S.$$

Also, for any two permutations σ_1 and σ_2,

$$(\sigma_1\sigma_2)(S) = \{(\sigma_1\sigma_2)(a) \mid a \in S\} = \{\sigma_1(\sigma_2 a) \mid a \in S\}$$
$$= \sigma_1(\{\sigma_2 a \mid a \in S\}) = \sigma_1(\sigma_2(S)).$$

So, we have a well-defined action of Σ_n on $P(A)$.

Note that any permutation takes distinct elements of A to distinct elements. This means that for all σ and S, $|\sigma(S)| = |S|$. On the other hand, if $|S_1| = |S_2|$ for $S_1, S_2 \in P(A)$, then there is a permutation σ such that $\sigma(S_1) = S_2$.

$$\sigma = \begin{pmatrix} \text{elements of } S_1 & \text{elements of } A \setminus S_1 \\ \text{elements of } S_2 & \text{elements of } A \setminus S_2 \end{pmatrix}$$

For any $S \in P(A)$, the orbit $O_S = \{T \in P(A) \mid |S| = |T|\}$. Thus, there are $n+1$ orbits $O_0, O_1, \dots, O_n$ indexed by possible cardinalities of subsets of A.

$$O_i = \{S \in P(A) \mid |S| = i\}.$$

$$|O_i| = \binom{n}{i} = \frac{n!}{i!\,(n-i)!}.$$

□

(iv) $$P(A) = \bigcup_{i=0}^{n} O_i = \bigcup_{i=0}^{n} \{S \in P(A) \mid |S| = i\}$$

and $$|P(A)| = \sum_{i=0}^{n} |O_i| \Rightarrow 2^n = \sum_{i=0}^{n} \binom{n}{i}.$$

By the binomial theorem,

$$2^n = (1+1)^n = \sum_{i=0}^{n} \binom{n}{i} 1^i 1^{n-i} = \sum_{i=0}^{n} \binom{n}{i}$$

□

<u>Problem 4</u> Let G be a finite group with exactly two conjugacy classes. Suppose that $|G| = n$. We know that the identity element of G is in its own conjugacy class and there is only one other conjugacy class. Then all non-identity elements of G must form a single conjugacy class of order $n-1$. Then $n-1$ must divide $n = (n-1)+1$. So, $n-1$ must divide 1. This is possible only if $n-1 = 1$ and $n = 2$. So, G must be isomorphic to $\mathbb{Z}_2$.

It is easy to see that $\mathbb{Z}_2$ has exactly two conjugacy classes.

<u>Conclusion</u>: $\mathbb{Z}_2$ is the only finite group with exactly two conjugacy classes.

□