Week 5

Eternity vs. Collective Practice

… modern logic offered such a neat account of mathematical proof that there was almost nothing left to do. Except, perhaps, one little thing: if mathematics amounts to deductive reasoning using the axioms and rules of set theory, then to ground the subject all we need to do is to figure out what sort of entities sets are, how we can know things about them, and why that particular kind of knowledge tells us anything useful about the world. Such questions about the nature of abstract objects have therefore been the central focus of the philosophy of mathematics from the middle of the 20th century to the present day.… The problem is that set-theoretic idealisation idealises too much. Mathematical thought is messy. … we have a lot to learn about how mathematics channels these wellsprings of creativity into rigorous scientific discourse. But that requires doing hard work and getting our hands dirty.

(Jeremy Avigad, *Aeon*, 2018)

For millenia, mathematics has been considered the acme of ahistorical, timeless, and usually certain knowledge. Mathematical certainty was decisively lost through the discoveries of modern logic and metamathematics, primarily Gödel's incompleteness theorems. But before Lakatos, there was no systematic account of modern mathematics and its rigor as a fallible form of knowledge built from its own history.

(John Kadvany, *Imre Lakatos and the Guises of Reason*)

...the history of mathematics and the logic of mathematical discovery... cannot be developed without the criticism and ultimate rejection of formalism.

(Lakatos, p. 4)

To treat the "logic of discovery" rather than the "logic of justification" as a subject worthy of philosophical consideration is heresy for logical positivists.

Alain Badiou does not want to see mathematical certainty as "decisively lost": the starting position of his philosophy is that some human creations must transcend human limitations.

Badiou calls these *oeuvres en vérité* and he sees them in four domains: art, politics, science, and love.

Mathematical logic — set theory — is the basis of his argument for their possibility and of the means by which they can be recognized.

He also sees some works in mathematics as *oeuvres en vérité*.

But he does not seek their permanence in the formal structure of the proof.

Euler's formula for polyhedra

First noticed by Francesco Maurolico in *Compaginationes solidorum regularium* (1537).

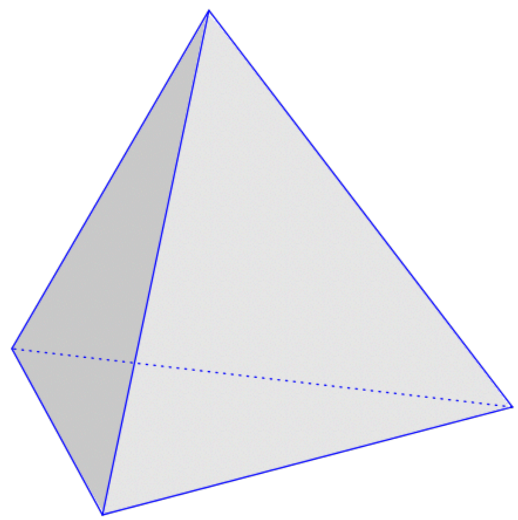
First stated by Leonhard Euler, mid 18th century.

Maurolico observed a surprising regularity in the *Platonic solids* (so named for the discussion in Plato's *Timaeus*).

Plato posited that each of the *four classical elements* is made up of one of the regular polyhedra:

Fire

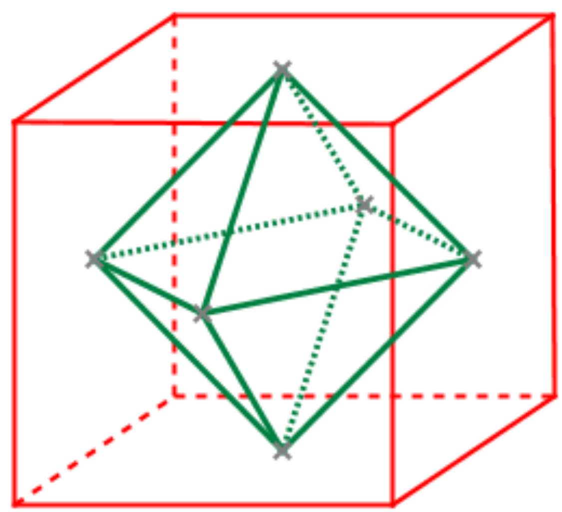
Fire is composed of *tetrahedra*:



V = 4, E = 6, F = 4.   V-E+F = 2

Earth, air

Earth is composed of *cubes* (obviously!), air of *octahedra*

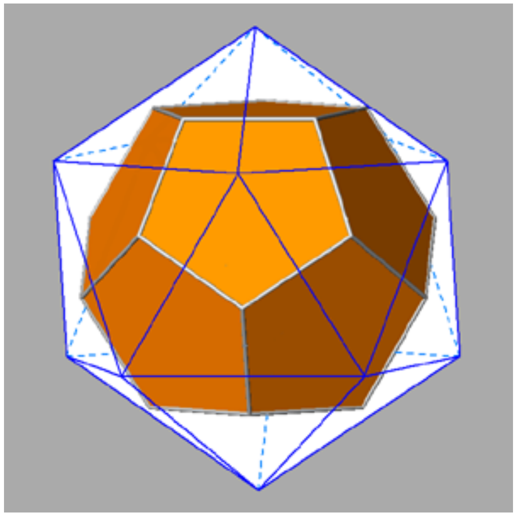


The cube: V = 8, E = 12, F = 6; V-E+F = 2.

The octahedron: V = 6, E = 12, F = 8: V-E+F = 2

Water, and ?

Water is made up of *icosahedra*. The *dodecahedron* has no element.

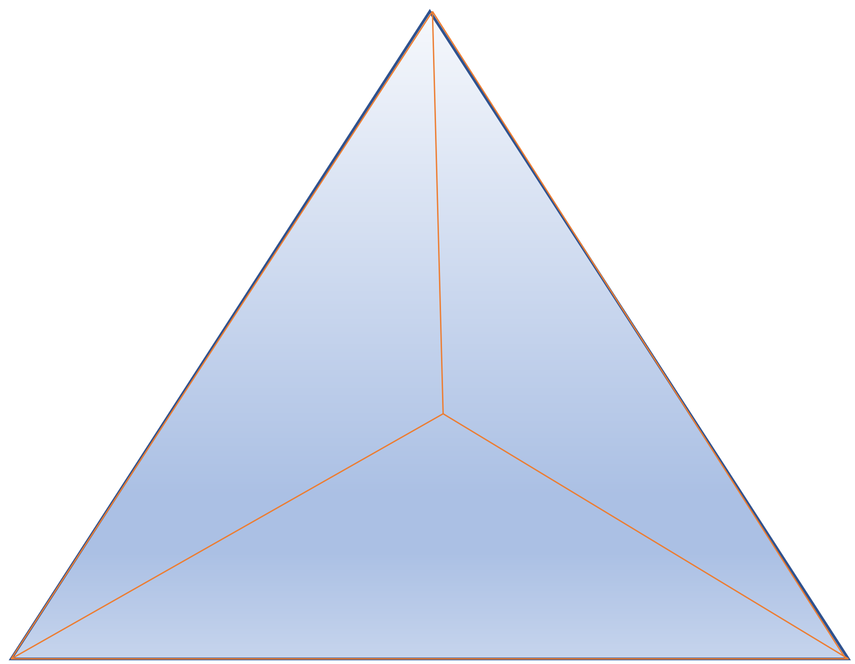


Dodecahedron: F = 12, E = 30, V = 20; V- E + F = 2

Icosahedron: F = 20, E = 30, V = 12; V - E + F = 2

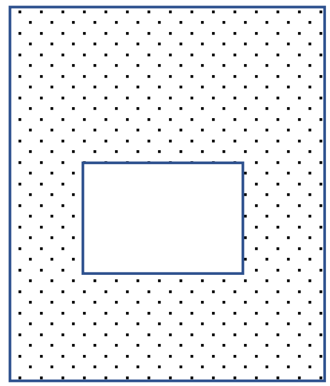
Coincidence?

As on p. 7 of Lakatos, we poke out one face and stretch the remaining figure out on the plane, so $V - E + F = 1$ for a plane polyhedron. For the tetrahedron:



Monster-barring

on pp. 13-23 Delta excludes several counterexamples, for example



On p. 21 Alpha has become so annoyed with this method that he leaves the room, after exclaiming:

…once upon a time it was a wonderful guess, full of challenge and excitement. Now, because of your weird shifts of meaning, it has turned into a poor convention, a despicable piece of dogma.

What an obstreperous bunch of students! Maybe this class can practice the roles from P&R and we can have a free-for-all on week 7!

Other methods

Exception-barring (Beta)
Teacher: You must admit that each new version of your conjecture is only an *ad hoc* elimination of a counterexample which has just cropped up…. How can you be sure that you have enumerated *all* exceptions?

Lemma incorporation, or the method of Proofs and Refutations (Teacher) The original attempt at proof was based on intermediate steps. In the example, the first step (lemma) is to remove a face and stretch the polyhedron onto the plane. Alpha asks: how do you know this is possible? The lemma is then extracted from the proof and incorporated into the theorem:

Euler's formula is true for ***simple*** *polyhedra*

where "simple" means you can remove a face and stretch the result on to a plane.

Lakatos as dialectician

Lakatos was a student of Karl Popper (whose *Conjectures and Refutations* was the reference of Lakatos's title) but previously had been a Communist – even a Stalinist – before the Hungarian uprising in 1956. Kadvany's book argues that his dialectical vision of "mathematical discovery" was influenced by his attendance at Lukács' weekly seminar as a student in Hungary.

Popper, like the American pragmatists, was an exponent of fallibilism (every scientific theory is open to revision upon refutation, and indeed always has to be falsifiable), and *Proofs and Refutations* is seen as the application of fallibilism in mathematics.

"Collective practice" is a more positive name for fallibilism, and also stresses the evolution of *values* as well as *beliefs* (since proofs are supposed to compel belief). New values often crystallize around new *concepts*.

Concept Formation

How to view the image under “Monster-barring” not as a counterexample to Euler's formula but an example of a different formula?

The number of faces F = 4, the number V = 8, and E = 12. So F-E+V = 0. For a general plane polyhedron P which has been drawn properly (to exclude monsters) we define the *Euler characteristic*

$$\chi(P) = F-E+V$$

Original Euler formula: $\chi(P) = 1$.
Square inside a square: $\chi(P) = 0$.
Two squares in the square: we get $\chi(P) = -1$.
g interior squares (holes): $\chi(P) = 1-g$
(Recall Brouwer/Lefschetz fixed point formula)

Is this a new concept?

Does this number g, which I call the *genus*, have a definition independent of the ad hoc construction?

Recall the Weil conjectures: the concept of which the genus is an instance ("number of holes") controls the number of solutions of equations.

The concept also occurs (in disguise) in Maxwell's equations where it distinguishes electricity from magnetism.

Topology in n dimensions

The number of n-dimensional holes in an m-dimensional polyhedron, $m \geq n$, is the *n-th Betti number* (named after Enrico Betti).

Theorem: the Euler characteristic calculated as an alternating sum of Betti numbers is the same as $\chi(P)$ defined by triangulations.

Proved over several centuries as a result of an extended negotiation over definitions and concepts. *Proofs and Refutations* models aspects of this process.

Weil conjectures: Grothendieck and collaborators extended these concepts in new situations, counting solutions to equations.

What proofs prove, according to Lakatos

Many working mathematicians are puzzled about what proofs are for if they do not prove. On the one hand they know from experience that proofs are fallible but on the other hand they know from their dogmatist indoctrination that genuine proofs must be infallible. Applied mathematicians usually solve this dilemma by a shamefaced but firm belief that the proofs of the pure mathematicians are 'complete', and so really prove. Pure mathematicians, however, know better–they have such respect only for the'complete proofs'of logicians.

(footnote 29, continued below)

According to working mathematicians

If asked what is then the use, the function, of their 'incomplete proofs', most of them are at a loss. For instance, G. H. Hardy ...characterise[d] mathematical proof 'as we working mathematicians are familiar with it',... in the following way: 'There is strictly speaking no such thing as mathematical proof; we can, in the last analysis, do nothing but point;... proofs are what Littlewood and I call gas, rhetorical flourishes designed to affect psychology, pictures on the board in the lecture, devices to stimulate the imagination of pupils'([1928], p. 18). R. L. Wilder thinks that a proof is'only a testing process that we apply to suggestions of our intuition'([1944], p. 318). G. Pólya points out that proofs, even if incomplete, establish connections between mathematical facts and this helps us to keep them in our memory: proofs yield a mnemotechnic system ([1945],pp. 190–1).

Lakatos on truth

For more than two thousand years there has been an argument between ***dogmatists*** *and* ***sceptics****. The dogmatists hold that — by the power of our human intellect and/or senses — we can attain truth and know that we have attained it. The sceptics on the other hand either hold that we cannot attain the truth at all (unless with the help of mystical experience), or that we cannot know if we can attain it or that we have attained it. In this great debate… mathematics has been the proud fortress of dogmatism. Whenever the mathematical dogmatism of the day got into 'a crisis', a new version once again provided genuine rigour and ultimate foundations, thereby restoring the image of authoritative, infallible, irrefutable mathematics…*

(*Proofs and Refutations,* Introduction)

Euler's original formula is a "special case" of many theorems in a number of branches of contemporary mathematics. The sense in which it represents an eternal **truth**, as in the quotation of Lakatos, is itself a matter of interpretation.

The situation has changed since that quotation was written. For one thing, mathematicians have read Proofs and Refutations…

Badiou bows down before mathematics

Je pense que le rapport de fond entre la philosophie et les mathématiques est effectivement un rapport de révérence, si je puis dire. Quelque chose dans la philosophie s'incline devant les mathématiques. (*In Praise of Mathematics,* p. 40)

As far as ontology is concerned, he continues :

…la philosophie… ne peut être saisie par les mathématiques qu'à son commencement même. En tant que science de l'être, les mathématiques sont cruciales dès le début, dès qu'on entre en philosophie. (p. 41).

Crucial terms for Badiou: *universal*, *absolute*, and *eternal*, as well as *truth* and *infinity* which are the philosophical concepts he wants to protect.

Badiou on absolute, eternal, truth and infinity

These two properties [absoluteness and eternity] require that truths – scientific, aesthetic, political, or existential – be infinite, without recourse to the idea of a God of any kind whatsoever.

(*In Praise of Mathematics,* p. 83)

The affect of a truth is the immanence of something infinite within finitude. I agree with ... Spinoza that sometimes 'we feel and know by experience that we are eternal'

(see Suture Press, Alain Badiou: *Sometimes we are Eternal*)

The classical position… mathematics was a condition for the existence of philosophy. (Badiou, speaking at Columbia last year)

the birth of philosophy… conditioned that of geometry.

(Derrida, also talking about the Greeks, p132)

Badiou and inaccessible cardinals

Mathematics appears in the third book (*Immanence of Truths*) of his *Being and Event* trilogy as one of the sciences, but it structures all three volumes.

For Badiou mathematics alone has invented a coherent and systematic way to talk about infinity: the logic of set theory. But the truth of the continuum hypothesis depends on your axioms!

Much contemporary set theory studies higher kinds of infinity, called *inaccessible cardinals* — whose existence is not implied by ZFC, and that therefore have to be added as axioms. Some inaccessible cardinals imply the continuum hypothesis, others do not, and they have various advantages and disadvantages.

Most of *Immanence of Truths* is about inaccessible cardinals.

From ***Number and Numbers***

0.3. *Firstly, number rules our political conceptions, with the currency … of suffrage, of opinion polls, of the majority. Every "political" assembly, general or local, municipal or international, voting-booth or public meeting, is settled with a count.*

Keep that in mind in connection with the upcoming election!

0.4. *Number rules over the quasi-totality of the "human sciences" … It is overrun by the statistical data of the entire domain of its disciplines.*

0.6. *… The ideology of modern parliamentary societies, if they have one, is not humanism, the rights of the subject. It is number, the countable, countability. Every citizen is today expected to be cognizant of foreign trade figures, of the flexibility of the exchange rate, of the developments of the stock market.*

1.15. *We must speak not of one unique age of modern thinking of number, but of what one might call the "first modernity" of the thinking* *of number. The names of this first modernity are not those of Proust and Joyce, they are those of Bolzano, Frege, Cantor, Dedekind and Peano.*

Here Badiou is clearly trolling his audience of littéraires.

3.18. *The "constructivist" thesis that makes of iteration, of succession, of passage, the essence of number, leads to the conclusion that very few numbers exist, since here "exist" has no sense apart from that effectively supported by some such passage. …Even a demi- intuitionist like [Émile] Borel thinks that the great majority of whole natural numbers "don't exist" except as a fictional and inaccessible mass. … the domain of number is rather an ontological prescription incommensurable to any subject, and immersed in the infinity of infinities.*

3.19. *Thus the problem becomes: how to think number whilst admitting, against Leibniz, that there are real indiscernibles; against the intuitionists, that number persists and does not pass; and against the foundational use of the subjective theme, that number exceeds all finitude?*

Fermat's Last Theorem

Fermat's last theorem : *Let n be an integer > 2. The equation*

$$\mathbf{a^n + b^n = c^n}$$

has no solutions where a, b, c are all positive integers.

Stated by Pierre de Fermat in 1637 without proof,
(contrast with the case n = 2: $\mathbf{3^2 + 4^2 = 5^2; 5^2 + 12^2 = 13^2}$, etc; known about 4000 years ago, see Week 11.)

Known for numbers up to 4,000,000 by 1980 by various arguments.
Proved by Andrew Wiles between 1987 and 1994 (helped in the last steps by Richard Taylor after), ending speculation that the question might be undecidable!

Badiou on contemporary mathematics

In June 2019 in Paris, speaking at a conference in honor of Badiou's *L'immanence des vérités*, I chose to reply to the brief paragraphs in the book where he asks whether or not Andrew Wiles's proof of Fermat's Last Theorem deserves to be considered an *oeuvre en vérité*. Badiou responds in the negative and says that it belongs in the *archives*; in Paris my talk aimed to save it from the status of *déchets*, refuse.

Badiou writes (p. 589) that Wiles's theorem is an example of *travail rétroactif*:
il mesure en quelque sorte la puissance des moyens nouveaux quant à la solution de problèmes anciens.

Contrasted with *travail technique* and *travail créatif*, qui *consiste d'inventer des perspectives, des concepts, des méthodes de démonstration*. Only *travail créatif* qualifies as *oeuvre*.

Parlera-t-on pour autant, en se basant sur ce seul tour de force, de 'l'oeuvre de Wiles'? C'est douteux.

Sketch of Wiles's proof

Suppose, contrary to Fermat's claim, there is a triple of positive integers a, b, c such that

(A) $$a^p + b^p = c^p$$

for some odd *prime* number p (it's enough to consider prime exponents). In 1985, Gerhard Frey had pointed out that a, b, and c could be rearranged into

(B) a new equation, called an *elliptic curve*,

$$y^2 = x(x - a^p)(x + b^p)$$

with properties that were universally expected to be impossible.

More precisely, it had long been known how to leverage an equation like (B) into

(C) a *Galois representation*,

which is an infinite collection of equations that are related to (B), and to each other, by precise rules.

The links between (A), (B), and (C) were all well-understood in 1985. But by that year, most number theorists were convinced — mainly thanks to the insights of the *Langlands program*, named after the Canadian mathematician Robert P. Langlands — that to every object of type (C) one could assign, again by a precise rule,

(D) a *modular form*,

which is a kind of two-dimensional generalization of the familiar sine and cosine functions. The final link was provided when Ken Ribet confirmed a suggestion by Jean-Pierre Serre that the properties of the modular form (D) entailed by the form of Frey's equation (B) implied the existence of

(E) another *modular form, this one of weight 2 and level 2*.

But there are no such forms!

Therefore there is no Galois representation (C),
therefore no equation (B),
therefore no solution (A).

(Logical puzzle: how can equation (B) not exist? We wrote it down…)
This is a classic *proof by contradiction* and it works provided the missing link between (C) and (D) — the *modularity conjecture* — could be established.

Wiles's proof is the beginning, not the end, of a story

Wiles proved the modularity conjecture — the link between (C) and (D)s, which (unlike Fermat's Last Theorem) is at the center of most of contemporary number theory. The paper with Taylor that completed the proof has been cited by 357 publications, which makes it the fifth most cited journal article in number theory of all time (the most cited article is the proof of FLT itself, with more than 600 citations!). These publications would not have existed without the "perspectives, concepts, methods of proof" introduced by Wiles in order to solve the (relatively marginal) question of FLT.

Quotation from IRCAM presentation

It is clear in any case that the work of Wiles et al is provisional, from the point of view of the ***Langlands program****. They have succeeded in solving burning questions which stopped burning when they were solved. Recall Weil's remark ...about the Bhagavad-Gita's teaching that "knowledge and indifference happens at the same time." Perhaps this expresses the "oppression of finiteness" as in Badiou's book. Still, solving a big conjecture leaves neither a void, nor a cold beauty, but rather* ***the feeling that one has not yet found the question that should have been asked in place of the conjecture****.*
This is certainly the case with the work of Wiles. There are the Langlands conjectures which go far beyond any specific case where they have been demonstrated. The resolution of Fermat's theorem within the framework of the Langlands program could never be a ***déchet****, because it confirms the depth of this program. But nor does it clarify the* ***deep reason*** *for this program.*

IRCAM talk, continued

There are three *travaux créatifs* to distinguish here:

- Le travail de Langlands et ses précurseurs : formulation du programme;

- Le travail de Wiles et ceux qui l'ont suivi : vérification du programme et conséquences

- Le travail d'un groupe de mathématiciens qui ne sont pas encore nés: détermination de la *raison profonde* du programme.

Quine-Putnam indispensability thesis (formulated most explicitly by Hilary Putnam): one needs to admit the existence of mathematical objects for the purposes of natural science.

quantification over mathematical entities is indispensable for science, both formal and physical; therefore we should accept such quantification; but this commits us to accepting the existence of the mathematical entities in question. This type of argument stems, of course, from Quine, who has for years stressed both the indispensability of quantification over mathematical entities and the intellectual dishonesty of denying the existence of what one daily presupposes. (Putnam, *Philosophy of Logic*, Chapter 8)

Putnam uses the continuum hypothesis to reject realism

Gödel's proof that the continuum hypothesis is consistent with ZFC is based on the principle called $V = L$.

Cohen's proof that the negation of the continuum hypothesis is also consistent with ZFC entails $V \neq L$.

the realist standpoint is that there is a fact of the matter—a fact independent of our legislation—as to whether $V = L$ or not.

(Putnam, "Models and Reality," 1980)

Putnam uses this and similar examples to reject "metaphysical realism" and thus to subordinate Aristotle's *first philosophy* to *science*:

The program of realism in the philosophy of science-of ***empirical*** *realism, not metaphysical realism-is to show that scientific theories can be regarded as better and better representations of an objective world with which we are interacting*... (*Ibid.*)

Question: In view of these competing attitudes to First Philosophy, what dialogue is possible between philosophy in the Quine-Putnam style and that exemplified by Husserl in Derrida's reading?

Ian Hacking on Wittgenstein's notion of cartesian proof

It is tempting to co-opt the apt words used by Wittgenstein's translators: 'perspicuous' and 'surveyable', and say that Descartes wanted proof to be both. Here is Wittgenstein's key sentence of the late 1930s.

*Perspicuity (**Übersichtlichkeit**) is part of proof. If the process by which I get a result were not surveyable (**übersehbar**) I might indeed make a note that this numbers is what comes out—but what fact is that supposed to confirm for me? I don't know what is **supposed** to come out. (RFM I §153, p. 45).*

(Hacking, *Why Is There Philosophy of Mathematics At All?*, p. 26)

A cartesian (surveyable, perspicuous, synoptic) proof

I see it and I believe it (Plato’s *Meno*)

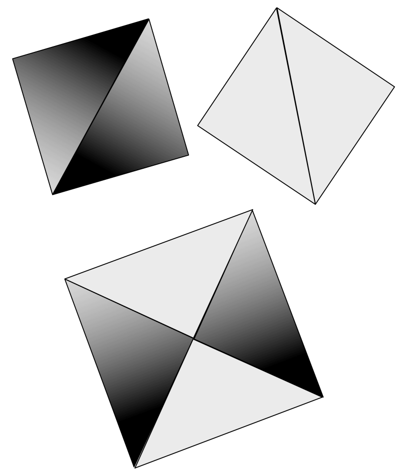


Figure: Dividing a square into two squares

Hacking on Wittgenstein, continued

If you are inclined to use Wittgenstein's words, you may find it useful to observe that he ***introduced*** *them, in connection with maths, in the quotation above. Both* ***Übersichtlichkeit*** *and* ***übersehbar*** *are used in §54. Thereafter he* ***quoted*** *those sentences, marked in quotation marks, and commented upon the words. There is a sense (Quine's) in which he hardly ever* ***used*** *the words in connection with mathematics after their first usage; rather he elucidated what he had meant.*

(Hacking, *Ibid.*)

Hacking contrasts *cartesian* proofs, like the one in *Meno*, with *leibnizian* proofs obtained by systematic calculation on the basis of rules, not necessarily guided by an idea. The terminology is due to Hacking, who suspects that most proofs are leibnizian. Wittgenstein is identified as a cartesian on the basis of the quotation from *RFM I*, where he talks about what is *supposed* to come out.

Where does "surveyability" fit in Wittgenstein's philosophy?

Elsewhere in *RFM* Wittgenstein speaks of "the hardness of the logical **must**" (an expression found in other collections of his comments). Then there is this image:

> What is the characteristic mark of 'internal properties'? That they persist always, unalterably, in the whole they constitute; as it were independently of any outside happenings. As the construction of a machine on paper does not break when the machine itself succumbs to external forces. — Or again, I should like to say that they are not subject to wind and weather like physical features of things; rather they are unassailable, like shadows. (RFM 74)

(From *RFM*, I, §102)

Although Wittgenstein, like Quine, stresses the social consensus necessary for mathematics, the undeniable experience of logical compulsion pervades *RFM.*

Are mathematics and logical compulsion identical?

The question arises: can the machine that does not break, the unassailable shadow, be imagined without an experience of mathematics? Or is the compelling quality of mathematics inherited from our ability to conceive of these metaphors?