ALGEBRAIC NUMBER THEORY W4043

Homework, week 3, due February 10

1. Let $\mathcal{O}$ be the ring of integers of a number field K. A *fractional ideal* of $\mathcal{O}$ is a non-zero finitely generated $\mathcal{O}$-submodule of K. Let $M \subset K$ be a fractional ideal of $\mathcal{O}$. Show that M^{-1}, defined to be the $\mathcal{O}$-submodule of $a \in K$ such that $a \cdot m \in \mathcal{O}$ for all $m \in M$, is again a fractional ideal.

2. Prove the following Proposition:

Proposition. Let $\mathcal{O}$ be the ring of integers of a number field, $\{\mathfrak{p}_i, i \in \mathbb{N}\}$ a sequence of two-by-two distinct prime ideals. Then $\cap_i \mathfrak{p}_i = \{0\}$.

3. Let R be an integral domain with fraction field K. A *multiplicative subset* $S \subset R$ is a subset such that,

- $1 \in S$, $0 \notin S$;
- If $s, s' \in S$ then $ss' \in S$.

The *localization* $S^{-1}R$ is the subset of K consisting of elements $\frac{r}{s}$ with $r \in R$ and $s \in S$. (Alternatively, it is the set of equivalence classes of pairs (r, s), with $r \in R$ and $s \in S$, with (r, s) equivalent to (r', s') if and only if $rs' = r's$). After convincing yourself that $S^{-1}R$ is a ring, show that

(a) If S is the set of non-zero elements of R, then $S^{-1}R = K$;

(b) If R is a Dedekind domain, then so is $S^{-1}R$ for any multiplicative subset $S \subset R$.

(c) If $I \subset R$ is an ideal, let $S^{-1}I \subset S^{-1}R$ be the ideal of $S^{-1}R$ generated by I. Show that the map

$$I \mapsto S^{-1}I$$

is a surjection from the set of ideals of R to the set of ideals of $S^{-1}R$. Use the proof to construct a bijection between the set of prime ideals of $S^{-1}R$ and the subset of prime ideals $\mathfrak{p} \subset R$ such that $\mathfrak{p} \cap S = \emptyset$.

(d) Let R be a Dedekind domain, $\mathfrak{p} \subset R$ be a prime ideal, let $S_{\mathfrak{p}} = R \setminus \mathfrak{p}$, and define $R_{\mathfrak{p}} = S_{\mathfrak{p}}^{-1}R$. Show that $R_{\mathfrak{p}}$ is a *discrete valuation ring*, i.e. a Dedekind domain with a unique non-zero prime ideal. In particular, show (using problem 2) that every non-zero element $a \in R_{\mathfrak{p}}$ has a unique factorization of the form $a = uc^b$, where c is a generator of the unique non-zero prime ideal of $R_{\mathfrak{p}}$, b is a non-negative integer, and u is an invertible element of $R_{\mathfrak{p}}$.

4. Show that the subgroup

$$L := \{(a, b, c) \in \mathbb{Z}^3 \mid a \equiv b \pmod 5, b \equiv a + c \pmod 2\} \subset \mathbb{R}^3$$

is a lattice. Find a fundamental domain for L in $\mathbb{R}^3$ and compute its volume.