ALGEBRAIC NUMBER THEORY W4043

Homework, week 9, due April 7

1. Hindry's book, p. 160, Exercise 6.9.

2. An *arithmetic function* is a function $f : \mathbb{N} \to \mathbb{C}$. An arithmetic function f is *multiplicative* if $f(ab) = f(a)f(b)$ whenever a and b are relatively prime.

Suppose f and g are two arithmetic functions. Define the *convolution*

$$(f * g)(n) = \sum_{d|n} f(d)g(\frac{n}{d}).$$

(a) Let $\tau(n)$ denote the number of integers dividing n. Let $\mathbf{1}$ be the function defined by $\mathbf{1}(n) = 1$ for all n. Show that $\mathbf{1} * \mathbf{1} = \tau$.

(b) Suppose f and g are multiplicative functions. Show that $f * g$ is also multiplicative.

(c) Define the *Möbius function* μ to be the unique multiplicative function such that $\mu(1) = 1$, $\mu(p) = -1$ for any prime p, and $\mu(n) = 0$ if n is not square-free. Let f be the function $f(n) = n$ for all n. Compute $f * \mu$.

(d) Define the *von Mangoldt* function Λ by $\Lambda(1) = 0$, $\Lambda(n) = \log(p)$ if $n = p^i$ for some prime p, $\Lambda(n) = 0$ if n is not a prime power. Let

$$D(s) = \sum_n \frac{\Lambda(n)}{n^s}.$$

Show that $D(s)$ converges absolutely for $Re(s) > 1$ and that, on the half plane $Re(s) > 1$, we have the equality

$$D(s) = -\frac{\zeta'(s)}{\zeta(s)}$$

where $\zeta(s) = \sum_n \frac{1}{n^s}$ is the Riemann zeta function.