ALGEBRAIC NUMBER THEORY W4043

Homework, week 8, due November 7

Let n be a positive integer. A *quadratic form* in n variables $x_1, \dots, x_n$ is a homogeneous polynomial Q of degree 2 in $x_1, \dots, x_n$.

1. For every $n > 0$, find a quadratic form Q_n in n variables with coefficients in $\mathbb{Z}$ such that the only rational solution to the equality

$$Q_n(a_1, \dots, a_n) = 0$$

is the zero solution $a_1, \dots, a_n$.

2. Let $n \geq 3$ and p be a prime number, and let Q be a quadratic form in n variables with coefficients in $\mathbb{Z}$. Show that the congruence

$$Q(x_1, \dots, x_n) \equiv 0 \pmod{p}$$

has a solution with $(a_1, \dots, a_n) \in \mathbb{Z}^n$ and not all a_i divisible by p.

3. Let $Q(x, y)$ be a quadratic form in 2 variables with coefficients in $\mathbb{Z}$, let p be a prime number, and $a \in \mathbb{Z}$ an integer not divisible by p. Show that the congruence

$$Q(x, y) \equiv a \pmod{p}$$

has a solution.

4. Find a homogeneous polynomial $F(X, Y, Z)$ of degree 3 with coefficients in $\mathbb{Z}$, with the property that, if

$$F(a, b, c) \equiv 0 \pmod{2}$$

with $a, b, c \in \mathbb{Z}$, then a, b, and c are all divisible by 2.

5. Hindry's book, Exercise 6.10, p. 24.