ALGEBRAIC NUMBER THEORY W4043

1. Homework, week 3, due September 26

1. Let $\mathcal{O}$ denote the ring of integers in $K = \mathbb{Q}(\sqrt{-14})$.

(a) Show that $3 + \sqrt{-14}$ is an irreducible element in $\mathcal{O}$.

(b) Show that 3 is not equal to $N_{K/\mathbb{Q}}(x)$ for any $x \in \mathcal{O}$.

(c) Show that 3 is an irreducible element in $\mathcal{O}$.

(d) Show that the principal ideal (3) is not a prime ideal and compute its factorization as a product of prime ideals.

2. Hindry's book, Exercise 6.16, p. 120. You can use Corollary 3-5.10 from Hindry's book; it will be proved later in the semester.

3. Let K/k be a cubic extension of fields of characteristic 0, of the form $K = k(\sqrt[3]{d})$ for some $d \in k$ that is not a cube in k. We assume $k \supset \zeta_3$, a primitive 3-rd root of 1.

(a). Show that $Gal(K/k)$ is cyclic of order 3.

Let $s : K \to K$ be a generator of $Gal(K/k)$,

$$s(\sqrt[3]{d}) = \zeta_3(\sqrt[3]{d}),$$

and let $Tr : K \to k$ be the k-linear trace map, $Tr(\alpha) = a + s(a) + s^2(a)$.

(b) Find a basis for $\ker Tr$.

(c) Let $f(X) \in \mathbb{Q}[X]$ be a cubic polynomial, and let $L/\mathbb{Q}$ denote its splitting field. Suppose $[L : \mathbb{Q}] = 3$. Prove that all the roots of f are real.

(d) Find a cubic polynomial $f(X) \in \mathbb{Q}[X]$ such that the splitting field $L/\mathbb{Q}$ is of degree 3, and such that the prime 5 is inert in L. What is the order of the residue field of L at the unique prime dividing 5?

(More difficult: replace 5 by 7.)