ALGEBRAIC NUMBER THEORY W4043

TAKE HOME FINAL, DUE DECEMBER 18, 2017

The final is to be handed to the teacher in Room 521 on Monday, December 18, between 10 AM and noon. All grading will be completed by the morning of December 19. For this reason, it will be impossible to accept late returns!

1. Let $K \subset \mathbb{C}$ be the splitting field over $\mathbb{Q}$ of the polynomial $X^3 - 15$. We admit without proof that neither 2 nor 7 is ramified in K.

(a) List the subfields of K in which 3 is unramified. List the subfields of K in which 5 is unramified.

(b) How many prime ideals of K lie above each of the following primes: 2, 3, 5, 7?

(c) What would you need to know in order to establish that neither 2 nor 7 is ramified in K?

2. (a) Find all the reduced binary quadratic forms of discriminant $\Delta = -68$.

(b) Find representatives for the ideal class group of the ring of integers of $\mathbb{Q}(\sqrt{-17})$. Prove carefully that they are all distinct. Which of them is a prime ideal?

3. Let p be an odd prime number.

(a) Show that the infinite series

$$log_p(1+x) = \sum_{i \geq 1} (-1)^{i-1} \frac{x^i}{i}$$

converges for $x \in p\mathbb{Z}_p$ to an element of $\mathbb{Z}_p$, and that

$$log_p((1+x)(1+y)) = log_p(1+x) + log_p(1+y).$$

Now let $x \in \mathbb{Z}_p$ and suppose $x \equiv 1 \pmod{p}$. Consider the sequence of elements of $\mathbb{Z}_p$:

$$y_n = 1 + 2 \cdot \sum_{i=1}^{n} \frac{(-1)^{i-1}}{i} \binom{2i-2}{i-1} (\frac{x-1}{4})^i.$$

(b) Prove that the sequence (y_n) converges to an element $y \in \mathbb{Z}_p$.

(c) Prove that $y^2 = x$. (Hint: use (a).)

4. Let p be an odd prime, and let $\zeta_p = e^{\frac{2\pi i}{p}} \in \mathbb{C}^\times$. If $q \neq p$ is a prime number, let $D_q \subset Gal(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ be the decomposition group of q. Let $c : \mathbb{N} \to \mathbb{C}$ be the arithmetic function defined by the following rule:

- $c(1) = 1$.
- $c(n) = 1$ if $n > 1$ and every prime factor q of n has the property that D_q is a group of order 2.
- $c(n) = 0$ otherwise.

(a) Is c a multiplicative function?

(b) Define $D(s) = \sum_{n \geq 1} \frac{c(n)}{n^s}$. Show that $D(s)$ converges absolutely on the half-plane $Re(s) > 1$.

(c) Show that there is no $\epsilon > 0$ such that $D(s)$ extends to a continuous function on the half-line $(1 - \epsilon, \infty)$.