Week 7

Mathematical genres

Who are our great authors?

Au cours de la réception de clôture… je croise une élégante inconnue : « ah », me dit-elle, « vous êtes mathématicien ! Quels sont vos grands auteurs ? »

J'ai oublié ma réponse, mais je me souviens d'avoir été frappé par l'étrangeté de la question. Une mathématicienne se serait enquise : « Quel est votre domaine ? … ». A. Weil, fidèle à sa légende, aurait peut-être demandé : « Quel est votre grand théorème ? » – et se serait amusé du bredouillis qu'une question si intimidante n'aurait manqué de susciter. Mais « vos grands auteurs ? »...

C'est moins l'étrangeté de la question, d'ailleurs, que l'étrangeté de cette étrangeté qui n'a cessé de m'interpeller : pourquoi semblait-il si saugrenu de s'enquérir des grands auteurs d'un mathématicien ? N'y aurait-il pas, n'y aurait-il plus, de grands auteurs ? La mathématique aurait-elle une littérature sans Auteur ?

(Yves André, "Réflexions sur l'écriture et le style en mathématique," 2017)

Some historical examples

André finds a list of examples in P. Mancosu's article "Mathematical Style" in the Stanford Encyclopedia of Mathematics:

L'analyse ne diffère du style d'Archimède que dans les expressions, qui sont plus directes et plus conformes à l'art d'inventer.

(Leibniz, 1701)

Style, in fact, is so intimately welded to the spirit of a methodology that it must advance in step with it…

(Chasles, 1837, writing about Monge)

Dedekind's legacy … consisted not only of important theorems, examples, and concepts, but of a whole style of mathematics that has been an inspiration to each successive generation.

(Edwards, 1980)

The Style of any mathematics which comes into being, depends wholly on the Culture in which it is rooted, the sort of mankind it is that ponders it. …The idea of the Euclidean geometry is actualized in the earliest forms of Classical ornament, and that of the Infinitesimal Calculus in the earliest forms of Gothic architecture, centuries before the first learned mathematicians of the respective Cultures were born.

(Spengler, *Decline of the West*, 1919)

the German **esprit** *is essentially* **esprit de géométrie***…The Germans are geometers, they are not subtle [*fin*]; the Germans completely lack* **esprit de finesse***.*

(Pierre Duhem, 1915)

[Erhard Schmidt's'] system is directed towards the objects, the construction is organic. By contrast, Landau's style is foreign to reality, antagonistic to life, inorganic.

(L. Bieberbach (a Nazi), 1934)

Die Logik bleibt »Stillos« wie die Mathematik. **(Broch, *Schlafwandler*, III, 34)**

Mathematical style, just like literary style, is subject to important fluctuations in passing from one historical age to another. Without doubt, every author possesses an individual style; but one can also notice in each historical age a general tendency that is quite well recognizable. This style, under the influence of powerful mathematical personalities, is subject every once in a while to revolutions that inflect writing, and thus thought, for the following periods.

(Claude Chevalley, 1935, the year he co-founded Bourbaki)

Il n'y a guère qu'en mathématique que la pensée vivante se moule encore dans un canon datant de 2300 ans (pas seulement une tradition, mais un canon d'écriture). ...

(André, *Ibid.*)

Exposition or argument?

Where should the mathematical text be placed among the four traditional modes of rhetoric: narration, description, exposition, argumentation?… in the first place it belongs in the mode of exposition, even if proof ties it as well, of course, to argumentation.

(André, *Ibid.*)

Contrast with:

*Mathematical writing is **argument**. It is argument in its purest form, one might say—argument that relies on axiomatically established, conventionally agreed-upon truth conditions. However, it can, and in today's practice most often does, take the form of narrative… as an abstract sequence of events recounted by a narrator.*

(Christina Pawlowitsch, "Making See", 2020)

It was not always thus

On dira …qu'une telle conception individuelle du style [he had just quoted Barthes] *n'a pas sa place dans un texte mathématique. De fait, les contraintes de l'écriture mathématique corsettent très sévèrement le style individuel; de manière plus draconienne encore dans le canon qui s'est imposé, au prétexte de rigueur, sous l'influence de l'écriture de Bourbaki (et avant lui Gauss, Dedekind et d'autres). C'est un genre institué qui formate le texte, et bannit désormais la prose coulante d'un Cauchy tout comme le laconisme visionnaire d'un Riemann ou le jaillissement disert et imagé d'un Poincaré. Double bannissement : non seulement on ne peut plus écrire comme ces grands auteurs, mais la plupart des mathématiciens d'aujourd'hui ne peuvent plus les lire sans truchement et ne les citent presque jamais « dans le texte ».*

(André, *Ibid.*)

Genres of mathematical expression

Mathematical expression has nevertheless developed in a surprisingly varied range of genres. A few years ago, three mathematicians independently and almost simultaneously had the idea of writing a book on the model of Queneau's *Exercises in Style*, to illustrate some of this variety. Queneau told the same story in 99 ways, and Philip Ording wrote 99 versions of the proof of the solution to a simple cubic equation. (The other two versions are by John McCleary and by a pseudonymous author in France, and the three books are all quite different.)

Ian Hacking on Wittgenstein's notion of cartesian proof

It is tempting to co-opt the apt words used by Wittgenstein's translators: 'perspicuous' and 'surveyable', and say that Descartes wanted proof to be both. Here is Wittgenstein's key sentence of the late 1930s.

*Perspicuity (**Übersichtlichkeit**) is part of proof. If the process by which I get a result were not surveyable (**übersehbar**) I might indeed make a note that this numbers is what comes out—but what fact is that supposed to confirm for me? I don't know what is **supposed** to come out. (RFM I §153, p. 45).*

(Hacking, *Why Is There Philosophy of Mathematics At All?*, p. 26)

A cartesian (surveyable, perspicuous, synoptic) proof

(Plato's *Meno*)

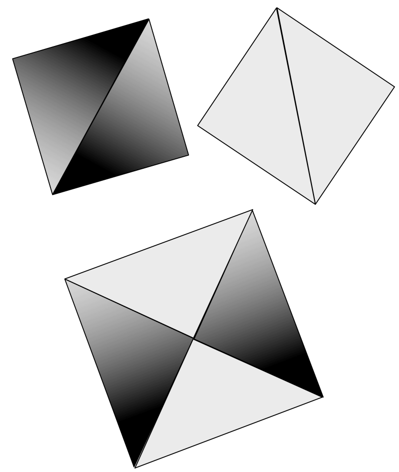

Figure: Dividing a square into two squares

Hacking on Wittgenstein, continued

If you are inclined to use Wittgenstein's words, you may find it useful to observe that he ***introduced*** *them, in connection with maths, in the quotation above. Both* ***Übersichtlichkeit*** *and* ***übersehbar*** *are used in §54. Thereafter he* ***quoted*** *those sentences, marked in quotation marks, and commented upon the words. There is a sense (Quine's) in which he hardly ever* ***used*** *the words in connection with mathematics after their first usage; rather he elucidated what he had meant.*

(Hacking, *Ibid.*)

Hacking contrasts *cartesian* proofs, like the one in *Meno*, with *leibnizian* proofs obtained by systematic calculation on the basis of rules, not necessarily guided by an *idea*. The terminology is due to Hacking, who suspects that most proofs are leibnizian. Wittgenstein is identified as a cartesian on the basis of the quotation from *RFM I*, where he talks about what is *supposed* to come out.

Close reading of a proof by Robert Thomason

Lemma 5.5.1. *Let X be a scheme with an ample family of line bundles, a fortiori a quasi-compact and quasi-separated scheme. Let $j : U \to X$ be an open immersion with U quasi-compact. Then for every perfect complex F' on U, there exists a perfect complex E' on X such that F' is isomorphic to a summand of j^*E in the derived category $D(\mathcal{O}_U - Mod)$.*

Proof. [1] Consider Rj_*F' on X.

[1] The function of this sentence is to reintroduce the protagonist F' in a new guise (Rj∗F') and in a new setting, namely, the scheme X. In its original form, the PC F' is native to U; the prefix Rj∗, one of *Grothendieck's six functors*, is the transitive verb that effects F''s migration from U to X. Think F' = *Guillaume* and Rj∗F' = *William* in 1066, U = Normandy, X = England… (with an ample supply of grain? trees? sheep?)

(Why is the sentence written in the *imperative mode*? Compare "Consider the lilies of the field, how they grow: they neither toil nor spin…")

William the Conqueror in England

[2] This complex is cohomologically bounded below with quasi-coherent cohomology (B.6),

[2] This is part of what it means for F' to be a PC, part of its heritage, a resource on which it can draw in its quest on X’s foreign soil.

If F' is *Guillaume* then the quasi-coherent cohomology might be his *cavalry*.

Hastings

[3] and so by 2.3.3 is quasi-isomorphic to a colimit of a directed

system of strict perfect complexes E'_α,

$$\varinjlim_{\alpha} E'_\alpha \simeq Rj_* F'. \qquad (5.5.1.1)$$

[3] This is the direct limit characterization, as suggested by Trobaugh's ghost. In the new world of the scheme X [*England*], the avatar Rj_*F' [*William*] is no longer itself a PC [*a native king*]. The result 2.3.3 details its relation to PCs [*William has conquered the native kings*].

This is the first instance of *discovery* (*anagnorisis*), in the sense of Aristotle's *Poetics*, to occur in this short narrative. As Thomason is at pains to explain, it also makes the turning point possible… Formula (5.5.1.1) is a diagrammatic representation of this discovery.

Back in Normandy

[4] We consider the induced isomorphism in $D^+(\mathcal{O}_U - Mod)$

$$\lim_{\substack{\longrightarrow \\ \alpha}} j^* E_\alpha = j^* \left(\lim_{\substack{\longrightarrow \\ \alpha}} E'_\alpha \right) \simeq j^* R j_*(F') \simeq F'. \qquad (5.5.1.2)$$

[4] The narrative is highly compressed at this point. Protagonist F''s quest is to redefine its status on U in terms of a PC E' native to X. The first steps have seen F' wandering to X in search of an E'; in [3], it has discovered a (potentially infinite) collection of E_α. The authors now consider what happens upon deploying a second transitive verb, the prefix j* [*762 ships*] another one of Grothendieck's six functors that mediate the transition from X [*England*] back to U [*Normandy*]: "isomorphism in $D^+(O_U\text{-Mod})$." The right-hand side of formula (5.5.1.2) reminds us that *Guillaume*, having wandered to *England* and become *William*, now returns to *Normandy* with the help of *ships* and returns to *his original status*. But j* transforms [*transports*] each E_α [*the English kings have their own ships, in this alternative history*] and indeed transforms them all simultaneously; this is the meaning of the left-hand side of (5.5.1.2).

(In [4] the authors rely on readers' knowledge of the folklore concerning Grothendieck's six functors [*ships*], especially how two of them applied in the correct order return the protagonist to its rightful form.)

[5] By 2.4.1(f), the map (5.5.1.3) is an isomorphism

$$\varinjlim_{\alpha} Mor_{D(U)}(F', j^* E'_\alpha) \cong Mor_{D(U)}(F', \varinjlim_{\alpha} j^* E'_\alpha). \qquad (5.5.1.3)$$

[5] The second instance of discovery. The horde of $E\alpha$ has followed F', disguised as Rj*F', back to *Normandy*, becoming j* $E\alpha$ in the process. Now *Guillaume* turns to confront the invaders. But the protagonist (and the authors) are prepared: 2.4.1(f) reassures us that F''s war with the entire army of *English kings* is nothing more nor less than a series of single combats. This "nothing more nor less than" is a translation of the symbol ≅ in the middle of formula (5.5.1.3).

[6] Thus in $D(\mathcal{O}_U - Mod)$ the inverse isomorphism to (5.5.1.2) must factor through some $j^*E'_\alpha$. [7] Thus F' is a summand of $j^*E'_\alpha$ in $D(\mathcal{O}_U - Mod)$, proving the lemma.

[6] This is the climax of the battle. Back on F''s home terrain of D(OU – Mod), F''s confrontation with the j* Eα comes down to the single decisive encounter with $j^*E'_\alpha$ [*Harold of Wessex*].

Implicit in this conclusion is the apparent paradox that, in seeking a new identity in the possibly infinite collection of Eα, it is F''s very finiteness, part of its very nature as a PC [*native king*], that allows it to single out one Eα, namely E'_α, to be the E' of the statement of the lemma.

[7] is what Aristotle called the *sumperasma*, the recapitulation of the conclusion of the lemma, purely to remind the reader that the goal has been accomplished.

Beginning, first version

"Our paper would have been impossible without Harris' tensor product trick and it is a pleasure to acknowledge our debt to him." (Barnet-Lamb, Gee, Geraghty, Taylor, circa 2010)

Q. (with easy answer): How did they know it was a trick?

A. "The principal innovation is a tensor product trick that converts an odd-dimensional representation to an even-dimensional representation." (M. Harris, circa 2007)

Some famous tricks

Q. (with hard answer): How did *I* know it was a trick? What did it have in common with...

- ... Cantor's diagonalization trick?
- ... Weyl's unitarian trick (due to Hurwitz and Schur)?
- ... Lieberman's trick, Rabinowitch's trick, the Eilenberg-Mazur swindle*...
- ... the tricks in the (now dormant) *Tricki* ("tricks wiki")?

* $1 = 1 + 0 + 0 + \ldots = 1 + (-1+1) + (-1+1) \ldots$
$= (1+-1) + (1+-1) + (1+-1) = 0 + 0 + 0 \ldots = 0$

What is not a trick

A straightforward calculation is certainly not a trick. Nor is a syllogism, a standard estimate of magnitude, or a reference to the literature. Can I be more precise? Probably not. Capital-M Mathematics is neatly divided among axioms, definitions, theorems, and proofs; the mathematics of mathematicians blurs taxonomical boundaries.

A mathematical trick, like a trickster, is a notorious crosser of conventional borders; a “lord of in-between” like the devil who taught Robert Johnson to play the guitar, who “dwells at the crossroads.” A mathematical trick simultaneously disturbs the settled order and “makes this world” (Hyde).

What may be a trick

I would suggest that a trick involves drawing attention to an intrinsic element of a mathematical situation that *appears to be* but *is not in fact* irrelevant to the problem under consideration.

Alternatively, since a trick need not be subordinated to a pre-existing problem, it provides an unexpected point of contact, like a play on words, between two domains not previously known to be related.

The semantic field

The ambivalence of tricks, the sense of getting something for nothing, persists in other languages:

- Dutch *truuk (truc)* "Mostly used in connection with magicians and card tricks ... a 'truuk' cannot be something very serious." (The word "serious" needs to be taken very seriously!)
- Russian *tryuk* in other settings can mean deceit or craftiness.
- French *astuce* positive since 19th cent.; but see Oresme (1370): *Et doncques se l'entention est malvese, tele puissance est appellée astuce ou malicieuseté*
- Germans often use the English word "trick", traditionally called a *Kunstgriff*
  (Cf. Schopenhauer's *Eristische Dialektik* where *Kunstgriff* means "dishonest trick for winning arguments") .

Magic vs. metaphysical realism

As idealized by logical empiricist philosophers, Mathematics is insensitive to the complex interplay of delight (a "neat trick") and disdain (a "cheap trick") that accompanies the revelation of a new mathematical trick and constitutes a privileged moment of pleasure, like García Márquez's reaction, nearly falling out of bed when he read the first sentence of Kafka's *Metamorphosis*: "I didn't know you were allowed to write like that."

High vs. low mathematics, I

"the use of projections in finding the perpendicular form of a line is undesirable because it seems to the student to be only another trick of the omniscient teacher." (*Am. Math. Monthly*, 1917)

"most of us ... teach ... in a way that discourages students by giving them the impression that excellence in mathematical science is a matter of trick methods and even legerdemain." (MAA, 1940)

In contrast, in its advice to prospective authors of mathematical articles, the AMS gives tricks a positive valuation:
"Omit any computation which is routine (i.e. does not depend on unexpected tricks). Merely indicate the starting point, describe the procedure, and state the outcome."

More Weyl tricks

The first tricks in the most prestigious US journals of the time were both due to Weyl:

Starting with a given solution ... of Legendre's equations,we make use of the same trick as in §20...
H. Weyl, *Annals of Math.* (1935).

I transform the expression to which our method immediately leads by a very simple trick.
H. Weyl, Mean Motion II, *Am. J. Math.* (1939)

Normal mathematics, 1

The following figures, unlike the trickster, remain safely within conventional borders. First the lower orders:

The lumberjack: “We are in a forest whose trees will not fall with a few timid hatchet blows. We have to take up the double-bitted axe and the cross-cut saw, and hope that our muscles are equal to them” (Langlands, 1979).

The *bâtisseur*: “Il n’a que deux mains comme tout le monde – mais deux mains qui ne répugnent ni aux plus grosses besognes, ni aux plus délicates” (Grothendieck, *Récoltes et sémailles*)

Normal mathematics, 2

The philosopher: "The word 'philosophy' was fashionable in 1967, no longer so by 1979. There were lots of philosophies in 1967... It was just the way people talked" (W. Casselman, in response to my question about the "Langlands philosophy")

The yogi: "Par "yoga" il [Grothendieck] entendait un point de vue unifiant, une piste dans la recherche des concepts et des démonstrations, une méthode qu'on pouvait réutiliser" (P. Cartier, 2011)

Normal mathematics, 3

In between the routine and the exalted one finds the level of normal mathematical problem-solving, descriptions of which are dominated by the vocabulary of *combat*, with words like strategy, attack, and brute force prominent.

Trifunctionalism

Mathematical ethnography collides with Georges Dumézil’s trifunctional theory of Indo-European mythology:

Varna	Mathematical equivalent
Brahmin	Philosophy, yoga
Kshatriya	Strategy, attack
Vaishya, Shudra	Lumberjack, builder, machine, tools

In case you find that far-fetched...

The 'structures' are tools for the mathematician; as soon as he [sic] has recognized ... relations which satisfy the axioms of a known type, he has at his disposal immediately the entire arsenal of general theorems... Previously... he was obliged to forge for himself the means of attack... One could say that the axiomatic method is nothing but the "Taylor system" for mathematics.

(Bourbaki, *The Architecture of Mathematics, 1950*)

Having exhibited the mathematician as blacksmith and assembly line worker as well as military strategist, Bourbaki reminds us in the next paragraph of the (charismatic) first function:

Bourbaki’s trifunctionalism, continued

This is however, a very poor analogy; the mathematician does not work like a machine, nor as the workingman on a moving belt; we can not over-emphasize the fundamental role played in his research by a special intuition... not the popular sense-intuition, but rather a kind of direct divination ... which orients at one stroke in an unexpected direction the intuitive course of his thought, and which illumines with a new light the mathematical landscape.

What about the trickster?

Perhaps the most cogent objection to the trifunctional model: mathematics is hardly the only activity to which a trifunctional analysis can be applied.
The distinctiveness of mathematics may lie in the nature of its characteristic *tricks*.
Unlike *business* (cf. management philosophy, commercial strategy, marketing tools) or *politics* (philosophy of government, political strategy, and techniques of communication), mathematics has no place for *dirty tricks*.

As mathematicians, we play and dream but we don't cheat.
(Marie-France Vignéras)

al-Farabi’s *Catalogue of the Sciences*

In the mathematical chapter of his Catalogue of the Sciences (*ihsa al-‘ulum*) the 10th century Baghdad philosopher al-Farabi listed algebra, not as a free-standing branch of mathematics like arithmetic and geometry (following Aristotle) but rather alongside mechanical devices, in a final section on *‘ilm al-hiyal* – “the science of *al-hiyal* (singular *hila’*) an equivalent of the Greek *mekhane*, variously translated as “ingenious devices,” “mechanics,” or “tricks.”
A century earlier the Banu Musa brothers had published *Kitab al-hiyal*, a celebrated catalogue of mechanical devices, including automata.

For more information on the Banu Musa, see Truitt, *Medieval Robots*, p. 20.

'ilm al-hiyal, from al-Farabi

The seven branches of mathematics: arithmetic, geometry, optics, astronomy, music, weights, and:

علم الحيل

وأما علم الحيل ـ فإنه علم وجه التدبير في مطابقة جميع ما يبرهن
وجوده في التعاليم التي سلف ذكرها بالقول والبرهان على الأجسام
الطبيعية وإيجادها ووضعها فيها . وذلك أن تلك العلوم كلها إنما تنظر
في الخطوط والسطوح والمجسمات وفي الأعداد وسائر ما تنظر على أنها
معقولة وحدها منتزعة من الأجسام الطبيعية . ويحتاج عند إيجاد هذه
وإظهارها بالإرادة والصنعة في الأجسام الطبيعية والمحسوسات التي قد
تبين أنه يتأتى إيجادها فيها ومطابقتها عليها من قبل أن للمواد والأجسام
المحسوسة أحوالاً تعوق أن توضع فيها كيف اتفق ، وبأي وجه ما اتفق ،
بل يحتاج أن توطأ الأجسام الطبيعية لقبول ما يلتمس من إيجاد هذه

al-Farabi on algebra

فمنها : الحيل العددية . وهي على وجوه كثيرة : منها العلم

المعروف عند أهل زماننا بالجبر والمقابلة (١) وما شاكل ذلك . على أن
هذا العلم مشترك للعدد والهندسة ،وهو يشتمل على وجوه التدابير في
استخراج الأعداد التي سبيلها أن تستعمل فيما أعطى أقليدس أصولها
من المنطقة والصم في المقالة العاشرة من كتابه في الاسطقسات وفيما لم
يذكر منها في تلك المقالة (٢) .

La ciencia de los ingenios una es aritmética, y tiene muchos respectos, y otra es la ciencia conocida entre nosotros por *Álgebra* y *Mocábala* y lo semejante a esto. Pues esta ciencia es común con la aritmética y la geometría y se ocupa de los modos de dirección en la invención de los números que se deben usar, según los

(More precisely: “among them are numerical tricks . . . and the science known to us as algebra and al-moqabala ”)

al-Khwarizmi and al-Khayyam

For Aristotelians, and for the medieval Arab philosophers, al-Khwarizmi’s algebra could not be a science because it applied indiscriminately to arithmetic and to geometry. One century after al-Farabi, Avicenna included geometry, astronomy, arithmetic, and music in the chapter of his Book of Science (*Daneš-nama*) devoted to mathematics but relegated algebra to the list of “secondary parts of arithmetic.”

Omar al-Khayyam thought otherwise:

Those who think algebra is a trick [*hila‘* – the only appearance of the word in his surviving works] to determine unknown numbers think the unthinkable; therefore you must not pay attention to those who judge by appearances.

From *mekhane* to *Kunstgriff*, 1

The Arabic *hila’* translates the Greek *mekhane* (as *‘ilm* – science – translates *episteme*).

If we dig a little deeper, we find Plutarch’s account of Plato’s rejection of *mechanical* methods in mathematics, “a sort of foundation myth for the science of mechanics”[1] which must have been familiar to al-Farabi and Avicenna, and “which explained the separation of mechanics from philosophy as the result of a quarrel between two philosophers.”

[1] Stanford Encyclopedia of Philosophy, *Archytas*

From *mekhane* to *Kunstgriff*, 2

Moving forward, Gherard of Cremona's Latin version of al-Farabi's catalogue translated *'ilm al-hiyal* by *ingeniorum scientia* (the science of *ingenium*, the Latin equivalent of *mekhane*); that *ingenium* also figured in Latin texts on mathematics as well as in the title of Descartes' early Rules for the Direction of the Mind [*ingenium*] [2] and that it admits a great variety of German translations, one of which is *Kunstgriff*.

The continuing associations of *ingenium* with machines (engineering) as well as genius, at the two ends of the trifunctional scale, neatly mirror mathematicians' ambivalence to tricks, and incidentally suggests that anything a mechanical theorem prover could invent would be assigned *ipso facto* the status of trick.

[2] in more than one relevant way an exact inversion of Aristotle's value system

The trickster as bridge between high and low

Compare:

- Krishna as Vishnu *and* his human avatar;
- Prometheus, who brought fire from heaven to earth;
- Hermes, messenger of the gods and the soul's guide to the underworld;
- Mephistopheles plays a similar role in the Faust legend;
- Esu, the Yoruba divine trickster, limped because his legs were of different lengths: "one anchored in the realm of the gods, ... the other ... in ... our human world."

The mathematical trick predates logicist or formalist idealizations and offers a short-cut bypassing the idealized route from human practice to inscription of theorems in the register of the eternals.

Bourbaki disdains tricks

Bourbaki's *Architecture of Mathematics* alludes to tricks only once:

[T]he axiomatic method has its cornerstone in the conviction that, not only is mathematics not a randomly developing concatenation of syllogisms, but neither is it a collection of more or less "astute" tricks, arrived at by lucky combinations, in which purely technical cleverness wins the day.

My provisional hypothesis is that the mathematical trickster serves as a bridge between high and low genres.

Why so serious?

(The Joker, in *The Dark Knight*)

Generic answer: the Professor represents the University, which is a locus of Power, and Power demands to be taken seriously.

Here are three hypothetical answers to the Joker's question that are *specific to mathematics*.

The theological answer

The word *mathematicus* was used primarily for astrologers, including Kepler (the Hapsburg emperor's *mathematicus*) and Galileo, both active just before Giordano's generation. Moreover,

Das gefiel Doctor Fausto wol speculiert vnnd studiert Tag vnnd Nacht darjnnen
Wolt sich hernach kein Theologum mehr nennen lassen
ward ein Weltmensch
Nennt sich ein Doctor Medicinae, ward ein Astrologus vnnd Mathematicus

(from the *Faustbuch*, circa 1580)

Luca Giordano, *Un matemático*, Museo Nacional de Bellas Artes, Buenos Aires

The ontological answer (High and low mathematics, II)

"The 'seriousness' of a mathematical theorem lies... in the significance of the mathematical ideas which it connects."
(G. H. Hardy)

Grothendieck made a special trip back to the IHES from his provincial exile to learn about Deligne's proof of the last of the Weil conjectures.

"If I had done it using motives," recalled Deligne, "he would have been very interested, because it would have meant the theory of motives had been developed. Since the proof used a trick, he did not care."

The ontological reason (High and low mathematics, II)

Serre reacted differently: “cela te choque,” he wrote to Grothendieck (about Deligne’s proof), “mais cela me ravit.”

And Langlands disagreed with Grothendieck’s ontological hierarchy: “perhaps [Grothendieck] could have drawn a different conclusion.” Deligne’s proof was based on “a profound understanding of the étale cohomology theory accompanied by an observation arising in the theory of automorphic forms.”

Which goes to show that trickiness is not an intrinsic, much less quantifiable property of a mathematical text.

The socio-political answer

H. Mehrtens stresses that aspirations to seriousness were integral to the modernization process. “As a mathematician [Till] Eulenspiegel transformed himself from an anarchist fool to a theologian.” Nor did mathematics merely leave its trickster outfits behind: the German modernizers were conscious of parallels between their goals and those of their artistic contemporaries. When the mathematician Heinrich Liebmann described mathematics as a “freie, schöpferische Kunst” in his Leipzig inaugural address, (explicitly citing the Berliner Secession painter Max Liebermann), the “freedom” he had in mind was to establish one’s own “Qualitätskriterien” and not to be subject to the criteria of engineers, teachers, and philosophers, just as Liebermann’s standards were not set by the Kaiser.

Sketch of Wiles's proof

Suppose, contrary to Fermat's claim, there is a triple of positive integers a, b, c such that

(A) $$a^p + b^p = c^p$$

for some odd *prime* number p (it's enough to consider prime exponents). In 1985, Gerhard Frey had pointed out that a, b, and c could be rearranged into

(B) a new equation, called an *elliptic curve*,

$$y^2 = x(x - a^p)(x + b^p)$$

with properties that were universally expected to be impossible.

More precisely, it had long been known how to leverage an equation like (B) into

(C) a *Galois representation*,

which is an infinite collection of equations that are related to (B), and to each other, by precise rules.

The links between (A), (B), and (C) were all well-understood in 1985. But by that year, most number theorists were convinced — mainly thanks to the insights of the *Langlands program*, named after the Canadian mathematician Robert P. Langlands — that to every object of type (C) one could assign, again by a precise rule,

(D) a *modular form*,

which is a kind of two-dimensional generalization of the familiar sine and cosine functions. The final link was provided when Ken Ribet confirmed a suggestion by Jean-Pierre Serre that the properties of the modular form (D) entailed by the form of Frey's equation (B) implied the existence of

(E) another *modular form, this one of weight 2 and level 2*.

But there are no such forms!

Therefore there is no Galois representation (C),
therefore no equation (B),
therefore no solution (A).

(Logical puzzle: how can equation (B) not exist? We wrote it down…)
This is a classic *proof by contradiction* and it works provided the missing link between (C) and (D) — the *modularity conjecture* — could be established.

Wiles's proof is the beginning, not the end, of a narrative

Wiles proved the modularity conjecture — the link between (C) and (D)s, which (unlike Fermat's Last Theorem) is at the center of most of contemporary number theory. The paper with Taylor that completed the proof has been cited by 357 publications, which makes it the fifth most cited journal article in number theory of all time (the most cited article is the proof of FLT itself, with more than 600 citations!). These publications would not have existed without the "perspectives, concepts, methods of proof" introduced by Wiles in order to solve the (relatively marginal) question of FLT.