Intro Modern Algebra I HW8 Solution

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Problem 1

(a)

$\ker(f \times g) = \{x \in \mathbb{Z} \mid f(x) = 0 \text{ and } g(x) = 0\} \overset{\text{def of f and g}}{=} \{x \in \mathbb{Z} \mid n|x \text{ and } m|x\} \overset{\text{def of lcm}}{=} \{x \in \mathbb{Z} \mid \text{lcm}(m,n)|x\} \overset{(m,n)=1}{=} \{x \in \mathbb{Z} \mid mn|x\} \overset{def}{=} \{mnk \mid k \in \mathbb{Z}\}$. equivalently this is the set $mn\mathbb{Z}$.

(b)

By first isomorphism theorem, $\mathbb{Z}/\ker(f \times g) \cong \text{Im}(f \times g)$, so $\mathbb{Z}/mn\mathbb{Z} \cong \text{Im}(f \times g)$ by (a). But $\mathbb{Z}/mn\mathbb{Z} = \mathbb{Z}_{mn}$ by definition(or, from the fact that $\mathbb{Z}/mn\mathbb{Z}$ is a cyclic group of mn elements), so $\mathbb{Z}_{mn} \cong \text{Im}(f \times g) \subseteq \mathbb{Z}_n \times \mathbb{Z}_m$. Since their cardinalities are equal we claim that indeed $\text{Im}(f \times g) = \mathbb{Z}_n \times \mathbb{Z}_m$ and $\mathbb{Z}_{mn} \cong \mathbb{Z}_n \times \mathbb{Z}_m$.

(c)

Since $f : \mathbb{Z} \to \mathbb{Z}_n$ is surjective, the image $\text{Im}(f \times f) = \{(f(x), f(x)) \mid x \in \mathbb{Z}\} = \{(x, x) \mid x \in \mathbb{Z}_n\} = \Delta \subseteq \mathbb{Z}_n \times \mathbb{Z}_n$ is the set of diagonal elements. Also, $\ker(f) = n\mathbb{Z}$ is the multiples of n. We note that this is indeed consistent with the first isomorphism theorem by observing that $\Delta \cong \mathbb{Z}_n \cong \mathbb{Z}/n\mathbb{Z}$.

Problem 2

(a)

Bezout's theorem says that $am + bn = d$ for some $a, b \in \mathbb{Z}$, so any $kd \in < d >$ can be written as $akm + bkn$ and thus $kd \in HN$. On the other hand, given any $am + bn \in HN$ we have $k|am + bn$ since $k|m$ and $k|n$, so indeed $am + bn \in < d >$. Therefore $HN = < d >$.

(b)

$H \cap N = \{x \in \mathbb{Z} \mid m|x, n|x\} = \{x \in \mathbb{Z} \mid c|x\}$ by definition of lcm, and the last set is evidently $< c >$.

(c)

The second isomorphism theorem says that $HN/N \cong H/(H \cap N)$ which translates to $< d > / < n > \cong < m > / < c >$. To show this is equivalent to $\mathbb{Z}_{c/m} \cong \mathbb{Z}_{n/d}$ we just need to prove the following statement:

For $a|b$, $< a > / < b > \cong \mathbb{Z}_{b/a}$.

Proof. $\mathbb{Z} \cong < a >$ by multiplication by a(from the hint), and the subgroup $< b/a > \subset \mathbb{Z}$ is mapped to $< b > \subset < a >$ via this isomorphism. Therefore $< b > \cong < b/a >$ and $\mathbb{Z}/ < b/a > \cong < a > / < b >$, and the left side is equivalent to $\mathbb{Z}_{b/a}$ as we have illustrated many times before. □

Problem 3

Proof. First noice that either $H \subseteq N$ or not. If $H \subseteq N$ we are done. If not then consider the group $HN \supsetneq N$ in G. Since HN is strictly larger than N and $G \supseteq HN \supsetneq N$ we have $[G : HN] \cdot [HN : N] = [G : N] = p$ and $[HN : N] > 1$. Because p is prime, it must be that $[HN : N] = p$ and $[G : HN] = 1$. This implies that $G = HN$, and by the second isomorphism theorem we know $[H : H \cap N] = [HN : N] = [G : N] = p$. □

Problem 4

(a)

Consider the quotient maps $f : G \to G/N$ and $g : G \to G/M$, using the same trick in problem 1 they induce a map

$$f \times g : G \to G/N \times G/M$$

and $\ker(f \times g) = \{x \in G \mid f(x) = e, g(x) = e\} = \ker(f) \cap \ker(g) = N \cap M$. This is a normal subgroup of G and is a subgroup of both N and M because it is contained in both.

Next we show that $f \times g$ is surjective. Since $G = MN$, it follows that any $x \in G$ has a (not necessarily unique) decomposition $x = a_x b_x$, $a_x \in M, b_x \in N$.

Therefore by the first isomorphism theorem we have $G/N \cap M \cong G/N \times G/M$ as desired. Now given any $(aN, bM) \in G/N \times G/M$, we claim $f \times g(ab) = (aN, bM)$. When $a = b = e$ this is trivial, so WLOG let $a \neq e$, so $a \notin N$. But since $G = MN$, $a \in M$ so $f \times g(ab) = (abN, bM)$. Now if $b = e$ the claim is satisfied trivially so we furthermore let $b \neq e$, then by the same reasoning $b \in N$ so indeed $f \times g(ab) = (aN, bM)$.

This tells us that the image of $f \times g$ is $G/N \times G/M$, and by the first isomorphism theorem $G/N \cap M \cong G/N \times G/M$.

(b)

If $G = MN$ and $M \cap N = \{e\}$, by the second isomorphism theorem we have $MN/N \cong M/M \cap N$ and $MN/M \cong N/N \cap M$ which means $G/N \cong M$ and $G/M \cong N$. Therefore $G \cong M \times N$ from part (a).

Problem 5

(14)

$\phi^{-1}(H)$ is indeed a subgroup: since H is a subgroup of G/N, $N \in H$ so $e \in \phi^{-1}(H)$ since $e \in N = \phi^{-1}(N)$. If $a \in \phi^{-1}(H)$ then $aN \in H$ so $a^{-1}N \in H$ and $a^{-1} \in \phi^{-1}(H)$. If $a, b \in \phi^{-1}(H)$ then $aN, bN \in H$ and $abN \in H$ so $ab \in \phi^{-1}(H)$.

Now $\phi|_{\phi^{-1}(H)} : \phi^{-1}(H) \to H$ is a surjective group homomorphism with kernel N, and therefore $|\phi^{-1}(H)| = |H| \cdot |N|$.

(17)

Let $G_1 = \mathbb{Z}$, $G_2 = \mathbb{Z}_2$ and ϕ be the canonical map. Let $H_1 = \{0\}$, then $H_2 = \phi(H_1) = \{0\}$, but $\mathbb{Z}/\{0\} = \mathbb{Z} \ncong \mathbb{Z}_2 = \mathbb{Z}_2/\{0\}$.