ALGEBRAIC NUMBER THEORY W4043

Homework, week 10, due November 21

Dirichlet characters, continued

Notation is as in Homework 7.

1. We show that $X(p)$ is a cyclic group of order $p-1$ and that, for any $a \in (\mathbb{Z}/p\mathbb{Z})^\times$, $a \neq 1$. there exists $\chi \in X(p)$ such that $\chi(a) \neq 1$.

(a) Bearing in mind that $(\mathbb{Z}/p\mathbb{Z})^\times$ is a cyclic group, show that $X(p)$ has at most $p-1$ elements.

(b) Show that $X(p)$ has the structure of abelian group.

(c) Let g be a cyclic generator of $(\mathbb{Z}/p\mathbb{Z})^\times$ and define a function $\lambda : \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}$ by

$$\lambda(g^k) = e^{\frac{2\pi i k}{p-1}};\ \lambda(0) = 0.$$

Show that $\lambda \in X(p)$ and that, if n is the smallest positive integer such that $\lambda^n = \chi_0$, then $n = p-1$. Conclude that λ is a cyclic generator of $X(p)$.

(d) If $a \in (\mathbb{Z}/p\mathbb{Z})^\times$ and $a \neq 1$ then $\lambda(a) \neq 1$.

2. Let $a \in (\mathbb{Z}/p\mathbb{Z})^\times$, $a \neq 1$. Show that $\sum_{\chi \in X(p)} \chi(a) = 0$.

3. Let d be a divisor of $p-1$. Show that the set of $\chi \in X(p)$ such that $\chi^d = \chi_0$ is a subgroup of order d.

4. Hindry's book, Exercise 7.10, pp. 68-69. You may assume the result of Exercise 7.9, or Dirichlet's theorem on primes in an arithmetic progression.