Week 9

Can mathematics be done by machine?

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If Amy Coney Barrett says her personal and religious views would not affect her rulings then why don't we just feed the constitution into a computer and have the next justice be a robot?

(October 13, 2020)

Summary

Proof, in the form of step by step deduction, following the rules of logical reasoning, is the ultimate test of validity in mathematics. Some proofs, however, are so long or complex, or both, that they cannot be checked for errors by human experts. In response, a small but growing community of mathematicians, collaborating with computer scientists, have designed systems that allow proofs to be verified by machine. The success in certifying proofs of some prestigious theorems has led some mathematicians to propose a complete rethinking of the profession, requiring future proofs to be written in computer readable code. A few mathematicians have gone so far as to predict that artificial intelligence will replace humans in mathematical research, as in so many other activities.

Mechanization of what mathematics?

One's position on the possible future mechanization of proof is a function of one's view of mathematics itself.

Is it a **means to an end** that can be achieved as well, or better, by a competent machine as by a human being? If so, **what is that end**, and why are machines seen as more reliable than humans?

Or is mathematics rather **an end in itself**, a human practice that is pursued for its intrinsic value? If so, **what could that value be**, and can it ever be shared with machines?

Questions to keep in mind

Most articles about controversies regarding the AI future of mathematics focus primarily on two questions — "Is it good or bad?" and "Will it work or not?" — while neglecting to reflect on the presuppositions that underlie these questions — what is the *good* of mathematics, and work to what end? — not to mention what should always be the first question to be addressed to any significant social development — *cui bono*, in whose interest?

The ethical evaluation of mechanized mathematics should remember that what pure mathematicians find valuable about what we do is precisely that it provides a kind of understanding **whose value is not determined by the logic of the market**. (But I suspect mechanical mathematicians will be programmed, or naturally inclined, to disregard that value!)

Mathematics and utility

…as more arts were invented, and some were directed to the necessities of life, others to its recreation, the inventors of the latter were naturally always regarded as wiser than the inventors of the former, because their branches of knowledge did not aim at utility. Hence when all such inventions were already established, the sciences which do not aim at giving pleasure or at the necessities of life were discovered, and first in the places where men first began to have leisure. This is why the mathematical arts were founded in Egypt; for there the priestly caste was allowed to be at leisure.

Aristotle, *Metaphysics*, 981b

Si les trois âges du concept sont l'encyclopédie, la pédagogie et la formation professionnelle commerciale, seul le second peut nous empêcher de tomber des sommets du premier dans le désastre absolu du troisième, désastre absolu pour la pensée, quels qu'en soient bien entendu les bénéfices sociaux du point de vue du capitalisme universel.

Gilles Deleuze et Félix Guattari, *Qu'est-ce que la philosophie?*

A ten-year challenge

Electronic digital computing was not yet 15 years old in 1958 when Alan Newell and Herbert Simon, in a widely-quoted article published in the journal *Operations Research*, predicted that within ten years, digital computers would reach four milestones, on the way to a "world in which [human] intellectual power and speed are outstripped by the intelligence of machines."

The goal of mechanizing aspects of what many took to be the essence of humanness meant that artificial intelligence was a controversial field from its earliest years on.

(Donald MacKenzie, *Mechanizing Proof*)

The reward

The Fredkin Foundation established three prizes in Automatic Theorem Proving (ATP). In the mid-1980s the Foundation asked the AMS to appoint a formal ATP prize committee and to take over the administration of the awards.

The *Leibniz Prize* was to be awarded "for the proof of a 'substantial' theorem in which the computer played a major role." The prize criterion:

"*The quality of the results should not only make the paper a natural candidate for publication in one of the better mathematical journals, but a candidate for one of the established American Mathematical Society (AMS) prizes ... or even a Fields Medal. The proofs should not be less sophisticated than those of classical theorems when they first made their appearance.... Though obviously difficult to define precisely, the role of the computer program in the argument should not be mere auxiliary. Novel techniques, meaningful and original definitions, suggestions of interesting intermediate results, perspectives of wider application--any one of these contributions, and others that cannot be foreseen today, would meet the criteria.*"

Deep Blue

"Since support for these prizes has been withdrawn, currently there are no plans to make future awards." (from the AMS website)

However:

Deep Blue Team Awarded $100,000 Fredkin Prize (*NY TIMES, 7/30/97*)

By JOANN LOVIGLIO

PROVIDENCE, R.I. (AP) -- Creators of IBM's Deep Blue, the computer that beat Garry Kasparov, on Tuesday received a $100,000 prize established 17 years ago to be given the first time a computer beat a world chess champion.

Feng Hsu, Murray Campbell and A. Joseph Hoane Jr. will split Carnegie Mellon University's Fredkin Prize.

"In case there's any question, that's the only prize money we've gotten," Hoane joked before the award ceremony at the American Association for Artificial Intelligence's national conference.

The men said they were surprised by the worldwide attention and debate about the man vs. machine battle that ended with Deep Blue beating Kasparov on May 11 in the final game of a tied, six-game match -- Kasparov's only chess defeat.

"Some people are apprehensive about what the future can bring," Hsu said. "But it's important to remember that a computer is a tool. The fact that a computer won is not a bad thing."

Kasparov won the first match against Deep Blue in February 1996. But after that loss, IBM engineers retooled it, returning with a machine that could think twice as fast.

After Big Blue's victory, an upset Kasparov shrugged and bolted from the table, and later criticized IBM for programming the computer specifically to beat him.

Kasparov was invited to the conference but declined to attend, saying he would be on vacation.

Grand Challenges in AI

The Unfinished Agenda for 21st Century

- Any Language to Any Language Translation among the top 100 languages with less than 5% error and
- Any Spoken Language to Any Spoken Language (Speech To Speech) Translation among the top 100 languages with less than 5% error
- Discovery of a Major Mathematical Result by AI
- Remote Repair in Space
- Self-Reproducing Robots

(from Raj Reddy, Talk at the Heidelberg Laureate Forum 9/27/19)

First Grand Challenge (Determinacy of computation meets indeterminacy of translation)

It is a truth universally acknowledged, that a single man in possession of a good fortune, must be in want of a wife.

(Jane Austen, *Pride and Prejudice*)

It is a universally recognized fact that a man who is lucky should have a wife.

(Google translate, three times via Arabic.)

This is a generally accepted fact. A lucky man must lack a wife.

(Google translate, twice via Chinese)

The "essence of humanness" (in red)

The ten-year timeline of Newell-Simon proved far too optimistic. But opinions differed as to whether the first three milestones — to "be the world chess champion" (unless forbidden to compete), to compose "music that will be accepted by critics as possessing considerable aesthetic value," and to "discover and prove an important new mathematical theorem" — had been reached by 2001, when MacKenzie's book appeared. IBM's supercomputer Deep Blue received no official title, though it had defeated "Classical" world chess champion Garry Kasparov in 1997. Most nevertheless agreed that AI had crossed a meaningful threshhold. Twenty years later, skeptics who continued to insist that human *Go* proficiency could never be mechanized were silenced when Google's DeepMind program AlphaGo defeated the world champion, and was in turn defeated by a self-taught version of the same program.

Has MacKenzie accurately characterized the essence of humanness?

Six reasons not to trust humans with mathematics

1. They are sentimental.

Is a proof no more than its verification of the truth of a proposition, which can therefore be translated unambiguously?

Or does a human proof contain more than a logical deduction, just as Austen’s sentence is more than its literal meaning?

2. They are mortal (and their minds are not open to inspection)

Kevin Buzzard worries that much mathematical knowledge is implicit (as many philosophers have recognized) but often so localized (even within the mind of a single mortal mathematician) that it cannot be considered reproducible. Buzzard is a number theorist who thoroughly understands every step but one in the proof of Fermat's Last Theorem. The exception is the *Langlands-Tunnell theorem*, (after Robert P. Langlands, Jr. and Jerrold Tunnell). Wiles took this theorem on faith, as the starting point for his argument, because it had been checked by the experts.

2. They are mortal (and their minds are not open to inspection).

The Langlands–Tunnell work is sufficiently important that there is little doubt that the experts follow the proof. My question is – "is the experiment reproducible"? Is it science? If there's a nuclear war tomorrow and then one day in the future the paper mathematical literature on our planet is discovered and translated, would the finders be able to put together a full proof of Langlands–Tunnell? Or are there some arguments which are merely "known to the experts"?

Figure: Kevin Buzzard, Professor at Imperial College, London

3. Humans make mistakes

The unreliability of human perceptions is one of the oldest themes in philosophy (a 14000-word entry on the Stanford Encyclopedia). Special properties of deductive reasoning are supposed to confer immunity on mathematics from this unreliability, but since most actual proofs contain mistakes we see that this is insufficient. We have seen several alternative paradigms proposed to account for this discrepancy.

Automated proof verification

This is an attempt to realize the *formalist* paradigm. Its prospects for success can be measured both technically (does it meet the expectations of the program?) and epistemically (to what extent does the mathematical community adhere to the formalist paradigm?) But it also raises a serious question: why are we ready to trust machines more than we trust human beings?

The underlying question is that of the different natures of machines, including possibly-existing-machines, and human beings. One view is of machines as embodiments of human reason, without the imperfections of the latter. Another is of machines as improvements on human beings in many or all respects. Machines don't get tired, machines don't have prejudices, machines don't lie.

4. Humans are political

In a report posted online today, Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt describe what Stix calls a "serious, unfixable gap" within a series of papers by S. Mochizuki.

Between 12 and 18 mathematicians who have studied the proof in depth believe it is correct, wrote Ivan Fesenko of the University of Nottingham in an email. But only mathematicians in "Mochizuki's orbit" have vouched for the proof's correctness, [Brian] Conrad commented in a blog discussion last December. "... nobody else ... has been willing to say even off the record that they are confident the proof is complete." When [Mochizuki] told colleagues the nature of Scholze and Stix's objections, he wrote, his descriptions "were met with a remarkably unanimous response of utter astonishment and even disbelief (at times accompanied by bouts of laughter!) that such manifestly erroneous misunderstandings could have occurred."

(Erica Klarreich, *Quanta*, 9/20/2018)

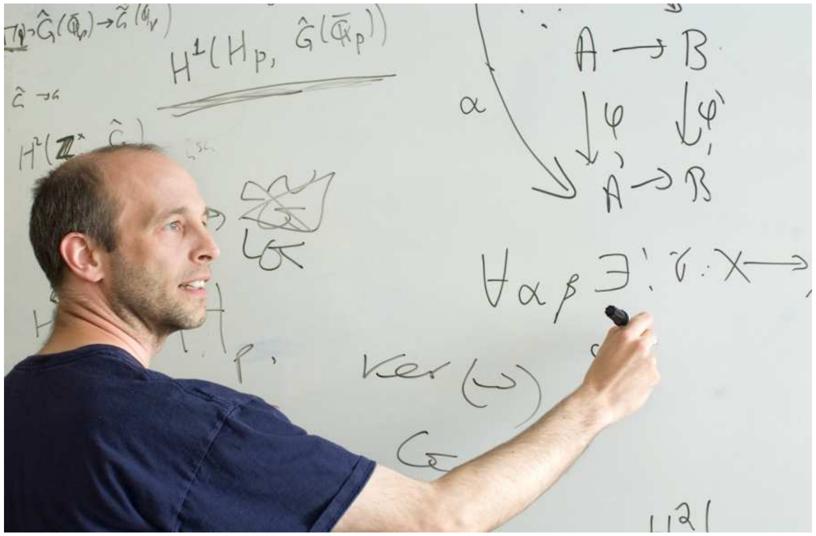
5. Humans get bored

[Voevodsky's new system] hastens the day when our mathematical literature has been verified mechanically and referees are relieved of the tedium of checking the proofs in articles submitted for publication.

(Dan Grayson, 2017)

(See the discussion of 19th century "computers" below, and in Lorraine Daston's article.)

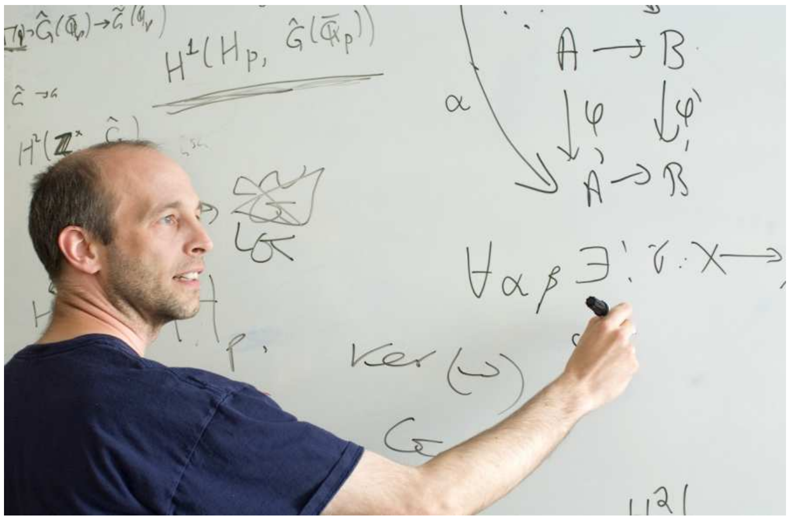
6. Humans are imprecise



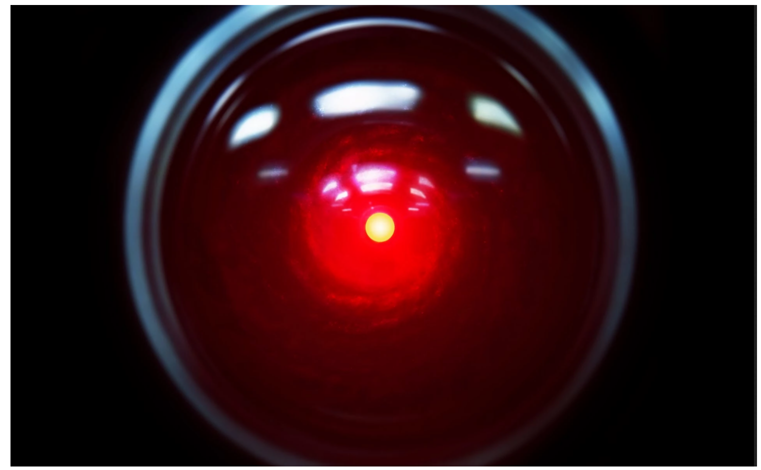
"OK, now note that the polynomial *X* has degree 1."

(Kevin is feeding a proof to a computer one line at a time, and the computer checks that each line follows from the preceding line and its built-in library of established results.)

Who is right: Kevin or HAL?



“X has degree 1, X^2 has degree 2, X^3 has degree 3 . . . ”



“I’m sorry Kevin, I’m afraid I don’t see that.”

Human mathematicians: good riddance!

On pp. 326-327 (of *Mechanizing Proof*), Donald MacKenzie quotes a computer scientist who argued that mathematicians' failure to perform proofs formally, "by manipulating uninterpreted formulae accordingly [sic] to explicitly stated rules" proves that "mathematics today is still a discipline with a sizeable pre-scientific component, in which the spirit of the Middle Ages is allowed to linger on" [Dijkstra 1988], a philosopher (Peter Nidditch) who claimed that "in the whole literature of mathematics there is not a single valid proof in the logical sense," and another philosopher who doubts "that mathematics is an essentially human activity" [Teller 1980].

The seven steps [refereeing, discussion, etc.] that it is asserted [by DeMillo, Lipton, and Perlis] lead to belief in theorems by mathematicians are also applied in other disciplines (e.g., sociology, English literature) where there is less general agreement and confidence in the results. Surely the reason mathematics has more credibility is that disputes can generally be resolved by very formal methods, rather than by appeal to authority, or intuition, or whatever. (J. Horning, 1977, quoted by MacKenzie, p. 206)

This is an old controversy

Already in 1829, Thomas Carlyle could complain of "the intellectual bias of our time," that "what cannot be investigated and understood mechanically, cannot be investigated and understood at all" and that "Intellect, the power man has of knowing and believing, is now nearly synonymous with Logic, or the mere power of arranging and communicating." But even mathematicians were not ready to mechanize:

The "cold algebraists," reduced the once-noble science of mathematics to mere mechanical calculation, without any deeper meaning. While Fergola "sees God behind the circle and the triangle," these atheists "see only the nothingness behind their formulas." ...
Moreover, for the anti-algebraists of Naples, "certainty" was on the side of ***intuition****, not mechanical calculation.*

(Massimo Mazzotti, writing about mathematics in Naples in the 1830s)

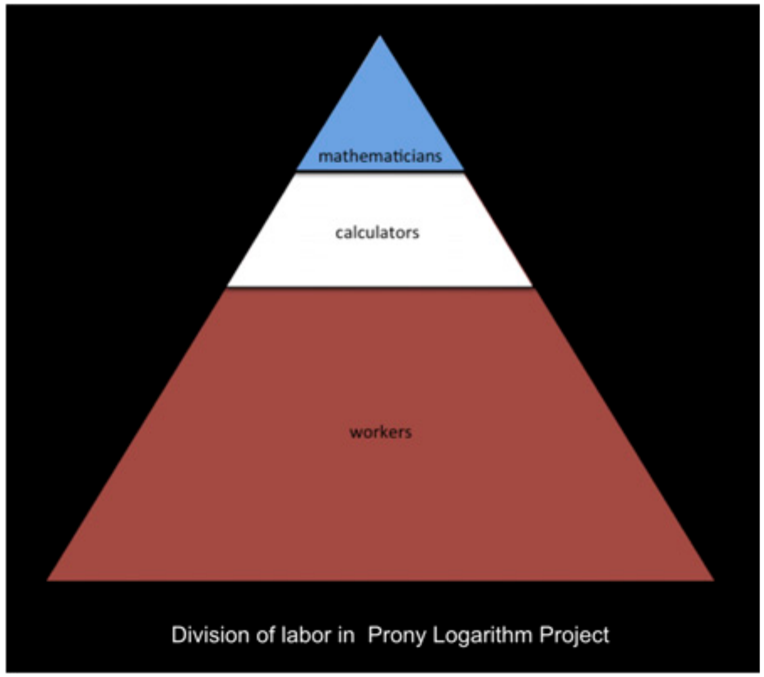
Mechanical reasoning vs. clear and distinct ideas

Analysis functioned as a shortcut; a sort of machinery of symbols that one need only "to arrange on paper, following certain very simple rules, in order to arrive infallibly at new truths." The very efficiency of the mechanical process, which obscured the route by which it achieved its results, rendered it suspect to mathematicians who believed that valid reasoning demanded clear and distinct ideas. ... The metaphor of the blind machine of analysis, which cranks out its results magically and mysteriously, recurs throughout the writings of [French] synthetic geometers.

(Lorraine Daston, on mathematics in Paris in the early 19th century)

I am also interested in the anthropomorphic language that many automated theorem proving practitioners used to talk about what kind of contributions they thought computers would be able to make to mathematics - some calling them future "mentors," "colleagues," and "co-workers," others comparing them to "high school students," or "apprentices," and still others calling them "assistants," "slaves," and "servants" - it seems to me that this anthropomorphic language has to do with what human faculties researchers expected would be automatable, and certain valuations and devaluations of mathematical labor. This, too, situates computing in the longer history of labor, human computing, and automation!

(Stephanie Dick, private communication about her work in progress)



In the 1820s Gaspard Riche de Prony, inspired by Adam Smith's account of the division of labor in pin making (itself inspired by an article in the *Encyclopédie*), created a pyramid of logarithm calculators (to 14 decimal places), divided into three categories: mathematicians "of distinction" at the top, workers at the bottom who actually did the arithmetic, and "algebraists" in the middle to translate the mathematical rules into mechanical algorithms for the unskilled.

Prony's method in turn inspired Babbage. "Human intelligence sunk to the mechanical level, kindling the idea of machine intelligence." (Daston, p. 11)

Daston calls the middle stratum "analytical intelligence" and traces the degradation of the reputation of calculating skill as versions of this division of labor was adopted during the 19th century.

In a passage that Babbage was to repeat like a refrain, Prony marveled that the stupidest laborers made the fewest errors in their endless rows of additions and subtractions: “I noted that the sheets with the fewest errors came particularly from those who had the most limited intelligence, [who had] an automatic existence, so to speak.” (Daston, p. 17)

Plans to mechanize mathematical research can be expected to have the same effect: which rung do today's "mathematicians of distinction" expect to occupy?

From Burroughs to Burroughs

What was optimal for human minds was not optimal for machines … at the level of the procedures required to mesh human and machines in long sequences of calculation, whether in the offices of the Nautical Almanac or the French Railways, tasks previously conceived holistically and executed by one calculator had to be analyzed into their smallest component parts, rigidly sequenced, and apportioned to the human or mechanical calculator able to execute that step most efficiently — where efficiently meant not better or even faster but cheaper.

In a sense, the analytical intelligence demanded by human-machine production lines for calculations was no different than the adaptations required by any mechanized manufacture… (Daston, p. 28)

[The] junk merchant does not sell his product to the consumer, he sells the consumer to his product. He does not improve and simplify his merchandise. He degrades and simplifies the client. W. Burroughs, *Naked Lunch*

[Cantor's paradise] is a place where nothing really happens (paraphrase of David Byrne)

Harris writes that to computers, every "step" is the same. Computers cannot, unless directed by a person, identify those "steps" in a proof that contain what he calls the ***key*** *- the particular insight that captures the* ***why*** *rather than the* ***that*** *of a theorem's truth ... some "key" or "main" or "crucial" or "fundamental" or "essential" insight that "hints that mathematical arguments admit not only the linear reading that conforms to logical deduction but also a topographical reading that more closely imitates the process of conception." He cites David Byrne's ... "heaven is a place where nothing ever happens" to introduce the world of the computer where all steps are created equal, all inferences are merely steps, and computers cannot show people what "key" insight grounds the* ***truth*** *or reveals the* ***why*** *of a mathematical theorem.*

The very possibility of automated theorem-proving, according to Harris, is predicated on the belief that "nothing really happens when a theorem is proved."

(Stephanie Dick, 2015 Harvard Dissertation)

Algorithms and dreams

…the typical strategy for automated theorem proving is a sophisticated version of the infinite-monkey scenario, with more or less intelligent guidance provided by the programmers but minus the monkeys. You begin with a collection of axioms defining the theory and add the negation of the theorem you want to prove. The program then applies logically valid transformations, possibly according to a predefined search strategy, until it arrives at a contradiction. … Another strategy Beeson discusses is quantifier elimination … What might be called recursive simplification includes both strategies mentioned above. It also underlies the principle of robot vacuum cleaner function, the task being completed recursively with the result guaranteed probabilistically. As far as I can tell, there is no key idea in either case. Trobaugh's intuition, by contrast, is nothing but a key idea. But I do not know how to characterize Trobaugh's intuition intrinsically, to show how it differs from the principles underlying the search strategies mentioned above.

Mathematics and narrative, 1

The android needs no semantics, by definition. The mathematician understands nothing without semantics. The ghost opens with a proposition about perfect complexes. One challenge in this article is to explain how this fits into the narrative without stopping to say what the terminology means. ...

To detect a narrative structure in a mathematical text, first look at the verbs. Apart from the verbs built into the formal language ("implies," "contains" in the sense of set-theoretic inclusion, and the like), nothing in a logical formula need be construed as a verb in order to be understood, and an automatic theorem prover can dispense with verbs entirely. One may therefore find it surprising that verbs and verb constructions, including transitive verbs of implied action, are pervasive in human mathematics. Trobaugh's ghost's single sentence consists of eighteen words, two of which are transitive verbs (shows, extends), one an intransitive verb (extends again); there is also a noun built on a transitive verb with pronounced literary associations (characterization).

Mathematics and narrative, 2

The word narrative lends itself to two misunderstandings. What for want of a better term I might call the "postmodern" misinterpretation is associated with the principle that "everything is narrative," so that mathematics as well would be "only" a collection of stories (so more or less any stories would do). The symmetric misunderstanding might be called "Platonist" and assumes a narrative has to be about something and that this "real" something[1] *is what should really focus our attention. The two misunderstandings join in an unhappy antinomy, along the lines that, yes, there is something, but we can only understand it by telling stories about it. The alternative I am exploring is that the mathematics is the narrative, that a logical argument of the sort an android can put together only deserves to be called mathematics when it can be inserted into a narrative. But this is just the point I suspect is impossible to get across to androids.*

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[1] *The real lover of learning* [**philomathes**] *... will not ... desist from eros until he lays hold of the nature of each thing in itself with [the rational] part of the soul ... drawing near to and having intercourse with the really real.* —Plato, *The Republic*, 490b

Can mechanical mathematics be *about* something?

Can we even say what human mathematics is *about*?

The Thomason-Trobaugh article is a contribution to the branch of mathematics known as K-theory, specifically algebraic K-theory. The name used to designate this branch of mathematics has two parts, each of which poses its own problems. The insider sees mathematics as a congeries of semi-autonomous subjects called "theories"—number theory, set theory, potential theory. The word was first used to delineate a branch of mathematics no later than 1798… In the examples given above the construction points to a discipline concerned with numbers, sets, and potentials, respectively; the word theory functions as a suffix, like -ology. But then what on earth could K-theory be about? Analyzing how the term is used, I am led to the tentative conclusion that it refers to the branch of mathematics concerned with objects that can be legitimately, or systematically, designated by the letter K. … I do not know how to answer the very interesting question whether the shape of K-theory, now a recognized branch of mathematics with its own journals … and an attractive two-volume Handbook, was in some sense determined by its name.