Intro Modern Algebra I HW11 Solution

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Problem 1

Exercise 13.12

Proof. Denote the descending central series of N and G/N by

$$N \triangleright N_1 \triangleright ... \triangleright N_n \triangleright \{1\}$$

and

$$G/N \triangleright H_1 \triangleright ... \triangleright H_m \triangleright \{1\}$$

By correspondence theorem we can lift the subgroups $H_1, ..., H_m$ of G/N uniquely to subgroups $G_1, ..., G_m$ of G such that

$$G \triangleright G_1 \triangleright ... \triangleright G_m \triangleright N$$

is a sequence of normal subgroups. Furthermore each $G_i/G_{i+1} = H_i/H_{i+1}$ is abelian. Therefore G has a subnormal series

$$G \triangleright G_1 \triangleright ... \triangleright G_m \triangleright N \triangleright N_1 \triangleright ... \triangleright N_n \triangleright \{1\}$$

where each quotient is abelian. □

Problem 2

Proof. Let G be a solvable group with subnormal series

$$G \triangleright G_1 \triangleright ... \triangleright G_n \triangleright \{1\}$$

Let H be any subgroup of G. For every $i = 1, 2, \ldots, n$ we have:

$$(H \cap G_i) \cap G_{i-1} = H \cap G_{i-1}$$

From the Second Isomorphism Theorem for Groups:

$$\frac{(H \cap G_i)\, G_{i-1}}{G_{i-1}} \cong \frac{H \cap G_i}{(H \cap G_i) \cap G_{i-1}} = \frac{H \cap G_i}{H \cap G_{i-1}}$$

In particular, $H \cap G_{i-1}$ is a normal subgroup of $H \cap G_i$. We have that:

$$(H \cap G_i)\, G_{i-1} \subseteq G_i$$

and so from the Correspondence Theorem:

$$\frac{(H \cap G_i)\, G_{i-1}}{G_{i-1}} \leq G_i / G_{i-1}$$

We have that G_i/G_{i-1} is abelian. Thus from Subgroup of Abelian Group is Abelian:

$$\frac{(H \cap G_i)\, G_{i-1}}{G_{i-1}} \text{ is abelian.}$$

Hence $\frac{H \cap G_i}{H \cap G_{i-1}}$ is abelian. Therefore, the series :

$$\{e\} = H \cap G_0 \lhd H \cap G_1 \lhd \cdots \lhd H \cap G_n = H$$

is a normal series with abelian factor groups for H. Therefore H is solvable.

□

Problem 3

$S_3 \rhd \mathbb{Z}_2 \rhd \{1\}$ is solvable but it is not abelian, and in particular the center of S_3 is just the identity element since any permutation that commutes with (1 2 3) is disjoint from (1 2 3) and thus can only be the identity.

Problem 4

(a)

Obviously the identity matrix is in H. Given any

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{pmatrix} \in H,$$

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+x' & z+z'+xy' \\ 0 & 1 & y+y' \\ 0 & 0 & 1 \end{pmatrix} \in H$$

so H is closed under multiplication. Also one can verify that

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -x & xy-z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix}$$

is also in H. Therefore H is a group.

(b)

For an element $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$ to be in the center,

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

so $z+z'+xy' = z'+z+x'y$ and $xy' = x'y$. For this to hold for all $x', y' \in \mathbb{R}$, it must be that $x = y = 0$, so the center is the set of

$$\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, z \in \mathbb{R}.$$

(c)

Let the subnormal series be $H \triangleright Z(H) \triangleright \{1\}$. Obviously $Z(H)/\{1\} \subseteq Z(H/\{1\})$, so we just have to show $H/Z(H) \subseteq Z(H/Z(H))$ or equivalently $[H,H] \subseteq Z(H)$. Indeed,

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -x & xy-z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -x' & x'y'-z' \\ 0 & 1 & -y' \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & xy'-xy-x'y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in Z(H)$$

So H is nilpotent.

(d)

The subgroup

$$\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, z \in \mathbb{Z}.$$

Is an abelian group that is different from $Z(H)$.

Problem 5

Proof. Let G be a group of order $2p$, then it has a p-Sylow subgroup H of order p, and since it has index 2 it is normal. Also since $|H| = p$ it is isomorphic to $\mathbb{Z}_p$. Write $H = \langle y \mid y^p = 1 \rangle$ and pick $x \in G \backslash H$. Since $[G : H] = p$ and $H \subsetneq \langle x, y \rangle \subset G$ it must be that $\langle x, y \rangle = G$, and since x does not have order p it must have order 2. So $G = \langle x, y \mid x^2 = y^p = 1 \rangle$. Since H is normal we know $xyx^{-1} \in H$ so $xyx^{-1} = y^t$ for some t.

Now

$$\begin{aligned} x = e^{-1}xe = \left(y^2\right)^{-1} xy^2 = y^{-1}\left(y^{-1}xy\right) y = y^{-1}x^t y \\ = \underbrace{\left(y^{-1}xy\right) \cdot \left(y^{-1}xy\right) \cdots \left(y^{-1}xy\right)}_{\text{t times}} = \left(x^t\right)^t = x^{t^2} \end{aligned}$$

So $p | t^2 - 1 = (t+1)(t-1)$ and either $p | t+1$ or $p | t-1$. The only possibilities then are $t = 1$ or $t = p-1$, which gives us $\mathbb{Z}_p$ or D_{2p} respectively. □

Optional Problem

Proof. We know any composition factor must have order p^k for some $k < r$. Now pick any composition factor H, since H is a p-group, $Z(H)$ has order at least p which is nontrivial. But since H has to be simple we know $H = Z(H)$ is abelian. Now take any $e \neq h \in H$ we know that $\langle h \rangle \triangleright H$ is nontrivial so $H = \langle h \rangle$ is cyclic, but a cyclic group is only simple when the order is prime, so it must be that $H \cong \mathbb{Z}_p$. □