Week 12

Why study the history of mathematics?

There is no one reason, of course. Here is one line of thinking, consistent (I think) with Gayatri's insistence (following Benjamin) that

studying in the present, we construct a past thing: epistemology at work.

In mathematics, much more than in the natural sciences, the disciplinary past is a reference that **takes the place of** the natural world. The introduction of nearly any mathematical text situates the work in a history. This history is almost never subjected to critical scrutiny (and then at most to check for appropriate attributions to work in the recent past).

So *one reason* to study the history of mathematics is critical, to set the record straight.

Another reason is for the sake of philosophical modesty. This is how **we** imagine the boundaries and goals of mathematics; other traditions have set them in different ways.

To take a relatively straightforward example, Reviel Netz is a leading historian of Greek mathematics. A prominent theme in his work is the reconstruction of the "cognitive history" of the ancient world, both as a corrective to the tendency of mathematicians to read the ancient sources abusively as stages on the way to contemporary mathematics ("Whig history") and as a contribution to the history of cognition.

Ording's chapter "Antiquity" is informed by a reading of Netz and others and (like other chapters) stresses the strangeness of the way of thinking with which conventional histories claim a continuity.

et j'ai vu quelquefois ce que l'homme a cru voir

(Rimbaud, *Le bateau ivre*)

And another reason to study the past is to remind ourselves that the present, including the mathematics of the present could have been *other*.

Pour une histoire des possibles. Analyses contrefactuelles et futurs non advenus (L'Univers historique) (French Edition) (French) Paperback – February 11, 2016

by Quentin Deluermoz (Author), Pierre Singaravelou (Author)

4 ratings

Deluermoz has more recently set the history of the Paris commune alongside the histories of less well-known communes of the same uprising, in Martinique, Algeria, or Lyon.

[when one reduces]thought to a mathematical apparatus… what is abandoned is the whole claim and approach of knowledge: to comprehend the given as such; not merely to determine the abstract spatio-temporal relations of the facts which allow them just to be grasped, but on the contrary to conceive them… as mediated conceptual moments which come to fulfillment only in the development of their social, historical, and human significance …

(Horkheimer and Adorno, *Dialectic of Enlightenment*)

…the market value of knowledge — its income-enhancing prospects for individuals and industry alike—is now understood as both its driving purpose and leading line of defense. Even when the humanities and interpretive social sciences are accounted as building the analytic thinkers needed by the professions or as building the mind and hence securing a more gratifying life for the individual, they align with the neoliberal notion of building human capital. In neither defense are the liberal arts depicted as ***representing****,* ***theorizing****,* ***interpreting****,* ***creating****, or* ***protecting the world****. They are not conceived as* ***binding****,* ***developing****, or* ***renewing*** *us as a people,* ***alerting*** *us to dangers, or providing* ***frames****,* ***figures****,* ***theories****, and* ***allegories*** *… Above all, they are not conceived as providing the various* ***capacities*** *required for democratic citizenship. Rather, they are conceived as something for individuals to imbibe like chocolate, practice like yoga, or utilize like engineering. … Even [neoliberalization's] critics cannot see the ways in which we have lost a recognition of ourselves as held together by literatures, images, religions, histories, myths, ideas, forms of reason, grammars, figures, and languages. Instead, we are presumed to be held together by technologies and capital flows. That presumption, of course, is at risk of becoming true, at which point humanity will have entered its darkest chapter ever.*

(Wendy Brown, *Undoing the Demos*)

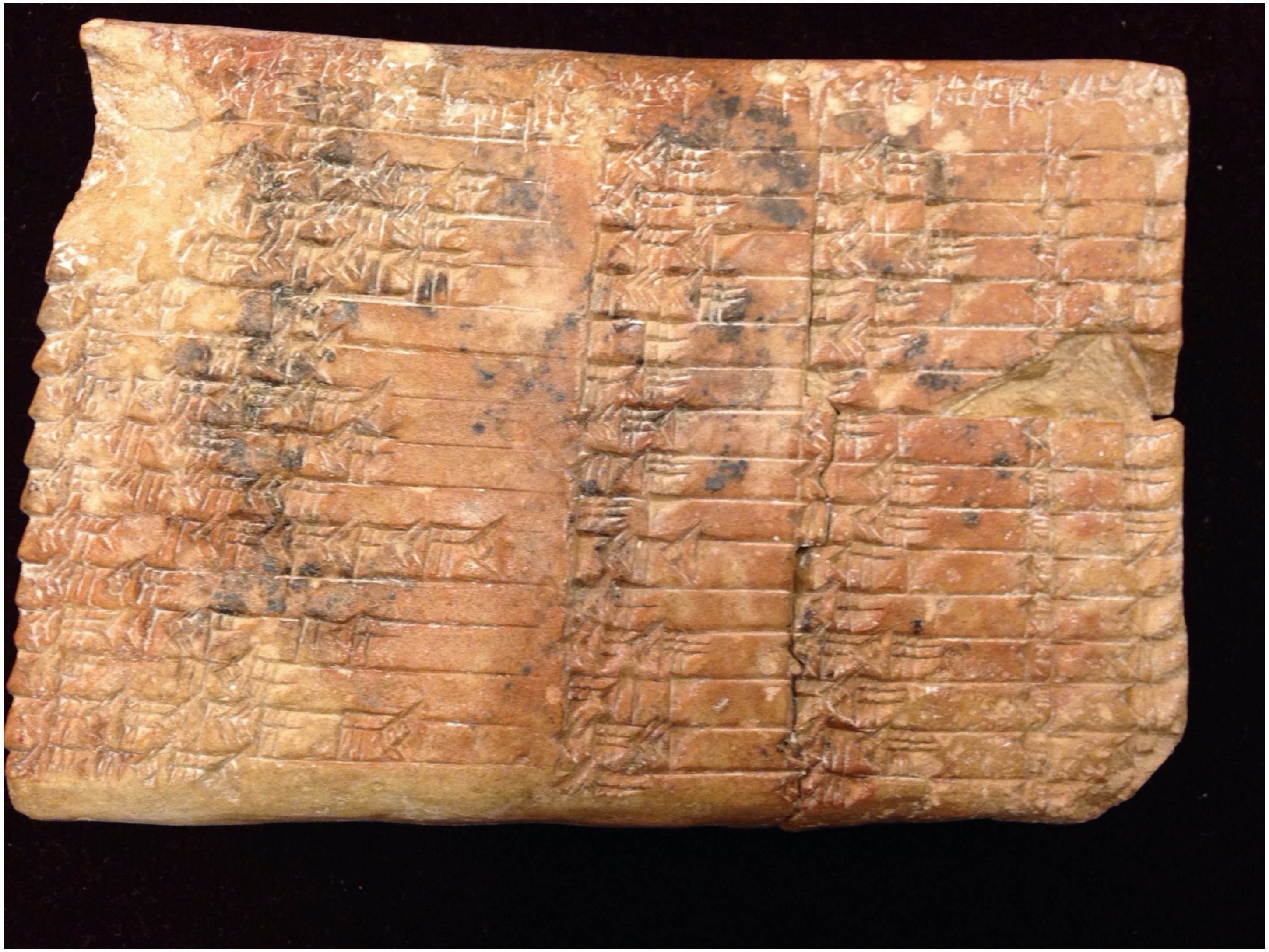
In neither defense are the ***LIBERAL ARTS*** *depicted as* ***representing****,* ***theorizing****,* ***interpreting****,* ***creating****, or* ***protecting the world****. They are not conceived as* ***binding****,* ***developing****, or* ***renewing*** *us as a people,* ***alerting*** *us to dangers, or providing* ***frames****,* ***figures****,* ***theories****, and* ***allegories*** *… Above all, they are not conceived as providing the various* ***capacities*** *required for democratic citizenship.*

So… where is the rhetoric that conveys what **MATHEMATICS** does?

(In case you are wondering why I would need to teach a course like this.)

The essence of mathematics lies entirely in its freedom.
(das Wesen der Mathematik liegt gerade in ihrer Freiheit)

Georg Cantor



Plimpton 322, ca. 1800 BC,
Columbia Rare Book and Manuscript Library

Surviving correspondence shows that [Plimpton] bought the tablet for $10 from a well-known dealer called Edgar J. Banks in about 1922. Banks told him it came from an archaeological site called Senkereh in southern Iraq, whose ancient name was Larsa.

(E. Robson, "Words and Pictures: New Light on Plimpton 322," 2002)

Robson explains that, in addition to numbers, the tablet includes an inscription in Akkadian and some abbreviations in Sumerian.

line	*α*	*p*	*q*
1	44.76°	12	5
2	44.25°	1 04	27
3	43.79°	1 15	32
4	43.27°	2 05	54
5	42.08°	9	4
6	41.54°	20	9
7	40.32°	54	25
8	39.77°	32	15
9	38.72°	25	12
10	37.44°	1 21	40
11	36.87°	2	1
12	34.98°	48	25
13	33.86°	15	8
14	33.26°	50	27
15	31.89°	9	5

All figures are in sexagesimal, so 2 05 on line 4 denotes $2 \times 60 + 5 = 125$. So

$x = (125)^2 - 54^2 = 12709$,

$y = 2(125)(54) = 13500$,

$z = (125)^2 + 54^2 = 18541$

$$12709^2 + 13500^2 = 18541^2$$ (Check it out!)

If p and q take on all whole values subject only to the conditions

(1) $p > q > 0$,

(2) p and q have no common divisor,

(3) p and q are not both odd,

then the expressions

$$x = p^2 - q^2,\ y = 2pq,\ z = p^2 + q^2$$

generate all reduced pythagorean triples $x^2 + y^2 = z^2$.

A. Aaboe, *Episodes from the Early History of Mathematics*, 1964.

From a Babylonian calculation of the inner diagonal of a rectangular gate in a wall:

this 26 40, 8 53 20 *the width*,
and 6 40, the thickness of the wall, you see.
26 40, the height of the wall, let your head retain, then
11 51 06 40 you see.
8 53 20, the width of the gate, let your head retain, then
1 19 ··· 44 26 40 you see.
6 40, the thickness of the wall, let your head retain, then
44 26 40 you see.
Heap them, 13 54 34 14 26 40 you see.
Its likeside let come up, then 28 53 20 you see
(for) the gate that (has) 26 40 (as its) height. So you do.

The expression you see translates the (Akkadian) word *ta-mar*.

No explanation is given for this peculiar calculation of a diagonal.

The earliest evidence of numeracy

Starting in the eighth millenium B.C. "number was first recorded, in the form of tiny 'tokens' made of clay or stone and shaped into simple geometrical forms." Robson reports the hypothesis that "[e]ven before writing… people from southern Iraq were recording numbers of stored commodities to protect them from theft or to document transactions." Some 5000 tablets from Uruk, dating to the late fourth millenium, contain the first literate documents and "the very large majority" contain accounting records "and little else," with a good deal of attention given to the brewing and distribution of beer.

The separation of mathematics from the humanities was far in the future.

Some 80% of extant mathematical tablets are pedagogical in nature.

Many of the *problems* of the problem texts have to do with relations between areas or volumes of physical objects and their dimensions—fields, say, or vessels for carrying grain—or relations between prices of goods, costs, quantities, and profits. In one direction the solution involves a straightforward application of rules for multiplication, given known formulas for areas or volumes of standard shapes; but in the other direction the solution requires techniques that would now be qualified as *algebra*.

Mespotamian homework

From the Old Babylonian collection of geometrical problems BM 15285, before 1500 BC. (Robson p. 47)

(7) The square-side is 1 cable long. Inside it I drew a second square side. The square-side that I drew touches the outer square-side. What is its area?

(8) The square side is 1 cable long. Inside it <I drew> 4 wedges and 1 square side. The square-side that I drew touches the second square-side. What is its area?

From PUL 31, the Sargon period. c. 2250 BC (Robson, p. 56)

The long side is 4(ĝeš) and 3 <rods>: <what is> the short side of a 1(iku) field? Its short side is to be found.

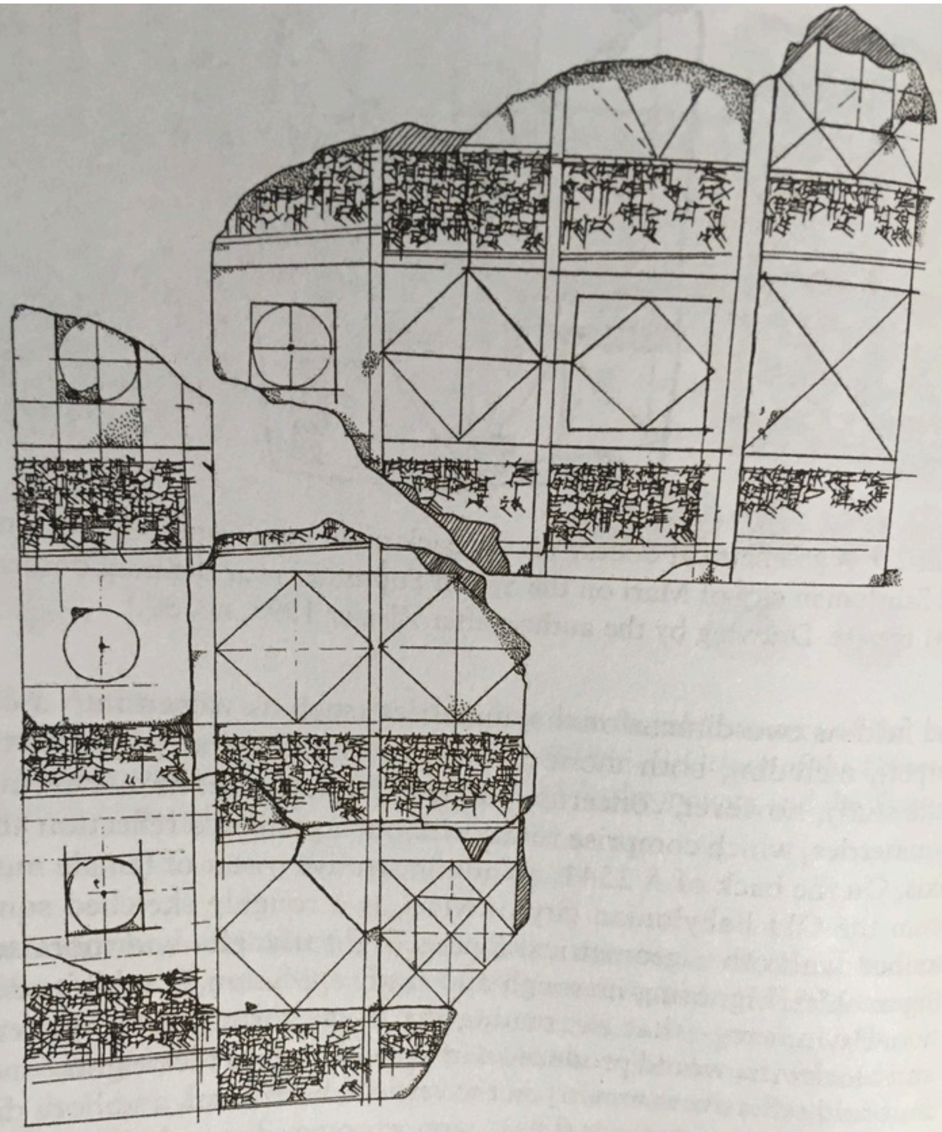


Figure 2.10 A compilation of geometrical problems from the Old Babylonian period. (BM 15285. Drawing by the author.)

An Old Babylonian word problem, solved

9 <shekels of> silver for a trench. The length exceeds the width by 3;30 (rods). Its depth is ½ rod, the work rate 10 shekels. Its wages are 6 grains. What are the length and width?

You, when you proceed: solve the reciprocal of the wages, multiply by 0;09, the silver, so that it gives you 4 30. Multiply 4 30 by the work rate, so that it gives you 45. Solve the reciprocal of ½ rod, multiply by 45, so that it gives you 7;30.

Break off ½ of that by which the length exceeds the width, so that it gives you 1;45. Combine 1;45, so that it gives you 3;03 45. Add 7;30 to 3;03 45, so that it gives <you> 10;33 45. Take its square-side, so that it gives you 3;15. Put down 3;15 twice. Add 1;45 to 1 (copy of 3;15), take away 1;45 from 1 (copy of 3;15), so that it gives you length and width.

The length is 5 rods, the width 1 ½ rods. That is the procedure.
From the tablet YBC 4692, unprovenanced. (Robson p. 89)

Recovering the history

The discovery and decipherment of the ancient cuneiform tablets was accompanied by a rewriting of the history of mathematics in which the practices of the Near East were treated as precursors to the "Greek miracle." Morris Kline's judgment in an influential popular history:

Mathematics as an organized, independent, and reasoned discipline did not exist before the classical Greeks…entered upon the scene.

…at its simplest the realist historical enterprise consists of identifying Platonic mathematical objects in the historical record and equating the terminology used to describe and manipulate them with their modern-day technical counterparts. The emphasis is on tracing mathematical ***sameness*** *across time and space.* (Robson, p. 273)

Robson adds that the study of Mesopotamian mathematics was long deformed by Orientalism:

If in the twentieth century mathematics was eternally unchanging then so was the ancient Orient.

Diagrams in Greek mathematics

But the history of Greek mathematics has also had to be rewritten to free it from the prejudices of modern European philosophy.

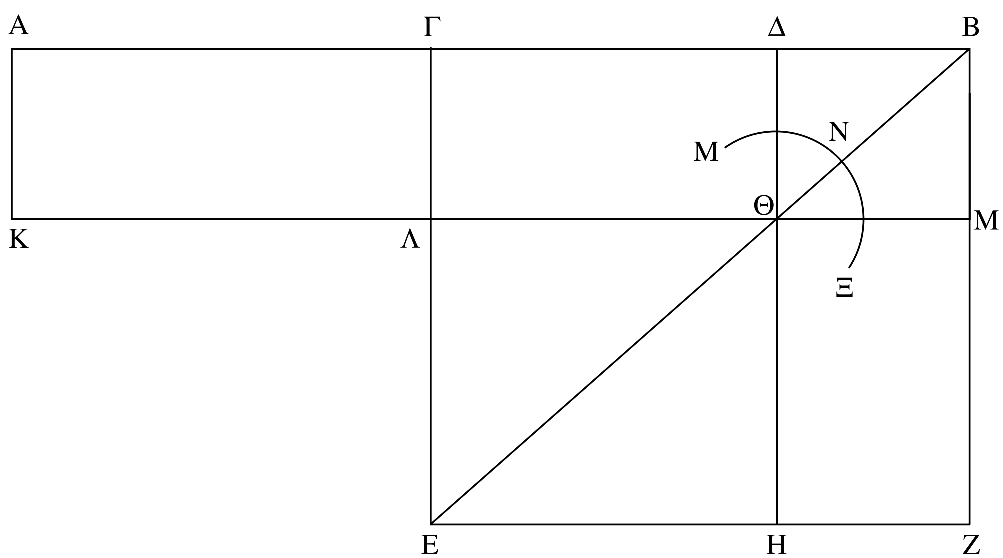
Reviel Netz, in his "cognitive history" of Greek mathematics, has argued for the centrality of visual intuition in this history, and claims that "the lettered diagram ... is a predominant feature" of Greek mathematics (Netz 1999, 14). In books V and VII–IX of Euclid's *Elements,* "all the propositions were accompanied by diagrams." Netz writes that "text and diagram are interdependent" (41), and his explanation combines the cartesian and the leibnizian: "[T]he visual presence [of the diagram] offers a synoptic view, an easy access to the contents; the verbalization limits the contents" (181).

In some situations, the visual dominates; for example, "the clarity of the concept" of the tangent in Apollonius "owes a lot to visual intuition ... the visual may fulfil, for the Greeks, what we expect the verbal to do" (Netz 1999, 102).

The historians' question

are the historically determined features of a given piece of mathematics significant to it as **mathematics***?*

(Netz, *The Transformation of Mathematics in the Early Mediterranean World : From Problems to Equations*)



This figure on p. 10 of Netz (1999) proves what we understand (since 1886) by the equation $(a + b)(a - b) + b^2 = a^2$.

In Euclid's *Elements* II.5 it says

If a straight line is cut into equal and unequal <segments>, the rectangle contained by the unequal segments of the whole, with the square on the <line> between the cuts, is equal to the square on the half .

The mathematicians said it was the same as the algebraic formula.

here arrives the seduction of a-historicism: mathematics is supposed to be compelling, it overpowers its readers by the incontrovertibility of its arguments. So, the a-historicist feels, unless one is overpowered by the argument, it is not really mathematical. The real form of the mathematical argument, then, is the form through which the reader feels its validity – that is, for a modern reader, the modern form. In its geometrical cloth, the Euclidean formulation is rendered inaccessible to the modern reader, so that it is no longer... a piece of mathematics.

Does mathematics change with time?

The historian Sabetai Unguru disagreed and got into a famous fight with the mathematicians. Netz interprets Unguru's reasoning as follows:

*By transforming the geometrical relations of **Elements** II into an algebraic equation, they are rendered trivial: so that, instead of allowing us to see better the significance of ancient argument, we, instead, lose sight of its importance for the ancient audience.*

This leads to a new problem:

But what if the very nature of mathematics had changed with time? In this case, there is a complicated process characterizing the history of mathematics, and the first task of the historian would be to uncover its dynamics.

Identity according to set theory (and Leibniz)

The question of identity would seem to have been settled long ago in mathematics by the adoption of the "=" sign as a standard item in the lexicon used to construct meaningful mathematical propositions. But there is a rich philosophical literature on this question. Leibniz's principle of the *identity of indiscernables* states roughly that A = B if everything that is true of (or can be predicated of) A is true of B and vice versa — if A and B have the same *attributes* in any sense that can be given to this term. Identity in Leibniz's sense is the property captured by the = sign in set theory. But since A is called A and B is called B this does not quite suffice to unravel A = B.

Identity according to Grothendieck

According to the principle that Grothendieck used to great effect, "knowing" a set A means "knowing" where A fits in the category of sets, which amounts to "knowing" all its relations (the morphisms in this case are just *set- theoretic functions*) to all other sets. This principle, valid in any category, is known as *Yoneda's Lemma* and it is so formulated as to be obvious to prove, but the experienced will be aware that we are skating on the edge of paradox — of Russell's paradox, more precisely — by talking about things like "all other sets."   One refers to the *category of sets* rather than the *set of all sets*; in the former, Russell's noxious deductions are not permitted. But this comes at a cost: unless one makes additional restrictions, we have lost the principle of identity; it is not appropriate to say that two sets A and B are *equal*.

Mathematics without identity

Branches of mathematics that cannot rely on the physical world for guidance are subject to the problems of identity. When we solve a problem we want to be able to point to the answer unambiguously. In the categorical framework, the best we can do is say that the solution is *unique up to unique isomorphism*. This means that, when you and I set out to solve the same problem, we know we will be making choices at the outset, based on our individual perspectives; *unique* means there is a way to translate my solution into yours, and *up to unique isomorphism* means there is only one way to do it. But finding that translation means solving another problem!

Problems of translation pile up as one climbs the n-categorical ladder:

...one seems caught at first sight in an infinite chain of ever 'higher,' and presumably, messier structures, where one is going to get hopelessly lost, unless one discovers some simple guiding principle....

(Grothendieck, letter to D. Quillen, 1983)

Grothendieck on two mathematical styles

Prenons par exemple la tâche de démontrer un théorème qui reste hypothétique (à quoi, pour certains, semblerait se réduire le travail mathématique). Je vois deux approches extrêmes pour s'y prendre. L'une est celle du **marteau et du burin**, quand le problème posé est vu comme une grosse noix, dure et lisse, dont il s'agit d'atteindre l'intérieur, la chair nourricière protégée par la coque. Le principe est simple : on pose le tranchant du burin contre la coque, et on tape fort. Au besoin, on recommence en plusieurs endroits différents, jusqu'à ce que la coque se casse - et on est content. Cette approche est surtout tentante quand la coque présente des aspérités ou protubérances, par où "la prendre". Dans certains cas, de tels "bouts" par où prendre la noix sautent aux yeux, dans d'autres cas, il faut la retourner attentivement dans tous les sens, la prospecter avec soin, avant de trouver un point d'attaque. Le cas le plus difficile est celui où la coque est d'une rotondité et d'une dureté parfaite et uniforme. On a beau taper fort, le tranchant du burin patine et égratigne à peine la surface - on finit par se lasser à la tâche. Parfois quand même on finit par y arriver, à force de muscle et d'endurance.

Je pourrais illustrer la deuxième approche, en gardant l’image de la noix qu’il s’agit d’ouvrir. La première parabole qui m’est venue à l’esprit tantôt, c’est qu’on plonge la noix dans un liquide émollient, de l’eau simplement pourquoi pas, de temps en temps on frotte pour qu’elle pénètre mieux, pour le reste on laisse faire le temps. La coque s’assouplit au fil des semaines et des mois - quand le temps est mûr, une pression de la main suffit, la coque s’ouvre comme celle d’un avocat mûr à point ! Ou encore, on laisse mûrir la noix sous le soleil et sous la pluie et peut-être aussi sous les gelées de l’hiver. Quand le temps est mûr c’est une pousse délicate sortie de la substantifique chair qui aura percé la coque, comme en se jouant - ou pour mieux dire, la coque se sera ouverte d’elle-même, pour lui laisser passage.

Category theory

(or, the mathematical imagination reduced to its most basic expression)

A category *C* consists of a collection of *objects* (sometimes written Ob(*C*)) and for each pair (c,c') of objects, a set *Mor*(c,c') of *morphisms*.

Warning: The **collection** Ob(*C*) is generally *not* a set! For example, in the category of sets, Ob(*C*) is the collection of all sets, which (as Russell explained) cannot be a set.

The morphisms in *Mor*(c,c') are often denoted by arrows f: c $\rightarrow$ c'. In the category of sets the morphisms are *functions* from c to c'. If you have an arrow f: c $\rightarrow$ c', and a second arrow g: c' $\rightarrow$ c" in *Mor*(c',c"), you can concatenate them to form an arrow $g \circ f$: c $\rightarrow$ c" called the *composition*. There is always a special arrow Id: c $\rightarrow$ c called the *identity morphism*. These data satisfy a short list of axioms (composition with the identity doesn't change anything, etc.)

In the category theory of Samuel Eilenberg and Saunders MacLane, a relation between two categories is called a *functor*. A functor from C to D takes the objects of C to objects of D and morphisms as well.

Topological spaces form a category, where the morphisms are continuous maps. Vector spaces form a category, where the morphisms are linear maps.

The proof of Brouwer's fixed point theorem (which implies that you can't comb the hair on a billiard ball) becomes very easy with the help of a functor from topological spaces to vector spaces: the *homology* functor, which is a direct outgrowth of the considerations in the proof of Euler's formula (and which appears in a later chapter of *Proofs and Refutations*, in Poincaré's proof).