INTRODUCTION TO MODERN ALGEBRA I, GU4041, SPRING 2020

Practice Midterm 1

For any positive integer m, we denote by $\mathbb{Z}_m$ a cyclic group with m elements.

1. True or False? If false, give a counterexample; if true, provide an explanation. The explanation can be brief but it is not enough to say that the statement was explained in the course.

(a) For any two sets A, B,

$$A \backslash (A \backslash B) = B.$$

(b) Let G be a group and $g \in G$. If $gh = hg$ for all $h \in G$ then $g^3 h = hg^3$ for all $h \in G$.

(c) $75 \equiv -7 \pmod{17}$.

(d) Let A, B, C be sets, and let $f : A \to B$ be injective and $g : B \to C$ be injective. Then $g \circ f : A \to C$ is injective.

(e) Let G be a group and let $\iota : G \to G$ be the function $\iota(g) = g^{-1}$. Then $\iota(gh) = \iota(g)\iota(h)$ for all $g, h \in G$.

2. (a) List all the generators of the groups $\mathbb{Z}_7$ and $\mathbb{Z}_8$.

(b) In arithmetic modulo 9 find the number x between 1 and 9 such that

$$991^{13} \equiv x \pmod{9}.$$

(c) Let $f : \mathbb{Z}_9 \to \mathbb{Z}_9$ be the function defined by

$$f([a]_9) = [3a]_9.$$

Show that f is a homomorphism of groups and list the elements of its kernel.

3. Describe the set of equivalence classes for those of the following relations that are equivalence relations, or explain why the relation is not an equivalence relation:

(a) X is the set of residents of the United States; we say $x \sim y$ if x and y live in the same state.

(b) $X = \mathbb{R}^2$ is the set of points in the cartesian plane; $P \sim Q$ if the distance between P and Q is at most 1.

(c) X is the set of 2×2 invertible matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{C}$. We say $A \sim B$ if $P_A(t) = \det(tI_2 - A)$ and $P_B(t) = \det(tI_2 - B)$ have the same roots, where I_2 is the identity matrix.

4. Which of these inclusions (i.e., injective homomorphisms) of groups are impossible? Justify your answer.

(a) $\mathbb{Z}_7 \subseteq \mathbb{Z}_{14}$.

(b) $\mathbb{Z}_6 \subseteq \mathbb{Z}_9$.

(c) The Klein group K_4 inside $\mathbb{Z}_{32}$.

5. (a) Show that the additive group $\mathbb{R}$ of real numbers has no finite subgroups.

(b) Show that $\mathbb{Z}$ is a subgroup of $\mathbb{Q}$ and find a subgroup $A \subseteq \mathbb{Q}$ such that $A \supset \mathbb{Z}$ but $A \neq \mathbb{Z}$.

6. Let G be a group with 5 elements, and let e denote the identity element. Prove that there is exactly one element $x \in G$ such that $x^2 = e$. *Do not use Lagrange's theorem* even if you know the statement.

(Hint: Suppose $x \neq e$ and $x^2 = e$. There is some element $y \in G$ such that $xy \neq e$ and $xy \neq y$.)