Week 1

Introduction

The history of mathematics is largely absent from the 'culture' of the educated public, historians and mathematicians included. The extent to which it is dismissed, abhorred, even derided, has to be experienced to be believed.

Ivor Grattan-Guinness, *The Rainbow of Mathematics*,1996

Here are the sciences along with their relations which Aristotle mentions in the *Analytics:*

Geometry		Stereometry (solid geometry)	Arithmetic
|		|	|
Optics (mathematical)	Astronomy (mathematical)	Mechanics	Harmonics (mathematical)
|	|		|
Concerning the Rainbow	Nautical Astronomy, Phenomena, Empirical		Acoustical Science

(from *Aristotle and Mathematics*, Stanford Encyclopedia of Philosophy)

"Powerful,… this naming the unknown: x, neither alive nor dead, took on an existence of its own, worming its way into an infinitude of equations, of propositions. The unknown as a **thing***."* (Olsson, p. 10)

Definition 1: *the practice of turning the unknown into a* **thing**, *as a first step toward making it known*.

(Compare Wittgenstein, *Remarks* I, 32: "The mathematician creates *essence*.")

When I considered what people generally want in calculating, I found that it always is a number.

(*Kitab al-jabr w'al-muqabala* of Muhammed Ibn Musa al-Khwārizmī.)

Divide [the inheritance] between the two sons; there will be for each of them three dirhems and a half plus two-fifths of ***thing****; and this is equal to one* ***thing****. Reduce it by subtracting two-fifths of* ***thing*** *from* ***thing****.*

Mathematics is “*that transcendent kingdom to which only the truly great have access and wherein truth abides.*” (Simone Weil)

Definition 2: a **yoga** that allows us a glimpse of that transcendent kingdom where truth abides.

Imaginary numbers

If you are looking for the solution to the equation

$$X^3 + bX + c \ = 0$$

where b and c are numbers, Scipione Dal Ferro gives you the answer.

$$X = \sqrt[3]{\frac{c}{2} + \sqrt{\frac{c^2}{4} - \frac{b^3}{27}}} + \sqrt[3]{\frac{c}{2} - \sqrt{\frac{c^2}{4} - \frac{b^3}{27}}},$$

The big puzzle for the 16th century Italians was to make sense of this even when the number

$$c^2/4 \ - \ b^3/27$$

under the radical is negative: how to take its square root?

Imagination

[I]n that kind of calculation [involving imaginary numbers] you have very solid figures at the beginning, which can represent metres or weights or something similarly tangible, and which are at least real numbers. And there are real numbers at the end of the calculation as well. But they're connected to one another by something that doesn't exist. Isn't that like a bridge consisting only of the first and last pillars, and yet you walk over it as securely as though it was all there?

(Musil, *Törless*, p. 82)

tshshsh. . . .” Only then the lesson would follow. One day Plappa told us about irrational numbers, and I remember I wept and banged the table with my fist and cried, “I do not want that square root of minus one; take that square root of minus one away!” This irrational root grew into me as something strange, foreign, terrible; it tortured me; it could not be thought out. It could not be defeated because it was beyond reason.

(Y. Zamiatin, *We*, Record 8)

The stretching of the imagination to embrace an otherwise unembraceable fictum would…be unavailable to us as a felt experience. This would suggest, for example, that no matter how assiduously we study the three-century-long encounter with $\sqrt{-1}$, *we will not get any closer to an inner experience of this grand act of imagination-stretching—because there is no inner experience to understand. It would allow only the existence of a before (the imaginative act) and an after.*

(Mazur, p. 42)

Definition 3: *the act of collective imagination-stretching*.

Cardano writes in the first chapter:

> Since this art surpasses all human subtlety and the perspicuity of mortal talent and is a truly celestial gift and a very clear test of the capacity of men's minds, whoever applies himself to it will believe that there is nothing that he cannot understand.[8]

one point in *Ars Magna*, Cardano finds himself forced to invoke a square root of −15. He says to the reader, giving no further justification, "You will have to imagine $\sqrt{-15}$," and then he goes on to calculate with it, even though he says that he is doing this by "dismissing mental tortures."[12] The colorful Latin phrase Cardano used for this is *dimissis incruciationibus*, and the translator notes that Cardano might very well be playing on a possible double meaning of this phrase in the sentence, which can be read either as "Dismissing mental tortures, multiply $5 + \sqrt{-15}$ by $5 - \sqrt{-15}$ . . ." or as "Cancelling out cross-multiples, multiply $5 + \sqrt{-15}$ by $5 - \sqrt{-15}$ . . ."[13]

Definition 4: *is the viewpoint that enables mental ease rather than mental torture.*

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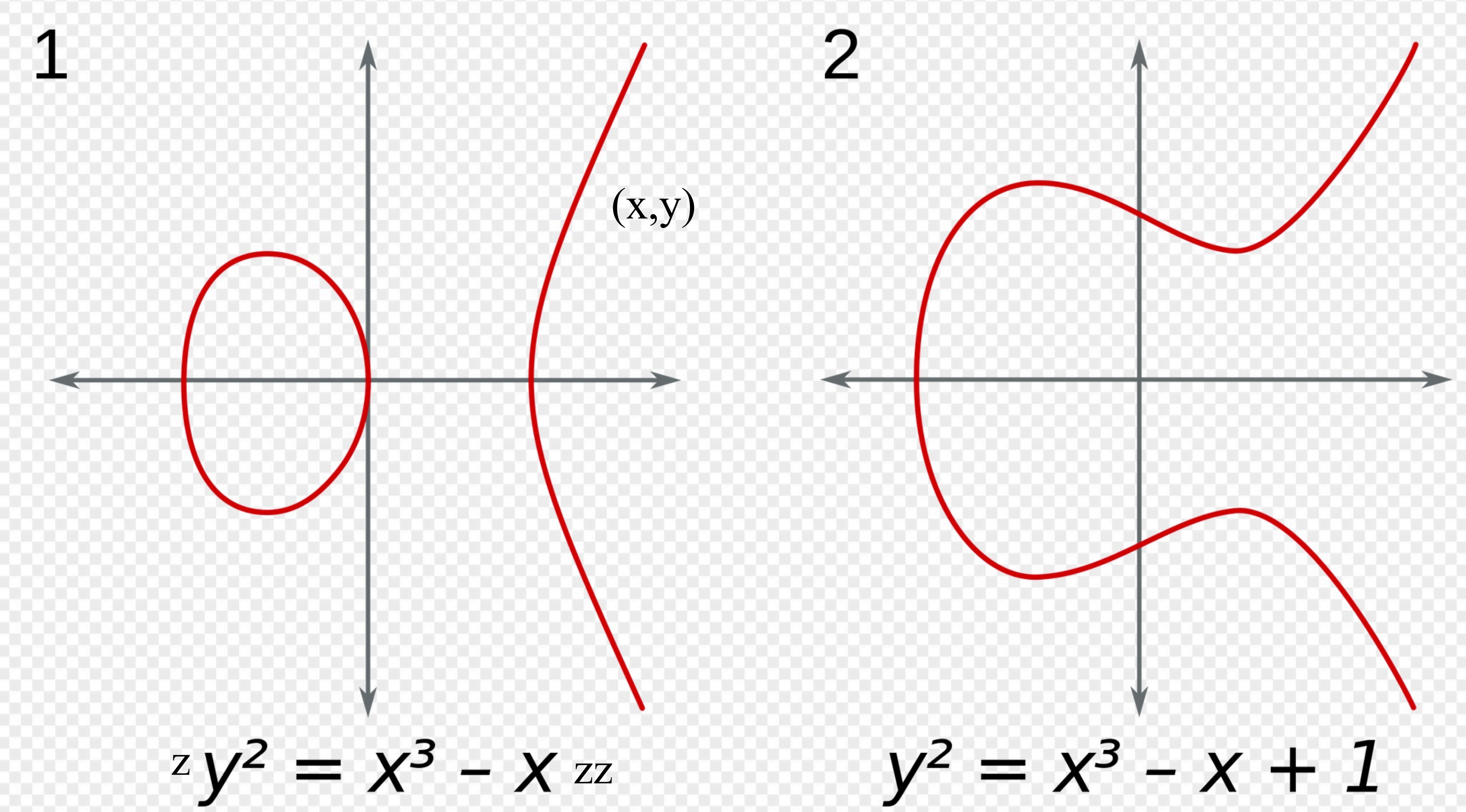
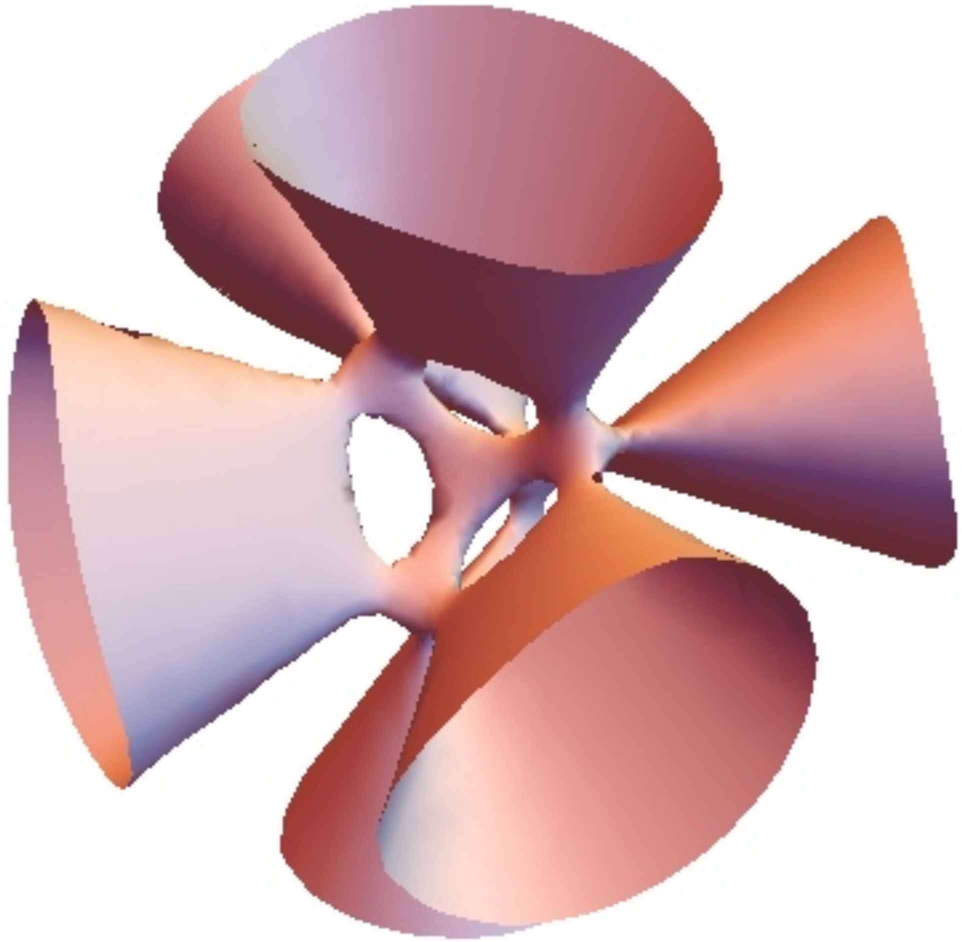
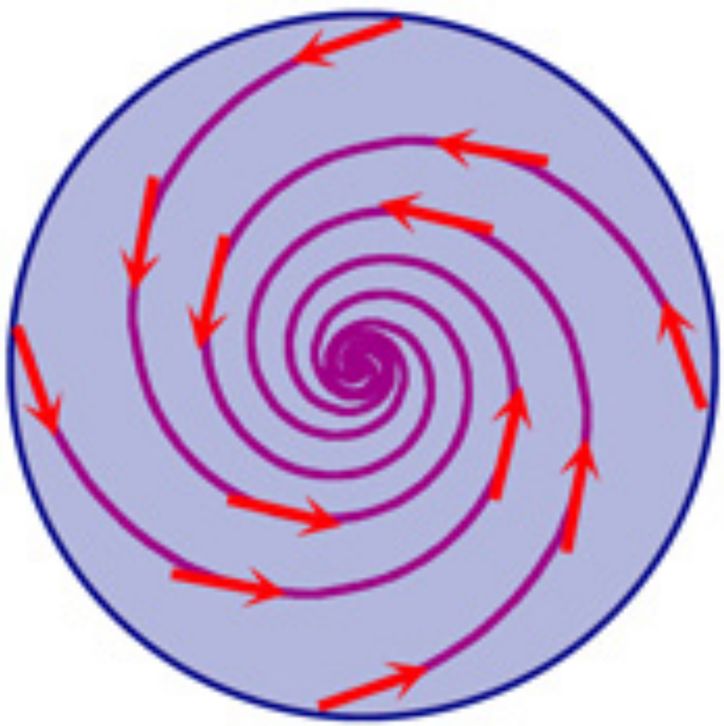
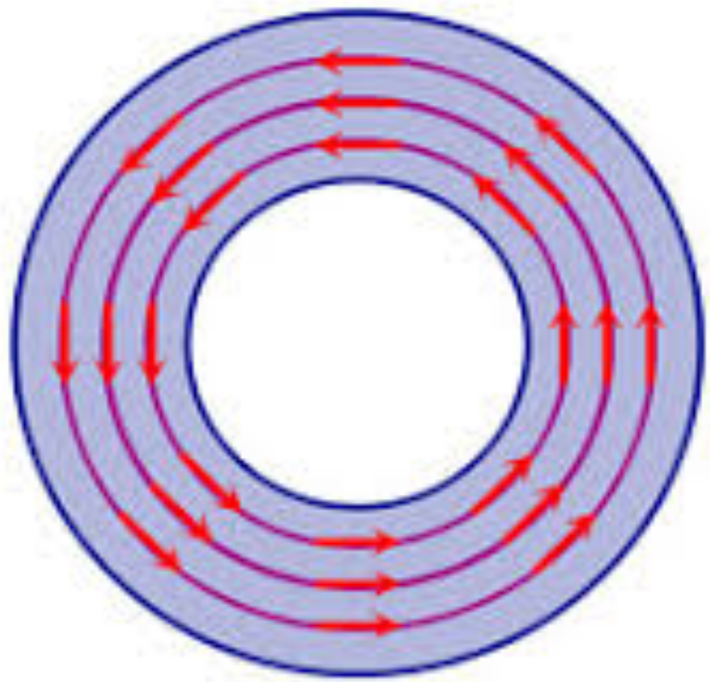


1
(x,y)
z y² = x³ – x zz
2
y² = x³ – x + 1







Definition 5. *the science of structures.*

"My dear sister," he writes, when he is back in his cell, "Telling nonspecialists of my research or of any other mathematical research, it seems to me, is like explaining a symphony to a deaf person. It could be attempted, you could talk of images and themes, of sad harmonies or triumphant dissonances, but in the end what would you have?

"A kind of poem, good or bad, unrelated to the thing it pretends to describe."

You might compare mathematics to an art, he goes on, to a type of sculpture in a very hard, resistant material. The grains and countergrains of the material, its very essence, limit the mathematician in a manner that gives his work the aspect of objectivity. But just like any work of art, it is inexplicable: the work itself is its explanation.

(Olsson, p. 86 quoting Weil's letter to his sister from prison in Finland)

…around 1820, mathematicians (Gauss, Abel, Galois, Jacobi) permitted themselves, with anxiety and delight, to be guided by the analogy [between an algebraic and a geometric theory]... [Now] *gone are the two theories, their conflicts and their delicious reciprocal reflections, their furtive caresses, their inexplicable quarrels; alas, all is just one theory, whose majestic beauty can no longer excite us. Nothing is more fecund than these slightly adulterous relationships; nothing gives greater pleasure to the connoisseur, whether he participates in it, or even if he is an historian contemplating it retrospectively.*

Naturally the mode of mathematical thinking varies by thinker. According to Hadamard's informal survey of colleagues in the United States, George Birkhoff would visualize algebraic symbols. Norbert Wiener would think either with or without words. For George Pólya, one word might appear in his head, and ideas would precipitate around it. Hadamard goes so far as to specify the "strange and cloudy imagery" that arises in his own mind as he follows a simple proof about prime numbers, listing the steps of the proof on the left-hand side of the page, his mental images on the right. The images are, for example, "a confused mass," or "a point rather remote from the confused mass." In fact, he writes, every time he undertakes mathematical research he develops a set of such images, which helps hold everything together.

(Olsson, section 7)

In the twenty-first century, everyone can benefit from being able to think mathematically. This is not the same as "doing math." The latter usually involves the application of formulas, procedures, and symbolic manipulations; mathematical thinking is a powerful way of thinking about things in the world -- logically, analytically, quantitatively, and with precision. It is not a natural way of thinking, but it can be learned. Mathematicians, scientists, and engineers need to "do math," and it takes many years of college-level education to learn all that is required. Mathematical thinking is valuable to everyone, and can be mastered in about six weeks by anyone who has completed high school mathematics. Mathematical thinking does not have to be about mathematics at all, but parts of mathematics provide the ideal target domain to learn how to think that way, and that is the approach taken by this short but valuable book.

(Blurb for K. Devlin, *Introduction to Mathematical Thinking*)

Definition 6: No definition of mathematical thinking is available!

Einstein is quoted by Hadamard: "*The words or the language, as they are written or spoken, do not seem to play any role in my mechanism of thought… The psychical entities which seem to serve… are certain signs and more or less clear images which can be 'voluntarily' reproduced and combined.*"

Olsson attempts to see writing the same way but (p. 110) "*the longer I go on writing, the more I sense its limitations, see the tiny word critters scuttling around an inexpressible landscape.… Moreover, after many years of writing I find I have traveled down so many forking paths, motivated by a sensation that something like the truth awaits at the end of a path… only to learn that the path either doesn't end or that it leads… into a cul-de-sac … where I come upon a moss-streaked stone pedestal that once supported a statue, the statue having been for some reason taken away.*"

Kovalevskaya's letter ...highlights her two achievements. First she set a new case of the motion of a rigid body, and gave a solution in terms of hyperelliptic functions. Furthermore, ...with the exception of three cases it is impossible to find a general solution of the problem of motion of a rigid body in terms of analytic functions. The achievements of Kovalevskaya and Poincaré were their realization that in general one cannot find analytical solutions that would describe the position of the rigid bodies or planets at all times. Traditionally, a differential equation is solved by finding a function that satisfied the equation; a trajectory is then determined by starting the solution with a particular initial condition. Before the discoveries of Poincaré and Kovalevskaya, it was thought that a nonlinear system would always have a solution; we just needed to be clever enough to find it. .... Kovalevskaya and Poincaré showed that no matter how clever we are, we will not be able to solve most of the differential equations. The belief in determinism, that the present state of the world determines the future precisely, was shattered.

(Frank Y. Wang, Pioneer women in chaos theory, 2009).

Definition 7: the science of finding qualitative classifications of the problems one can't solve.

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