# Modern Algebra I Problem Set 6 Answer Key

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## 1 Problem 1

(a). The cosets of $\{0, 2, 4\}$ in $(\mathbb{Z}_6, +)$ are:

- $\{0, 2, 4\} + 0 = \{0, 2, 4\}$
- $\{0, 2, 4\} + 1 = \{1, 3, 5\}$

So the index of $\{0, 2, 4\}$ in $(\mathbb{Z}_6, +)$ is 2.
(b). The cosets of $\mathbb{R}$ in $(\mathbb{C}, +)$ are of the form

$$\mathbb{R} + bi = \{a + bi | a \in \mathbb{R}\},$$

where $b$ is any real number. So the index of $\mathbb{R}$ in $(\mathbb{C}, +)$ is infinity.
(c). The cosets of $3\mathbb{Z}$ in $(\mathbb{Z}, +)$ are:

- $3\mathbb{Z} + 0 = \{\cdots, -6, -3, 0, 3, 6, \cdots\}$
- $3\mathbb{Z} + 1 = \{\cdots, -5, -2, 1, 4, 7, \cdots\}$
- $3\mathbb{Z} + 2 = \{\cdots, -4, -1, 2, 5, 8, \cdots\}$ So the index of $3\mathbb{Z}$ in $(\mathbb{Z}, +)$ is 3.

## 2 Problem 2

### 2.1 Exercise 1

(a). This is a subgroup of $S_4$, because it is closed under multiplication. However, it is not a normal subgroup because

$$(1\ 2) \cdot (1\ 3\ 4) \cdot (1\ 2) = (2\ 3\ 4).$$

(b). This is a subgroup of $S_4$ because it is closed under multiplication. Moreover, it is normal subgroup because it is closed under conjugation. Indeed, for any $\sigma \in S_4$, we have

$$\sigma(i\ j)(k\ l)\sigma^{-1} = \Big(\sigma(i)\ \sigma(j)\Big)\Big(\sigma(k)\ \sigma(l)\Big),$$

where $i, j, k, l$ is some permutation of $\{1, 2, 3, 4\}$. Note that the right hand side of the equality is a product of two disjoint transpositions, and therefore lies in the subgroup.

(c). This is a subgroup, since one can check that it is closed under multiplication. However, it is not a normal subgroup because

$$(1\ 2) \cdot (1\ 2\ 3\ 4) \cdot (1\ 2) = (2\ 1\ 3\ 4),$$

i.e. it is not closed under conjugation.
(d). This is not a subgroup at all, because

$$(1\ 2\ 3) \cdot (2\ 3\ 4) = (1\ 3)(2\ 4),$$

i.e. it is not closed under multiplication.

In conclusion, (b) satisfies condition (i); (a) and (c) satisfy condition (ii); and (d) satisfies condition (iii).

### 2.2 Exercise 2

(a) The image of $f$ is

$$Im(f) = \{1, i, -1, -i\}.$$

(b). The kernel of $f$ is

$$Ker(f) = \{0, 4, 8\}.$$

(c). The quotient group is

$$\mathbb{Z}_{12}/Ker(f) = \{0 + Ker(f), 1 + Ker(f), 2 + Ker(f), 3 + Ker(f)\}.$$

You may write down the Cayley table for $\mathbb{Z}_{12}/Ker(f)$ and $Im(f)$ yourself and see that they are isomorphic. Indeed, the two groups above are both isomoprhic to the cyclic group $\mathbb{Z}_4$.

## 3 Problem 3

Choose the order 2 subgroup of $S_3$ to be $H = \{\text{id}, (1\ 2)\}$.
(a). Let $g = (2\ 3)$. Then

$$(2\ 3) \cdot (1\ 2) \cdot (2\ 3) = (1\ 3) \notin H.$$

Therefore, $H$ is not a normal subgroup.
(b) The left cosets $S_3/H$ are:

- $\text{id} \cdot H = \{\text{id}, (1\ 2)\}$
- $(1\ 3) \cdot H = \{(1\ 3), (1\ 2\ 3)\}$
- $(2\ 3) \cdot H = \{(2\ 3), (1\ 3\ 2)\}$

The representatives of the left cosets are $\{\text{id}, (1\ 3), (2\ 3)\}$.

Furthermore, the right cosets $H \backslash S_3$ are:

- $H \cdot \text{id} = \{\text{id}, (1\ 2)\}$

- $H \cdot (1\ 3) = \{(1\ 3), (1\ 3\ 2)\}$
- $H \cdot (2\ 3) = \{(2\ 3), (1\ 2\ 3)\}$

The representatives of the right cosets are $\{\text{id}, (1\ 3), (2\ 3)\}$.

# 4 Problem 4

We want to show the following statements are equivalent:

(i) $g_1 H = g_2 H$

(ii) $H g_1^{-1} = H g_2^{-1}$

(iii) $g_1 H \subset g_2 H$

(iv) $g_1 \in g_2 H$

(v) $g_1^{-1} g_2 \in H$.

To proceed, we will show that $(i) \implies (iii) \implies (iv) \implies (v) \implies (ii) \implies (i)$.

- $(i) \implies (iii)$: Obvious.
- $(iii) \implies (iv)$: If $g_1 H \subset g_2 H$, then $g_1 = g_1 \cdot 1 \in g_2 H$.
- $(iv) \implies (v)$: If $g_1 \in g_2 H$, then $g_1 = g_2 \cdot h$ for some $h \in H$. Hence $g_1^{-1} g_2 = h^{-1} \in H$.
- $(v) \implies (ii)$: If $g_1^{-1} g_2 \in H$, then
$$g_1^{-1} g_2 = h \tag{1}$$
for some $h \in H$. We then have $g_1^{-1} = h g_2^{-1}$, implying that $g_1^{-1} \in H g_2^{-1}$, and therefore $H g_1^{-1} \subset H g_2^{-1}$. On the other hand, taking inverses on both sides of equation (1) gives us $g_2^{-1} g_1 = h^{-1}$, so $g_2^{-1} \in H g_1^{-1}$. This shows that $H g_2^{-1} \subset H g_1^{-1}$, and therefore $H g_1^{-1} = H g_2^{-1}$.
- $(ii) \implies (i)$: If $H g_1^{-1} = H g_2^{-1}$, then $g_1^{-1} = h g_2^{-1}$ for some $h \in H$. Taking inverses on both sides, we get $g_1 = g_2 h^{-1}$. This means that $g_1 \in g_2 H$, and therefore $g_1 H \subset g_2 H$. Using a similar argument where we switch $g_1$ and $g_2$, we get $g_2 H \subset g_1 H$. Therefore, $g_1 H = g_2 H$.

# 5 Problem 5

(a) We check that $G$ satisfies all the group axioms:

- Closure: If $A = \begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & \lambda \end{pmatrix}$ and $A' = \begin{pmatrix} a' & b' & e' \\ c' & d' & f' \\ 0 & 0 & \lambda' \end{pmatrix}$, then we compute that
$$A \cdot A' = \begin{pmatrix} aa' + bc' & ab' + bd' & ae' + bf' + e\lambda' \\ ca' + dc' & cb' + dd' & ce' + df' + f\lambda' \\ 0 & 0 & \lambda\lambda' \end{pmatrix},$$
and that
$$((aa' + bc')(cb' + dd') - (ab' + bd')(ca' + dc'))\lambda\lambda' = 1.$$

- Identity: The identity matrix $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ belongs to $G$.

- Inverse: Let $A = \begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & \lambda \end{pmatrix}$ be any element in $G$. Since $\det(A) = (ad - bc)\lambda = 1$, it is invertible and its inverse $A^-1$ also has determinant equal to 1. Moreover, the $(1,3)$ minor and the $(2,3)$ minor of $A$ are 0, so $A^{-1}$ has zeroes on its $(3,1)$ entry and $(3,2)$ entry. This shows that $A^-1$ also belongs to $G$.

- Associativity: Holds because matrix multiplication is associative.

Therefore, $G$ is a group.

(b). For any $B = \begin{pmatrix} 1 & 0 & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix} \in H$ and $B' = \begin{pmatrix} 1 & 0 & e' \\ 0 & 1 & f' \\ 0 & 0 & 1 \end{pmatrix} \in H$, we have

$$B \cdot B' = \begin{pmatrix} 1 & 0 & e + e' \\ 0 & 1 & f + f' \\ 0 & 0 & 1 \end{pmatrix} \in H,$$

so $H$ is closed under multiplication. Furthermore, note that

$$B^-1 = \begin{pmatrix} 1 & 0 & -e \\ 0 & 1 & -f \\ 0 & 0 & 1 \end{pmatrix} \in H,$$

so $H$ is also closed under inverses. This shows that $H$ is a subgroup of $G$, as desired.

(c) We need to check that for any $g \in G$ and $h \in H$, we have $ghg^{-1} \in H$. Using the block matrix notation, we write

$$g = \left(\begin{array}{c|c} X & * \\ \hline 0 & \lambda \end{array}\right), \quad h = \left(\begin{array}{c|c} \mathrm{id} & * \\ \hline 0 & 1 \end{array}\right),$$

where the top left corner of $g$ and $h$ is a 2 by 2 submatrix. Then

$$ghg^{-1} = \left(\begin{array}{c|c} X & * \\ \hline 0 & \lambda \end{array}\right) \left(\begin{array}{c|c} \mathrm{id} & * \\ \hline 0 & 1 \end{array}\right) \left(\begin{array}{c|c} X^{-1} & * \\ \hline 0 & \lambda^{-1} \end{array}\right) = \left(\begin{array}{c|c} \mathrm{id} & * \\ \hline 0 & 1 \end{array}\right) \in H.$$

Therefore, $H$ is a normal subgroup of $G$.

(d). For any $\begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & \lambda \end{pmatrix}$ and $A' = \begin{pmatrix} a' & b' & e' \\ c' & d' & f' \\ 0 & 0 & \lambda' \end{pmatrix}$ in $G$, we check that $\phi(A \cdot A') = \phi(A) \cdot \phi(A')$. On one hand,

$$\phi(A \cdot A') = \phi\left( \begin{pmatrix} aa' + bc' & ab' + bd' & ae' + bf' + e\lambda' \\ ca' + dc' & cb' + dd' & ce' + df' + f\lambda' \\ 0 & 0 & \lambda\lambda' \end{pmatrix} \right) = \begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix}.$$

On the other hand,

$$\phi(A)\cdot\phi(A') = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa'+bc' & ab'+bd' \\ ca'+dc' & cb'+dd' \end{pmatrix}.$$

Thus, $\phi(A\cdot A') = \phi(A)\cdot\phi(A')$ as desired, giving us that $\phi$ is a group homomorphism.

Moreover, $\phi(A)$ is the identity matrix if and only if $a = d = 1$ and $b = c = 0$, which is exactly the condition for which $\phi(A) \in H$.

# 6 Problem 6

The group of rotations of the regular $n$-gon is $S = \{e, s, s^2, \cdots, s^{n-1}\}$. Note that this is a subset of the dihedral group $D_{2n} = \{e, s, s^2, \cdots, s^{n-1}, f, fs, fs^2, \cdots, fs^{n-1}\}$. It is a subgroup because it contains the identity element, and for any elements $s^i, s^j \in S$, we have $s^i \cdot s^{-j} = s^{i-j} \in S$. To show that it is a normal subgroup, it suffices to check that for any $g = fs^k \in D_{2n}$, we have $gHg^{-1} \subset H$. Indeed, for every $s^i \in S$,

$$gs^ig^{-1} = fs^ks^i(fs^k)^{-1} = fs^ks^is^{-k}f = s^{-k}s^is^kff = s^{-k}s^is^k \in H.$$

Therefore, $S$ is a normal subgroup of $D_{2n}$. The quotient group $D_{2n}/S$ consists of two elements: $eH = \{e, s, s^2, \cdots, s^{n-1}\}$, and $fH = \{f, fs, fs^2, \cdots, fs^{n-1}\}$, with representatives $e$ and $f$. It is isomorphic to $\mathbb{Z}_2$.