ALGEBRAIC NUMBER THEORY W4043

Homework, week 5, due February 24

1. Let $K = \mathbb{Q}(\sqrt{-3})$.

(a) Find the discriminant of K.

(b) Use (a) to show that K has class number 1.

2. Exercises 6.13 and 6.16 on pp. 119-20 in Hindry's book. You will need to use Minkowski's constant, as in Corollary 5.10 on p. 110, rather than the simpler estimate given in the course.

3. Let K be the splitting field over $\mathbb{Q}$ of the polynomial $X^5 - 1$.

(a) List the intermediate fields K_i between K and $\mathbb{Q}$ and for each i, use Galois theory to find $\alpha_i \in K$ such that $K_i = \mathbb{Q}(\alpha_i)$.

(b) Show that there is a unique subfield $K' \subset K$ such that $[K' : \mathbb{Q}]$ is quadratic. Determine the set of primes $p \in \mathbb{Q}$ that ramify in K', and use this to write $K' = \mathbb{Q}(\sqrt{d})$ for some integer d. What are r_1 (the number of real embeddings) and r_2 (the number of pairs of complex conjugate embeddings) for K'?

(c) For any $n > 1$, show that $\phi(n)$ is divisible by 2. Let $n = p_1 \cdot p_2$ be the product of two odd primes and let K_n be the splitting field over $\mathbb{Q}$ of the polynomial $X^n - 1$. List all the subfields $L \subset K_n$ such that $[L : \mathbb{Q}] = 2$ and determine r_1 and r_2 for each such L.