MODERN ALGEBRA I GU4041

Homework 9, due April 3: Classification of abelian groups

1. List the isomorphism classes of abelian groups of the following orders: 27, 200, 605, 720.

2. Judson, section 13.4, exercises 6, 8. In problem 8, "not true in general" means "not necessarily true if G, H, K are not assumed to be abelian.

3. Find the smallest integer $n > 42$ such that there is exactly one isomorphism class of abelian groups of order n and exactly one isomorphism class of abelian groups of order $n+1$. Justify your answer, including why there is no smaller n.

4. Let $n > 1$ and $m > 1$ be integers. In the next question, we recall that if $a \in \mathbb{Z}$ and $x \in \mathbb{Z}_n$, we can define $ax \in \mathbb{Z}_n$ by letting $\tilde{x}$ be any element of $\mathbb{Z}$ with residue class x modulo n and letting ax denote the residue class of $a\tilde{x}$ modulo n.

(a) Show that if a and d are integers such that $(a, n) = (d, m) = 1$, then there is an automorphism

$$\alpha_{a,d} : \mathbb{Z}_n \times \mathbb{Z}_m$$

such that, for all $(x, y) \in \mathbb{Z}_n \times \mathbb{Z}_m$,

$$\alpha_{a,d}((x, y)) = (ax, dy).$$

(b) Suppose $(n, m) = 1$. Show that the group $\mathbb{Z}_{nm}$ has a unique subgroup A_n of order n and a unique subgroup A_m of order m. Write down an isomorphism

$$A_n \times A_m \xrightarrow{\sim} \mathbb{Z}_{nm}.$$

(c) If $(n, m) = 1$, show that any automorphism of $\mathbb{Z}_n \times \mathbb{Z}_m$ is of the form $\alpha_{a,d}$ where a and d are as in part (a).

(d) Write down an automorphism of $\mathbb{Z}_3 \times \mathbb{Z}_9$ that is *not* of the form $\alpha_{a,d}$.

(e) Suppose $a, b, c, d \in \mathbb{Z}$. Let $M : \mathbb{Z}_3 \times \mathbb{Z}_3$ be the function

$$M(x, y) = (ax + by, cx + dy).$$

For what a, b, c, d is this M an automorphism?

Recommended reading

Judson, Section 13.1.