Modern Algebra I HW 7 Solutions

Problem 1 • Suppose that $n \geq 10$. Then the subgroup of S_n generated by two disjoint 5-cycles $\sigma_1 = (12345)$ and $\sigma_2 = (6,7,8,9,10)$ is isomorphic to $\mathbb{Z}_5 \times \mathbb{Z}_5$. The isomorphism is given by $\sigma_1 \mapsto ([1],[0])$ and $\sigma_2 \mapsto ([0],[1])$.

• Now suppose that S_n has a subgroup isomorphic to $\mathbb{Z}_5 \times \mathbb{Z}_5$. Then S_n has a subgroup of order 25. So, by Lagrange's Theorem, 25 must divide $|S_n| = n!$. This is possible only if $n \geq 10$. $\square$

Problem 2 (a) $(14356) = (16)(15)(13)(14)$ $\qquad (a_1, \dots, a_n) = (a_1, a_n) \cdots (a_1, a_2)$

even permutation

(b) $(156)(234) = (16)(15)(24)(23)$ even permutation

(c) $(1426)(142) = (1246) = (16)(14)(12)$ odd permutation

$\square$

Problem 3 Exercise 8: An n-cycle can be written as a product of $(n-1)$ transpositions →

$(a_1, \dots, a_n) = (a_1, a_n) \cdots (a_1, a_2) = (a_1, a_2)(a_2, a_3) \dots (a_{n-1}, a_n)$

So, cycles of odd length are even permutations and belong in alternating groups. Then products of odd cycles are also in alternating groups.

In particular, $(12345)(678) \in A_{10}$. Here, (12345) and (678) are commuting elements (disjoint cycles) of orders 5 and 3 respectively.

The order of $(12345)(678)$ is $\text{LCM}(5,3) = 15$. $\square$

Exercise 9: Let $\sigma \in A_8$. Write σ as a product of disjoint cycles. Then the order of σ is the LCM of the orders of these cycles. If $|\sigma| = 26 = 2 \times 13$, then σ must be a multiple of a cycle of length at least 13. This is impossible since A_8 contains permutations of only 8 elements. A_8 does not have an element of order 26.

Alternatively, $|\sigma|$ must divide $|S_8| = 8!$ and $8!$ is not divisible by 13. $\square$

Problem 4 Exercise 22: Suppose that $\sigma = \sigma_1\sigma_2 \cdots \sigma_k = \sigma_1'\sigma_2' \cdots \sigma_{k'}'$ where σ_i, σ_j' are transpositions. Then identity $= \sigma\sigma^{-1} = \sigma_1 \cdots \sigma_k \sigma_{k'}' \sigma_{k'-1}' \cdots \sigma_2' \sigma_1'$ is a product of $k+k'$ transpositions. So, $k+k'$ must be even. (Lemma 5.14 in Judson). If k is odd, so is k'. $\square$

Exercise 23: Suppose that $\sigma = (a_1\ a_2\ \cdots\ a_{2k-1})$ where $k \in \mathbb{N}$. Then σ^2 takes $a_1 \mapsto a_3$, $a_3 \mapsto a_5, \ldots, a_{2k-3} \mapsto a_{2k-1}$, $a_{2k-1} \mapsto a_2$ and $a_2 \mapsto a_4, \ldots, a_{2k-4} \mapsto a_{2k-2}$, $a_{2k-2} \mapsto a_1$. So, $\sigma^2 = (\underbrace{a_1\ a_3 \cdots a_{2k-1}}_{\text{odd indices}}\ \underbrace{a_2\ a_4 \cdots a_{2k-2}}_{\text{even indices}})$ is a cycle of length $2k-1$. $\square$

Exercise 24: Any 3-cycle $(a\ b\ c)$ can be written as $(a\ b)(b\ c)$. $\square$

Exercise 25: Any permutation in A_n is a product of an even number of transpositions. It's enough to show that a product of two transpositions can be written as a product of 3-cycles.

Case I $\rightarrow$ $e = (a\ b)(a\ b) = (a\ b\ c)(a\ c\ b)$ for any $c \neq a, b$. Such c exists because $n \geq 3$.

Case II $\rightarrow$ $(a\ b)(a\ c) = (a\ c\ b)$
$b \neq a \neq c$

Case III $\rightarrow$ $(a\ b)(c\ d) = (c\ a\ d)(a\ b\ c)$
disjoint transpositions
$\square$

Exercise 26:

(a) It suffices to show that any transposition in S_n can be written as a finite product of the given permutations.

$(1\ a)$ with $a \neq 1$ is one of the given permutations.

$(a\ b)$ with $a, b \neq 1$ is the product $(1\ a)(1\ b)(1\ a)$

$1 \mapsto a \mapsto a \mapsto 1$
$a \mapsto 1 \mapsto b \mapsto b$
$b \mapsto b \mapsto 1 \mapsto a$
$\square$

(b) It suffices to show that any transposition $(1\ a)$ can be written as a finite product of the given permutations. We proceed by induction. We already have $(1 2)$ and $(1\ k{+}1) = (1\ k)(k\ k{+}1)(1\ k)$ for any $2 \leq k \leq n-1$. $\square$

(c) It is enough to show that any transposition $(a\ a{+}1)$ can be written as a finite product of (12) and $(12\cdots n)$. Again, we use induction. Note that $(12\cdots n)^{-1} = (12\cdots n)^{n-1} = (n\cdots 21)$.

$$(12) = (12).$$

$$(k{+}1,\ k{+}2) = (12\cdots n)(k\ k{+}1)(n\cdots 21) \quad \text{for any } 1 \le k \le n-2.$$

$$\downarrow$$

$$\begin{array}{lllllll} 1 & \mapsto & n & \mapsto & n & \mapsto & 1 \\ \text{under } (n\cdots 21) & & (k\ k{+}1) & & (12\cdots n) & & \end{array}$$

$$i \mapsto i-1 \mapsto i-1 \mapsto i \quad \text{if } 1 < i \le k \text{ or } k+2 < i \le n$$

$$k+1 \mapsto k \mapsto k+1 \mapsto k+2$$

$$k+2 \mapsto k+1 \mapsto k \mapsto k+1$$

$\square$