Simplicity of A_n

GU4041

Columbia University

April 27, 2020

We have already proved the case $n = 5$ of the following theorem:

Theorem (Camille Jordan, 1875)

For any $n \geq 5$, the alternating group $A_n \subset S_n$ is a simple group of order $\frac{n!}{2}$.

Some conjugacy classes in A_n

Lemma

The group A_n is generated by 3-cycles.

Proof: We know that A_n is generated by products $\sigma \cdot \tau$ where σ and τ are transpositions. So it suffices to show that any such product is also a product of 3-cycles.
First case: $\sigma = (ab)$, $\tau = (cd)$ disjoint. Then

$$(ab)(cd) = (dac)(abd)$$

Indeed, the second product is $\begin{pmatrix} a & b & c & d \\ c & b & d & a \end{pmatrix} \cdot \begin{pmatrix} a & b & c & d \\ b & d & c & a \end{pmatrix}$, so $a \to b \to b$; $b \to d \to a$; $c \to c \to d$; $d \to a \to c$. And this is exactly $(ab)(cd)$.

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Second case: $\sigma = (ab)$, $\tau = (ac)$, $c \neq b$. Then

$$(ab)(ac) = (acb).$$

If $\{a, b\} = \{c, d\}$ then $(ab)(cd) = (ab)^2$ is the identity. So there is no third case.
This completes the proof.
Note: this is a proof inside A_4.

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More conjugacy classes in A_n

Lemma

Suppose $n \geq 5$. Then A_n is generated by elements of the form $\sigma \cdot \tau$, where σ and τ are disjoint transpositions.

Proof.

By the previous result, we need to show that any 3-cycle in A_n can be written as a product $g_1 \cdot g_2$, where $g_1 = \sigma_1 \cdot \tau_1$; $g_2 = \sigma_2 \cdot \tau_2$, in each case disjoint.

This is a calculation in S_5:

$$(abc) = [(ab)(de)][(de)(bc)].$$

Check: the right hand side $b \to c \to c$; $c \to b \to a$; $d \to e \to d$; $e \to d \to e$; $a \to a \to b$. □

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All 3 cycles are conjugate in A_n, $n \geq 5$

Lemma

Let $n \geq 5$. Then any two 3-cycles in A_n are conjugate in A_n.

Proof.

Let $g = (abc)$, $h = (ijk)$. We know there is $\sigma \in S_n$ such that

$$\sigma g \sigma^{-1} = h.$$

If σ is even, we're done. If not, σ is odd. So choose $d, e \notin \{a, b, c\}$, and let $\sigma' = \sigma \cdot (de)$. This is an element of A_n, and (de) commutes with g. So

$$\sigma' g \sigma'^{,-1} = \sigma g \sigma^{-1} = h.$$

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Note that (123) and $(132) = (123)^{-1}$ are **not** conjugate in A_4.

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All permutations with cycle decomposition $(4, 2)$ *are conjugate in* A_6.

It's the same argument: if $g = (abcd)(ef)$ is conjugate to h in S_6 by τ, then either τ is even or $\tau \cdot (ef)$ is. And

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Strategy of the proof

We assume $n \geq 5$. Let $N \subset A_n$ be a normal subgroup. Suppose N contains a 3-cycle. Then N contains every 3-cycle, because N is normal and $n \geq 5$. But then N generates A_n, so $N = A_n$.

We thus have to prove that any normal subgroup of A_n contains a 3-cycle.

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Conjugacy classes in S_6

The conjugacy classes in S_6 are determined by their cycle decomposition, The partitions of 6 are

- $6 = 6$; a 6-cycle is the product of 5 transpositions, hence is odd.
- $6 = 5 + 1$; a 5-cycle is even.
- $6 = 4 + 2$; a 3-cycle is the product of 3 transpositions, hence its product with a disjoint transposition is even.
- $6 = 3 + 3$: even.
- $6 = 3 + 2 + 1$: odd; $6 = 3 + 1 + 1 + 1$: even
- $6 = 2 + 2 + 2$: odd.
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There are thus 6 S_6-conjugacy classes contained in A_6, listed with the number of elements.

- $6 = 1 + 1 + 1 + 1 + 1 + 1$; (1)
- $6 = 3 + 3$; $2 \cdot \binom{6}{3} = 40^*$.
- $6 = 5 + 1$; $(6 \cdot 4! = 144)$.
- $6 = 4 + 2$; $(\binom{6}{4} \cdot 3! = 90)$
- $6 = 3 + 1 + 1 + 1$: $(\binom{6}{3} \cdot 2 = 40)$
- $6 = 2 + 2 + 1 + 1$: $(\frac{1}{2}(\binom{6}{2})(\binom{4}{2}) = 45$

And $1 + 40 + 144 + 90 + 40 + 45 = 360 = |A_6|$.
*20 choices for $\{a, b, c\}$, then $(abc)(def)$ has four signs; but each one is counted twice because $(abc)(def) = (def)(abc)$.

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As in the case of A_5, we see that 144 does not divide 360, so there are two conjugacy classes of 5-cycles, each with 72 elements.

On the other hand, we have seen all 3 cycles, and all $(4,2)$ permutations, are conjugate in A_6. Thus the possible sizes of conjugacy classes (without checking the $(3,3)$ permutations are all conjugate) are:

$$(1,45,72,72,90,40,40); (1,45,72,72,90,20,20,40)$$

The divisors of $360 = 2^3 \cdot 3^2 \cdot 5$ with more than 21 elements (we need the identity) are

$$24, 30, 36, 40, 45, 60, 72, 90, 120, 180.$$

The only odd one is 45, but we need the identity. Any even order must be bigger than 46, thus at least 66. But we cannot reach any of $72, 90, 120, 180$ as a sum of a subset of the above numbers.
So A_6 is simple.

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Corollary

There are two conjugacy classes of 5-cycles in A_5, and one conjugacy class of products of disjoint 2-cycles.

Proof of corollary: Since 24 does not divide 60, the 5 cycles form more than 1, thus 2 conjugacy classes; but 15 is not even, so it is a single conjugacy class.

Conjugacy classes in A_5

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There are two conjugacy classes of 5-cycles in A_5, and one conjugacy class of products of disjoint 2-cycles.

Proof of corollary: Since 24 does not divide 60, the 5 cycles form more than 1, thus 2 conjugacy classes; but 15 is not even, so it is a single conjugacy class.

Simplicity of A_n, $n \geq 7$

Let $n \geq 7$ and let $N \subset A_n$ be a normal subgroup. It suffices to show that N contains a 3 cycle. Let $\sigma \neq e$ be an element of N. Up to relabeling the numbers, we may assume $\sigma(1) \neq 1$. Suppose $\sigma(1) \in \{i, j, k\}$ with all the i, j, k distinct from 1 and let $\tau = (ijk) \in A_n$. Then

$$\tau\sigma\tau^{-1}(1) = \tau(\sigma(1)) \neq \sigma(1).$$

So $\tau\sigma\tau^{-1} \neq \sigma$ and both are in N.
Let $g = \tau\sigma\tau^{-1}\sigma^{-1} \neq e$. We see that $g =\in N$. But $g = \tau \cdot \sigma\tau^{-1}\sigma^{-1}$ is a product of two 3-cycles, so it moves at most six numbers.
Thus g belongs to a subgroup $H \subset A_n$ isomorphic to S_6; but g is even, so it belongs to a subgroup isomorphic to A_6. Moreover, $g \in H \cap N$, which is normal in H. Since H is simple, $H \cap N = H$.
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