Eisenstein Series on Reductive Groups

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In this talk, we will focus on defining Eisenstein series on reductive groups and introducing some of their properties, including the meromorphic continuation and the computation of its constant term.

1 Motivation

Why do we care about Eisenstein series and their constant terms? The theory of Eisenstein series is one of the fundamental tools for the study of automorphic forms and an indispensable part of the Langlands program[1]. The Langlands-Shahidi method is that of studying L-functions that appear in the constant terms of Eisenstein series. This method has its roots in the classical Eisenstein series on the upper half plane $\mathfrak{h}$. In the classical setting it accounts for the continuous spectrum for Riemann surfaces $\Gamma\backslash\mathfrak{h}$, as Γ ranges over conguruence subgroup of $SL_2(\mathbb{Z})$.

Let $s \in \mathbb{C}$ and $z \in \mathfrak{h}$. Consider the following Eistenstein series defined at the cusp of infinity by

$$E(z,s) = \sum_{\gamma \in B_2(\mathbb{Z})\backslash\Gamma} \mathrm{Im}(\gamma z)^s$$

Here $\Gamma = SL_2(\mathbb{Z})$ and $B_2(\mathbb{Z}) = \{\pm\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \mid b \in \mathbb{Z}\}$. The Fourier expansion at ∞ is given by

$$E(z,s) = y^s + \pi^{1/2}\frac{\Gamma\left(\frac{2s-1}{2}\right)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}y^{1-s} + \pi^s\Gamma(s)^{-1}\zeta(2s)^{-1}\sum_{n\neq 0}|n|^{-1/2}\left(\sum_{ab=|n|}\left(\frac{a}{b}\right)^{s-1/2}\right)W_s(nz)$$

Here $z = x + iy$ and $W_s(z)$ is the Whittaker function defined by

$$W_s(x+iy) = 2y^{1/2}K_{s-1/2}(2\pi y)\exp(2\pi i x)$$

and $K_v(z)$ is the K-Bessel function

$$K_\nu(z) = \int_0^\infty e^{-z\cosh t}\cosh(\nu t)dt,$$

where $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(\nu) > -1/2$.
Recall the completed L-function for the Riemann zeta function $\zeta(s)$:

$$L(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

[1]It determines the continuous spectrum of the group as well as all the non-cuspidal representations appearing discretely in the spectrum as ressidues of these series (which are meromorphic functions of several variables)

With a suitable normalization, we can see that the coefficient of y^{1-s} in the expansion of $E(z,s)$ is

$$\frac{L(2s-1)}{L(2s)}$$

On the other hand, the non-constant Fourier coefficients of $E(z,s)$ are

$$\frac{1}{L(2s)}$$

and they control the constant term via the quotient. The poles of $E(z,s)$ as a function of s are controlled by those of $L(2s-1)$ and are precisely those of the constant term.

This phenomenon is quite general and in fact constant terms of general Eisenstein series are ratios of L-functions, while their non-constant Whittaker Fourier coefficients are inverses of these L-functions. We will see the connection spelled out in the following lecture.

2 Preliminaries

We will introduce some notations and recall some facts about reductive groups.
Let k be a number field and G be a connected reductive group over k. Fix a minimal parabolic P_0 of G. Let P be a parabolic of G such that $P_0 \subset P$ (we call such a P standard). Write $P_0 = M_0N_0$, where N_0 is the unipotent radical of P_0 and M_0 is a Levi factor of P_0. Note that any two Levi subgroup are conjugate by a unique element in N_0. Fix a Levi decomposition $P = MN$ of P. Then we have $N_0 \subset N$. We can also fix M uniquely by requiring $M_0 \subset M$.
The adelization of G satisfying $G(\mathbb{A}_k) = P_0(\mathbb{A}_k)K$, which follows from the Iwasawa decomposition of $G(k_v)$ at each places.
Given an algebraic group H over k, let $X(H)_k$ denote the group of k-rational characters of H, i.e., all k-morphisms of H into $\mathbb{G}_m$. We write $X(H)$ for $X(H)_{\bar{k}}$.
We can assume by restriction of scalar that $k = \mathbb{Q}$. Let $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$ be the adeles. Set

$$\mathfrak{a}_G = \mathrm{Hom}(X(G)_{\mathbb{Q}}, \mathbb{R}).$$

One can define a homomorphism

$$H_G : G(\mathbb{A}) \to \mathfrak{a}_G$$

by[2]

$$\exp\langle H_G(g), \chi\rangle = |\chi(g)| = \prod_v |\chi(g_v)|_v.$$

Let $G(\mathbb{A})^1 = \ker(H_G)$. We also define the duals:

$$\mathfrak{a}_G^* = X(G) \otimes_{\mathbb{Z}} \mathbb{R}, \quad \mathfrak{a}_{G,\mathbb{C}}^* = a_G^* \otimes_{\mathbb{R}} \mathbb{C} = X(G) \otimes_{\mathbb{Z}} \mathbb{C}$$

Given an irreducible unitary representation π of $G(\mathbb{A})$, define an action of $i\mathfrak{a}_G^*$ on the representation by

$$\pi_\nu(x) = \pi(x)\exp\langle \nu, H_G(x)\rangle, \quad \nu \in i\mathfrak{a}_G^*.$$

[2] see the details of this definition can be found on p12 of Shahidi

Note that[3] $G(\mathbb{A}) = G(\mathbb{A})^1 \times A_G(\mathbb{R})^0$, where A_G is the split component of G.
Now we would like to recall the induced representation on $G(\mathbb{A})$ from $P(\mathbb{A})$. Let $\nu \in \mathfrak{a}^*_{P,\mathbb{C}} = \mathfrak{a}^*_{M,\mathbb{C}}$. The induced representation is defined by

$$I_P(\nu) = \mathrm{Ind}_{P(\mathbb{A})\uparrow G(\mathbb{A})}(R_{M,disc,\nu} \otimes \mathbf{1}_N)$$

Similar to notation introduced above,

$$R_{M,disc,\nu}(x) = R_{M,disc}(x) \exp\langle \nu, H_M(x)\rangle.$$

The representation $R_{M,disc} : M(\mathbb{A}) \to L^2_{disc}(M(\mathbb{Q})\backslash M(\mathbb{A})^1)$ is given by right regular action. In fact, $M(\mathbb{A})$ acts by projecting onto $M(\mathbb{A})^1$ first. L^2_{disc} is called the discrete spectrum. In the classical case of $\Gamma\backslash\mathfrak{h}$, roughly speaking, it is spanned by the constant functions and weight zero Maass cusp forms.

Let V_P denote the underlying vector space of $I_P(v)$. It consists of all measurable functions

$$\phi : N(\mathbb{A})M(\mathbb{Q})A_M(\mathbb{R})^0\backslash G(\mathbb{A}) \to \mathbb{C}$$

such that

1. The function $\phi_x : M(\mathbb{Q})\backslash M(\mathbb{A})^1 \to \mathbb{C}$ defined by $\phi_x(m) = \phi(mx)$ belongs to $L^2_{disc}(M(\mathbb{Q})\backslash M(\mathbb{A})^1)$ for all $x \in G(\mathbb{A})$.
2. $||\phi||^2 = \int_K \int_{M(\mathbb{Q})\backslash M(\mathbb{A})^1} |\phi(mk)|^2 dmdk = \int_K \int_{M(\mathbb{Q})\backslash M(\mathbb{A})^1} |\phi_k(m)|^2 dmdk < \infty.$

Note that both conditions are needed as no one implies the other. Let $y \in G(\mathbb{A})$. The action of $I_P(\nu)$ on V_P is defined by

$$\left(I_P(v)(y)\phi\right)(x) = \phi(xy) \exp\langle \nu + \rho_P, H_P(xy)\rangle \exp\langle -(\nu + \rho_P), H_P(x)\rangle.$$

Note that in this definition, the space V_P is independent of ν but the dependence on ν is reflected in the representation $I_P(\nu)$. The rationale of this weird-looking definition is to preserve the unitarity and be compatible with the right regular action.

3 Eisenstein series

Now we are finally ready to define the Eisenstein series attached to ϕ.

Definition 3.1. Given $x \in G(\mathbb{A})$, $\phi \in V_P$ and $\nu \in \mathfrak{a}^*_{P,\mathbb{C}} = \mathfrak{a}^*_{M,\mathbb{C}}$, let

$$E(x, \phi, \nu) = \sum_{\delta \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \phi(\delta x) \exp\langle \nu + \rho_P, H_P(\delta x)\rangle$$

Here ρ_P denotes half the sum of roots in $\mathfrak{n} = Lie(N)$.

We are also gonna define when two parabolics $P, P' \supset P_0$ are called "associated."

[3] the superscript 0 means connected component

Definition 3.2. Let A_0 be the maximal split subtorus of a maximally split torus T in G defined over $k = \mathbb{Q}$. Let $\mathfrak{a}_0$ denote $\mathfrak{a}_{A_0}$. The Weyl set $W(\mathfrak{a}_P, \mathfrak{a}_{P'})$ is the set of linear isomorphisms between $\mathfrak{a}_P$ and $\mathfrak{a}_{P'}$ that are obtained by restricting elements of the Weyl group $W(\mathfrak{a}_0) := W(A_0, G)$. We say P and P' are associated if $W(\mathfrak{a}_P, \mathfrak{a}_{P'}) \neq \phi$.

Suppose P and P' are associated. Given $s \in W(\mathfrak{a}_P, \mathfrak{a}_{P'})$, choose its representative $w_s \in G(\mathbb{Q})$ as an element in $W(\mathfrak{a}_0)$. Now we define the corresponding global intertwining operator[4]

$$M(s, \nu) : V_P \to V_{P'}$$

by (the GL_2 definition can be found in Bump p.227 (6.7))

$$(M(s,\nu)\phi)(x) = \int_{N'(\mathbb{A}) \cap w_s N(\mathbb{A}) w_s^{-1} \backslash N'(\mathbb{A})} \phi(w_s^{-1} n x) \exp\langle \nu + \rho_P, H_P(w_s^{-1} n x)\rangle \exp\langle -(s\nu + \rho_{P'}), H_{P'}(x)\rangle dn$$

This operator intertwines $I_P(\nu)$ with $I_{P'}(s\nu)$. We have

$$E(x, I_P(\nu)(y)\phi, \nu) = E(xy, \phi, \nu)$$
$$M(s,\nu)\left(I_P(\nu)(y)\right) = \left(I_{P'}(s\nu)(y)\right) M(s,\nu)$$

The convergence of the Eisenstein seriesso defined has been established in a suitable domain for a dense subset $V_P^0 \subset V_P$. The space V_P^0 consists of all the $\phi \in V_P$ satisfying

1. ϕ is K-finite, i.e. the space spanned by $\{I_P(\nu)(k)\phi | k \in K\}$ is finite dimensional
2. It is a subspace of a finite sum of irreducible sub-representations of V_P under the action $I_P(\nu)$ of $G(\mathbb{A})$.

Given P, set

$$(\mathfrak{a}_P^*)^+ = \{\Lambda \in \mathfrak{a}_P^* | \langle \Lambda, \alpha^\vee \rangle > 0\}.$$

Here α is the unique simple root of A in $\mathfrak{n}$, where A is the split component of M. Therefore, $(\mathfrak{a}_P^*)^+$ is the interior of the fundamental Weyl chamber associated to α.

Lemma 3.1 (Langlands). Suppose $\phi \in V_P^0$ and ν belongs to the open subset

$$\{\nu \in \mathfrak{a}_{P,\mathbb{C}}^* | \operatorname{Re}(\nu) \in \rho_P + (\mathfrak{a}_P^*)^+\}.$$

Then $E(x, \phi, \nu)$ and $M(s,\nu)\phi$ both converge absolutely to an analytic function of ν on this open subset.

We can analytically continue both $E(x, \phi, \nu)$ and $M(s,\nu)\phi$ to all of $\mathfrak{a}_P^*$.

Theorem 3.1 (Langlands). *Suppose $\phi \in V_P^0$. Then $E(x,\phi,\nu)$ and $M(s,\nu)\phi$ can be extended to meromorphic functions of $\nu \in \mathfrak{a}_P^*$ satisfying*

$$E(x, M(s,\nu)\phi, s\nu) = E(x, \phi, \nu).$$

If we have another $P'' \supset P_0$ associated to P', then for $t \in W(\mathfrak{a}_{P'}, \mathfrak{a}_{P''})$

$$M(ts, \nu) = M(t, s\nu) M(s, \nu).$$

When $\nu \in i\mathfrak{a}_P^*$, both $E(x,\phi,\nu)$ and $M(s,\nu)$ are analyticand $M(s,\nu)$ extends to a unitary operator from V_P to $V_{P'}$.

[4] overloading the notation s?

4 Computation of the constant term

We first calculate the constant term assuming P is a maximal parabolic subgroup. Let P' be another standard maximal parabolic subgroup. Write the Levi decomposition $P' = M'N'$ with $M' = M_{\theta'} \supset M_0$. Here the simple foots $\Delta(A_0, M) - \theta'$ generates $\mathfrak{n}'$ and θ' is the set of simple roots of A_0 in M'. By definition, the constant term of $E(\cdot, \phi, \nu)$ along P' is

$$E_{P'}(x, \phi, \nu) = \int_{N'(\mathbb{Q})\backslash N'(\mathbb{A})} E(nx, \phi, \nu)dn = \int_{N'(\mathbb{Q})\backslash N'(\mathbb{A})} \sum_{\delta \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \phi(\delta nx) \exp\langle \nu + \rho_P, H_P(\delta nx)\rangle dn$$

If we write $\phi_\nu(x) = \phi(x) \exp\langle \nu + \rho_P, H_P(x)\rangle$, then it becomes

$$\begin{aligned} E_{P'}(x, \phi, \nu) &= \int_{N'(\mathbb{Q})\backslash N'(\mathbb{A})} \sum_{\delta \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \phi_\nu(\delta nx)dn. \\ &= \int_{N'(\mathbb{Q})\backslash N'(\mathbb{A})} \sum_{\delta \in P(\mathbb{Q})\backslash G(\mathbb{Q})/N'(\mathbb{Q})} \sum_{\gamma \in N'(\mathbb{Q})\cap(\delta^{-1}P(\mathbb{Q})\delta)\backslash N'(\mathbb{Q})} \phi_\nu(\delta\gamma nx)dn. \end{aligned}$$

It is invariant from left because if for $\eta \in P(\mathbb{Q})$ we have $\delta\gamma = \eta\delta$, then $\gamma = \delta^{-1}\eta\delta$ lies in $N'(\mathbb{Q}) \cap (\delta^{-1}P(\mathbb{Q})\delta)$. After absorbing γ into n, we can rewrite the integral as

$$\int_{N'(\mathbb{Q})\cap(\delta^{-1}P(\mathbb{Q})\delta)\backslash N'(\mathbb{A})} \sum_{\delta \in P(\mathbb{Q})\backslash G(\mathbb{Q})/N'(\mathbb{Q})} \phi_\nu(\delta nx)dn.$$

By the Bruhat decomposition for $G(\mathbb{Q})$, we know $\delta \in P(\mathbb{Q})\backslash G(\mathbb{Q})/N'(\mathbb{Q})$ can be written as

$$\delta = w\gamma'.$$

Here $\gamma' \in P'(\mathbb{Q})/N'(\mathbb{Q})$ and $w \in N_G(A_0) \cap G(\mathbb{Q})$. Recall that

$$G(\mathbb{Q}) = \bigcup_{w \in N_G(A_0)} P(\mathbb{Q})wP'(\mathbb{Q}).$$

Using these facts we can rewrite one term of the integral by

$$\begin{aligned} \int_{N'(\mathbb{Q})\cap(\gamma'^{-1}w^{-1}P(\mathbb{Q})w\gamma')\backslash N'(\mathbb{A})} \phi_\nu(w\gamma' nx)dn &= \int_{N'(\mathbb{Q})\cap(\gamma'^{-1}w^{-1}P(\mathbb{Q})w\gamma')\backslash N'(\mathbb{A})} \phi_\nu(w\gamma' n{\gamma'}^{-1}\gamma x)dn \\ &= c(\gamma') \int_{N'(\mathbb{Q})\cap(w^{-1}P(\mathbb{Q})w)\backslash N'(\mathbb{A})} \phi_\nu(wn\gamma' x)dn \end{aligned}$$

The last step follows from the fact that γ' normalizes $N'(\mathbb{A})$ and $N'(\mathbb{Q})$ and we can apply the change of variable $\gamma' n{\gamma'}^{-1} \mapsto n$. Here $c(\gamma')$ is a constant depending on γ'.

Next, we continue to simplify this integral.

Let $\tilde{w}$ be an element of the double coset $P(\mathbb{Q})\backslash G(\mathbb{Q})/N'(\mathbb{Q})$ represented by w, which can be regarded as an element of the Weyl group $W(A_0, G)$.

Let $\Phi^+ = \Phi^+(A_0, G)$ denote the positive roots of A_0 in G and Φ^+_θ the positive roots of A_0 in M. In fact, the set of all the positive roots of A_0 in M is a subset of Φ^+. Define the sets

$$\begin{aligned} \Phi^1_\theta &= \{\alpha \in \Phi^+_\theta | \alpha = \tilde{w}(\beta), \beta \in \Phi^+ - \Phi^+_{\theta'}\} \\ \Phi^2_\theta &= \{\alpha \in \Phi^+_\theta | \alpha = \tilde{w}(\beta), \beta \in \Phi^+_{\theta'}\} \end{aligned}$$

These sets have the following properties:

- If α, α' in Φ^1_θ and $\alpha+\alpha'$ is a root, then $\alpha+\alpha'\in\Phi^1_\theta$.
- If $\alpha\in\Phi^1_\theta$, $\alpha'\in\Phi^2_\theta$ and $\alpha+\alpha'$ is a root, then $\alpha+\alpha'\in\Phi^1_\theta$.

Let N_1 be the unipotent subgroup of M whose Lie algebra is generated by root vectors X_α with $\alpha\in\Phi^1_\theta$. Then N_1 is the unipotent radical of a parabolic subgroup of M.
Since[5] $w^{-1}N_1w\subset N'$ and

$$w^{-1}P(\mathbb{Q})w\cap w^{-1}N_1(\mathbb{Q})w = w^{-1}M(\mathbb{Q})w\cap w^{-1}N_1(\mathbb{Q})w = w^{-1}N_1(\mathbb{Q})w,$$

we can break the integration over $w^{-1}P(\mathbb{Q})w\cap N'(\mathbb{Q})\backslash N'(\mathbb{A})$ as a product over

$$\left(w^{-1}P(\mathbb{Q})w\cap N'(\mathbb{Q})\right)(w^{-1}N_1(\mathbb{A})w)\backslash N'(\mathbb{A})$$

and

$$w^{-1}P(\mathbb{Q})w\cap N'(\mathbb{Q})\cap(w^{-1}N_1(\mathbb{A})w)\backslash w^{-1}N_1(\mathbb{A})w = w^{-1}N_1(\mathbb{Q})w\backslash w^{-1}N_1(\mathbb{A})w = N_1(\mathbb{Q})\backslash N_1(\mathbb{A})$$

Thus, the integral above becomes

$$\int_{(w^{-1}P(\mathbb{Q})w\cap N'(\mathbb{Q}))(w^{-1}N_1(\mathbb{A})w)\backslash N'(\mathbb{A})}\left(\int_{N_1(\mathbb{Q})\backslash N_1(\mathbb{A})}\phi_\nu(n_1wn\gamma'x)dn_1\right)dn.$$

Assume that ϕ is chosen such that ϕ_y is in $L^2_{cusp}(M(\mathbb{Q})\backslash M(\mathbb{A})^1)$. The inner integral equals to

$$\exp\langle\nu+\rho_P, H_P(wn\gamma'x)\rangle\int_{N_1(\mathbb{Q})\backslash N_1(\mathbb{A})}\phi(n_1wn\gamma'x)dn_1.$$

For $y=wn\gamma'x$, ϕ_y is in $L^2_{cusp}(M(\mathbb{Q})\backslash M(\mathbb{A})^1)$. Thus[6], the integral vanishes unless Φ^1_θ is empty. In that case, N_1 is trivial. Up to this point, we have not yet use the fact that P, P' are maximal. With this assumption, we can conclude $M'=w^{-1}Mw$.
Since $\gamma'\in P'(\mathbb{Q})/N'(\mathbb{Q})\cong M'(\mathbb{Q})$ and $M'(\mathbb{Q})=w^{-1}M(\mathbb{Q})w$, we can assume $\gamma'=1$ in the Bruhat decomposition $\delta=w\gamma'$. There are two different possibilities:

1. $\tilde{w}(\alpha)>0$, which gives $P=P'$.
2. $\tilde{w}(\alpha)<0$, which gives $wP'w^{-1}$ is the parabolic subgroup of G opposed to G. Consequently, $w^{-1}P(\mathbb{Q})w\cap N'(\mathbb{Q})=\{1\}$. Since $\gamma'=1$, the original integral we have been working on becomes

$$\int_{N'(\mathbb{A})}\phi_\nu(wnx)dn$$

which is a meromorphic function of ν on all of $\mathfrak{a}^*_{P,\mathbb{C}}$.

We summarize the computation above with the following:

Theorem 4.1. *Assume for every $y\in G(\mathbb{A})$ we have $\phi_y\in L^2_{cusp}(M(\mathbb{Q})\backslash M(\mathbb{A})^1)$, then the constant term*

$$E_{P'}(x,\phi,\nu)=\int_{N'(\mathbb{Q})\backslash N'(\mathbb{A})}E(nx,\phi,\nu)dn$$

[5] this is by definition of θ' and Φ^1_θ

[6] if Φ^1_θ is not empty, then we are integrating a cusp form against a nilpotent, which gives 0

is zero unless $\theta' = \tilde{w}(\theta)$ *for some* $w \in W(A_0, G)$, *i.e.* $M' = w^{-1}Mw$.
If $P = P'$, *then* $E_P(x, \phi, \nu) = \phi_\nu(x)$.
If P' *is opposite to* $w^{-1}Pw$, *then for* $\mathrm{Re}(\nu) \in \rho_P + (\mathfrak{a}_P^*)^+$

$$E_{P'}(x, \phi, \nu) = \phi_\nu(x)\delta_{\theta,\theta'} + \int_{N'(\mathbb{A})} \phi_\nu(wnx)dn$$

(The equivalent statement for GL_2 can be found in Bump p353 (7.15)) The general case involves the global intertwining operator $M(s, \nu)$ we introduced earlier. If P, P' are two standard parabolic subgroups of G of the same rank, then under the same assumptions,

$$E_{P'}(x, \phi, \nu) = \sum_{s \in W(\mathfrak{a}_P, \mathfrak{a}_{P'})} (M(s, \nu)\phi)(x) \exp\langle s\nu + \rho_{P'}, H_{P'}(x)\rangle.$$

Note that one can deduce the maximal case from this. The proof of the general case needs to take into consideration of the contribution of $s \in W(\mathfrak{a}_P, \mathfrak{a}_{P'})$ that's not $\{\pm 1\}$.