§Notation

Let G = red. group $/\mathbb{Q}$ (works for gn'l F)

Write $\mathbb{A} = \mathbb{A}_{\mathbb{Q}} = \mathbb{R} \times \mathbb{A}^{(\infty)}$

Fix a minimal parabolic P_0 of G over $\mathbb{Q}$.

E.g. $G = GL(2)$, $P_0 = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

$G = GL(n)$, $P_0 = \begin{pmatrix} \boxed{*} & & & \\ & \boxed{*} & * & \\ & & \ddots & \\ & & & \boxed{*} \end{pmatrix}$ $n = n_1 + \dots + n_r$

Then, $P_0 \underset{/\mathbb{Q}}{=} M_0 \ltimes U_0$ w/ M_0 = Levi
U_0 = unip. rad

Let $\underline{\mathcal{P}}$ = {standard parabolics P of G}

$\Rightarrow$ $P = M \ltimes U$, $P \supset P_0$, $M \supset M_0$, $U \subset U_0$

Let Z_M = Center(M) & S_M = max'l split torus of Z_M

<u>Def</u> $\mathfrak{a}_P := X_*(S_M) \otimes \mathbb{R}$, $\mathfrak{a}_P^* := X^*(S_M) \otimes \mathbb{R}$

If $P = P_0$, we write $\mathfrak{a}_0 := \mathfrak{a}_{P_0}$.

We can write $\mathfrak{a}_0 = \mathfrak{a}_P \oplus \mathfrak{a}_0^P$, $\mathfrak{a}_0^* = \mathfrak{a}_P^* \oplus (\mathfrak{a}_0^P)^*$

E.g. $G = GL(2)$, $P_0 = \begin{pmatrix} * & * \\ & * \end{pmatrix}$, $P = G$,

$\begin{pmatrix} a & \\ & b \end{pmatrix} = \begin{pmatrix} ac^{-1} & \\ & bc \end{pmatrix} \begin{pmatrix} c & \\ & c^{-1} \end{pmatrix}$ $\mathfrak{a}_0^* = \mathbb{R}.(e_1 + e_2) \oplus \mathbb{R}.(e_1 - e_2)$

$= \begin{pmatrix} ab & \\ & ab \end{pmatrix}^{1/2} \begin{pmatrix} c & \\ & c^{-1} \end{pmatrix}$

Given $P, Q \subset \underline{\mathcal{P}}$, let $W(P,Q)$ be the finite set of cosets

$$wM_P,\ w \in G(\mathbb{Q}) \text{ such that}$$

$$wM_P w^{-1} = M_Q$$

E.g. $\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\left(\begin{array}{cc|c} a & b & \\ c & d & \\ \hline & & e \end{array}\right)\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \left(\begin{array}{c|cc} e & & \\ \hline & a & b \\ & c & d \end{array}\right)$

In particular, $W = W(P_0, P_0) = N_{G(\mathbb{Q})}(M_0)/M_0(\mathbb{Q})$

Any $w \in W(P,Q)$ induces an iso $\mathfrak{a}_P \to \mathfrak{a}_Q$

Let $H_M : M(\mathbb{A}) \to \mathfrak{a}_P$ be the continuous g'p hom. given by

$$e^{\langle \chi, H_M(m)\rangle} = |\chi(m)|_{\mathbb{A}} \quad \forall \chi \in X^*(M),$$

where $\chi : M(\mathbb{A}) \to \mathbb{A}^\times$. Let $M(\mathbb{A})^1 := \ker H_M$

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Let $X_P :=$ g'p of C^0 quasi-char of $M(\mathbb{A})/M(\mathbb{A})^1$

w/ additive notation & $m^\lambda := \lambda(m)$

<u>Fact</u> H_M is onto & $X_P = X^*(M) \otimes_{\mathbb{Z}} \mathbb{C} = \mathfrak{a}_P^* \otimes_{\mathbb{R}} \mathbb{C}$

$\Rightarrow$ We have $\mathrm{Re} : X_P \to \mathfrak{a}_P^*$

Finally, fix a max'l compact $\mathbb{K} \subset G(\mathbb{A})$ in "good position" w.r.t. M_0, i.e.

$$G(\mathbb{A}) = M(\mathbb{A})\, U(\mathbb{A})\, \mathbb{K}$$

$$M(\mathbb{A}) \cap \mathbb{K} \text{ max'l cpct in } M(\mathbb{A})$$

E.g. $G = GL(2)$, $K = O_2(\mathbb{R}) \times GL_2(\hat{\mathbb{Z}})$

Consider the proj $M(\mathbb{A}) \to M(\mathbb{A})/M(\mathbb{A})^1$ & its left-$U(\mathbb{A})$, right-$\mathbb{K}$ invariant ext'n

$$m_P : G(\mathbb{A}) \to M(\mathbb{A})/M(\mathbb{A})^1$$

Def $H_P := H_M \circ m_P : G(\mathbb{A}) \to \mathfrak{a}_P$ $\quad (H_0 = H_{P_0})$

Write $X = G(\mathbb{Q}) \backslash G(\mathbb{A})$ & $X_P = U(\mathbb{A}) P(F) \backslash G(\mathbb{A})$

Let $\mathcal{A}_P$ = Space of autom. forms on X_P.

Given $\varphi \in \mathcal{A}_P$, $\lambda \in X_P$, let $\varphi_\lambda(g) := \varphi(g)\, m_P(g)^\lambda$

Then, $\varphi_\lambda \in \mathcal{A}_P$ again

Def $$E(g, \varphi, \lambda) := \sum_{\gamma \in P(\mathbb{Q}) \backslash G(\mathbb{Q})} \varphi_\lambda(\gamma g) \qquad (g \in G(\mathbb{A}))$$

converges abs. & loc. unif. in g & λ for $\mathrm{Re}(\lambda)$ suff. reg in pos. Weyl chamber of $\mathfrak{a}_P^*$

E.g. $G = SL_2$, $P = \begin{pmatrix} a & * \\ & a^{-1} \end{pmatrix}$, φ = adelization of $\mathrm{Im}(z)$

$\Rightarrow \lambda \leftrightarrow$ complex number $s \in \mathbb{C}$ & $\varphi_\lambda = \varphi^{s + 1/2}$

$\mathbb{C} \simeq \hat{X}_P$

Main Result: Extend E to $\lambda \in X_P$ meromorphically

§3 Principle of Meromorphic Continuation

Let M = complex analytic manifold (e.g. X_p)

Let E be a complex, Hausdorff, locally convex topo. v. sp (LCTVS) (e.g. $\mathcal{S}_{\text{unmg}}(X)$)

Def Let $(\mathcal{F}_i, \mu_i, c_i)_{i \in I}$ be a family of triples s.t.

(1) $\mathcal{F}_i$ are Hausdorff $\mathcal{F}_i$

(2) $(\mu_i)_{s \in M}$ is an analytic family of operators $E \to \mathcal{F}_i$

"weak" analytic def'n

(3) $c_i : M \to \mathcal{F}_i$ is an anylitic fct

It induces a system $\boxed{-}(s)$ of linear equation

$$(\mu_i)_s(v) = c_i(s) \qquad (i \in I, v \in E)$$

We call it an analytic family of linear syst.

Def Let $A_s \subseteq E$ be a family of subsets.

(1) It is of finite type if $\exists$ f. dim'l L & an analytic family $\lambda_s : L \to E$ s.t. $A_s \subset \operatorname{Im} \lambda_s$

(2) Obvious def of loc. f.t.

(3) $\boxed{-} = (\boxed{-}(s))_s$ is (loc.) f.t. if $\operatorname{Sol}(\boxed{-})$ is (loc) f.t.

<u>THM</u> (Principle of Mero. Continuation)

Let $\boxminus = (\boxminus(s))_{s \in \mathcal{R}}$ be an ASLE that is l.f.t.

Let $\mathcal{M}_{unq} = \{ s \in \mathcal{M} \mid Sol(\boxminus(s)) = \{v(s)\} \}$

Suppose $\mathcal{M}$ is conn'd & $\mathcal{M}_{unq}^{\circ} \neq \emptyset$. Then,

(1) $\mathcal{M}_{unq}$ is open & its complement is analytic

(2) v is holom. on $\mathcal{M}_{unq}$

(3) v is merom. on $\mathcal{M}$

Proof of this is easy. Hard part:

(1) Show $E(g, \lambda, \varphi)$ solves such an ASLE (param $s = \lambda \in X_p$)

(2) Show its unique sol^n

§3 Cuspidal Exponents along Parabolics

Recall $\mathfrak{a}_P^*$, $\mathfrak{a}_P$ $\ni \rho_P = \frac{1}{2}|\det \circ \mathrm{Ad}_P|$

E.g. $P = G = GL(n)$, $\mathfrak{a}_G = \mathbb{R}\cdot(e_1^\vee + \dots + e_n^\vee)$
$\mathfrak{a}_0^G$ = coroot space of $SL(n)$

$$\mathfrak{a}_0 = \mathfrak{a}_0^P \oplus \mathfrak{a}_P, \quad \mathfrak{a}_0^* = \dots$$

$$\mathfrak{a}_P = \mathfrak{a}_P^Q \oplus \mathfrak{a}_Q, \quad \mathfrak{a}_P^* = \dots \quad (P \subset Q)$$

where $\mathfrak{a}_P^Q = \mathfrak{a}_P \cap \mathfrak{a}_0^Q$. Write $\lambda = \lambda_P^Q + \lambda_Q$

Let $\Delta_0 \subset X^*(S_0) \subset \mathfrak{a}_0^*$ be the simple roots of S_0 on U_0, $\Delta_0^\vee$ = simple coroots

$\Rightarrow$ Δ_0 is a basis of $(\mathfrak{a}_0^G)^*$ $(\Delta_0^\vee \to \mathfrak{a}_0^G)$

For gen'l $P \in \mathcal{P}$, let $\Delta_0^P \subset \Delta_0$ be the set of simple roots of S_0 on $U_0 \cap M$, i.e. a basis of $(\mathfrak{a}_0^P)^*$

Let Δ_P = (bijective) Image of $\Delta_0 \setminus \Delta_0^P$ under $\mathfrak{a}_0^* \to \mathfrak{a}_P^*$
$\Delta_P^\vee$ = similar def^n

Denote $\alpha \mapsto \alpha^\vee$ the classical bij $\Delta_P \xrightarrow{\sim} \Delta_P^\vee$

Lemma. *Let $P \in \mathcal{P}$ and $\alpha \in \Delta_P$. Then,*

(1) All the coefficients of α with the respect to the basis Δ_0 are non-negative.
(2) $\langle \alpha, \alpha^\vee \rangle > 0$.

Def $A_0 = S_0(\mathbb{R})^0 \subset S_0(\mathbb{R}) \subset S_0(\mathbb{A}_{\mathbb{Q}})$

$$A_P = A_0 \cap S_M(\mathbb{A}),\quad A_P^Q = A_P \cap M_Q(\mathbb{A})^1$$

for $P, Q \in \mathcal{P}$ w/ $Q \supset P$

$H_P|_{A_P} : A_P \xrightarrow{\sim}$ grp of top. g'p

We have $M(\mathbb{A}) \simeq A_P \times M(\mathbb{A})^1 \Rightarrow X_P$ is the g'p of quasi-char of A_P

The space $\mathcal{A}_P$ admits a left action by A_P. We twist it as

$$(a \cdot \phi)(g) = \delta_P(a)^{-1/2} \phi(ag)$$

$$\Rightarrow \mathcal{A}_P = \bigoplus_{\lambda \in X_P} \mathcal{A}_{P,\lambda} \quad (\lambda\text{-gen'ized eigenspace})$$

For any $Q \in \mathcal{P}$ w/ $Q \subset P$, the constant term

$$C_{P,Q}\phi(g) := \int_{U_Q(\mathbb{Q})\backslash U_Q(\mathbb{A})} \phi(ug)\, du$$

defines a linear map $\mathcal{A}_P \to \mathcal{A}_Q$

If $Q \not\subset P$, set $C_{P,Q}\phi = 0$

Def $\mathcal{A}_P^{cusp} = \{\phi \in \mathcal{A}_P \mid C_{P,Q}\phi = 0, \forall Q\}$

$$\Rightarrow \mathcal{A}_P^{cusp} = \bigoplus_{\lambda} \mathcal{A}_{P,\lambda}^{cusp} \quad \& \quad \exists (\ )^{cusp} : \mathcal{A}_P \to \mathcal{A}_P^{cusp}$$

This cuspidal proj is char'd by $\quad C_c^\infty(\mathfrak{a}_P) \quad \mathcal{A}_P^{cusp}$

$$(\phi^{cusp}, \psi)_{X_P} = (\phi, \psi)_{X_P} \text{ if } \psi = (f \circ H_P)\cdot \psi'$$

Here, $(f_1, f_2)_{X_P} = \int_{X_P} f_1(g)\overline{f_2(g)}\,dg$

Def $C_{P,Q}^{cusp}\phi := (C_{P,Q}\phi)^{cusp} \in \mathcal{A}_Q^{cusp}$

$$\tilde{\tau}_P^{cusp} : \mathcal{A}_P \to \bigoplus_Q \mathcal{A}_Q^{cusp}, \quad \phi \mapsto (C_{P,Q}^{cusp}\phi)_Q$$

Def $\mathcal{E}_Q^{cusp}(\phi) \subset X_Q$ be the set of cusp. exponents of ϕ along Q, i.e.

$$\lambda \in \mathcal{E}_Q^{cusp}(\phi) \iff 0 \neq \mathrm{proj}_\lambda(C_{P,Q}^{cusp}\phi) \in \mathcal{A}_{Q,\lambda}^{cusp}$$

$$\Rightarrow \mathcal{E}^{cusp}(\phi) = \{(Q,\lambda) \mid Q \in \mathcal{P},\ Q \subset P,\ \lambda \in \mathcal{E}_Q^{cusp}(\phi)\}$$

Prop $\tilde{\tau}_G^{cusp}$ is injective

Not enough for us $\Rightarrow$ Want "leading" cusp. exp.

Let $\mathfrak{a}_{0,+}^* = \{\langle\lambda, \alpha^\vee\rangle \geq 0,\ \forall \alpha \in \Delta_0\}$

Def $\lambda \in X_P$ is leading if $\mathrm{Re}\,\lambda + \rho_P \in \mathfrak{a}_{0,+}^*$

$$\Rightarrow \mathcal{A}_P^{cusp,ld} = \bigoplus_{\lambda \text{ lead.}} \mathcal{A}_{P,\lambda}^{cusp}, \quad \mathrm{proj}_P^{ld} : \mathcal{A}_P^{cusp} \to \mathcal{A}_P^{cusp,ld}$$

$$\Rightarrow \mathcal{E}_P^{cusp,ld}(\phi) \subset \mathcal{E}_P^{cusp}(\phi) \ \&\ \mathcal{E}^{cusp,ld}(\phi) = \{(P,\lambda) \mid \dots\}$$

Def Lead cusp. comp of $\phi \in \mathcal{A}_P$ along Q is

$$\mathrm{proj}_Q^{ld}\left(C_{P,Q}^{cusp}\phi\right)$$

$\Rightarrow$ $\mathcal{L}_P : \mathcal{A}_P \to \bigoplus_{Q \subset P} \mathcal{A}_Q^{cusp, ld}$ as $\left(\bigoplus_{Q \in P} \mathrm{proj}_Q^{ld}\right) \circ \tilde{1}_P^{cusp}$

THM $\mathcal{L} = \mathcal{L}_G$ is injective.

Lemma (Special Case)

Let $\phi \in \mathcal{A}_G$. Assume every $(P, \lambda) \in \mathcal{E}^{cusp}(\phi)$ satisfies

(1) P is maximal (proper), i.e. $\Delta_P = \{\alpha\}$

(2) $\langle \mathrm{Re}\,\lambda + \rho_P, \alpha^\vee \rangle < 0$ (i.e. none of its exp are leading)

Then, $\phi = 0$.

Proof [Sketch] $C_{G,P}^{cusp}\phi \equiv 0$, $\forall P$ non-max'l
$\overset{\text{prev. prop}}{\Rightarrow}$ $C_{G,P}\phi \equiv 0$, $\forall P$ non-max'l

For P max'l, $C_{G,P}\phi$ is bdd on $G(\mathbb{A})^1$ $\Rightarrow$ its cusp. & we can control growth of its left translates $\Rightarrow$ $C_{G,P}\phi \equiv 0$ again $\Rightarrow$ ϕ is cuspidal.

By assumption, $\mathcal{E}_G^{cusp}(\phi) = \emptyset \Rightarrow \phi = 0$. $\square$

Corollary Let $\phi \in \mathcal{A}_G$, $(P,\lambda) \in \mathcal{E}^{cusp}(\phi)$, $\alpha \in \Delta_P$

Let P_α be std parab. s.t. $\Delta_0^{P_\alpha} = \Delta_0^P \cup \{\beta\}$ lift of α

Assume P is min'l w.r.t. ϕ (i.e. $\mathcal{E}_P^{cusp}\phi \neq \phi$ but $\mathcal{E}_Q^{cusp}\phi = \phi$ $\forall Q \subset P$) & $\langle \mathrm{Re}\,\lambda + \rho_P, \alpha^\vee \rangle < 0$.

Then, $\exists (Q,\mu) \in \mathcal{E}^{cusp}(\phi)$ (possibly $Q=P$) s.t.

- Q is max'l in P_α
- Q is min'l w.r.t. ϕ
- $\mu_{P_\alpha} = \lambda_{P_\alpha}$ (proj. $\mathfrak{a}_Q, \mathfrak{a}_P \twoheadrightarrow \mathfrak{a}_{P_\alpha}$)
- $\langle \mathrm{Re}\,\mu + \rho_P, \beta^\vee \rangle \geq 0$, where $\Delta_Q^{P_\alpha} = \{\beta\}$

Proof [Sketch] Make statement relative to P, Q both max'l in P_α (i.e. replace G by P_α).

Substract ϕ^{cusp} from $\phi \Rightarrow \mathcal{E}_G^{cusp}(\phi) = 0$

$\Rightarrow$ Use above to say at least one (Q,μ) satisfies last condition. $\square$

Proof of THM Let $0 \neq \phi \in \mathcal{A}_G$. Fix $\varpi^\vee \in \mathfrak{a}_0$

$$\langle \alpha, \varpi^\vee \rangle > 0, \text{ for all } \alpha \in \Delta_0$$

Let $(P,\lambda) \in \mathcal{E}^{cusp}(\phi)$ s.t. P is min'l w.r.t. ϕ and $\langle \mathrm{Re}\,\lambda + \rho_P, \varpi^\vee \rangle$ is max'l. Claim: λ is ld.

[Let's write $\mathrm{Re}\,\lambda$ instead of $\mathrm{Re}\,\lambda + \rho_P$ from now on.

Doesn't change anything since $(\rho_P)_{P_\alpha} = \rho_{P_\alpha} = (\rho_Q)_{P_\alpha}$

If not, $\exists\, \alpha \in \Delta_P$ s.t. $\langle \mathrm{Re}\,\lambda\ (+\rho_P), \alpha^\vee \rangle < 0$

Let $(Q, \mu) \in \mathcal{E}^{\mathrm{cusp}}(\phi)$ as above.

$$\Rightarrow \langle \mathrm{Re}\,\lambda, \varpi^\vee \rangle = \langle \underbrace{(\mathrm{Re}\,\lambda)_{P_\alpha}}_{= (\mathrm{Re}\,\mu)_{P_\alpha}}, \varpi^\vee \rangle + \langle (\mathrm{Re}\,\lambda)_0^{P_\alpha}, \varpi^\vee \rangle$$

(same for μ)

Also, $\langle (\mathrm{Re}\,\lambda)_0^{P_\alpha}, \varpi^\vee \rangle = \langle \mathrm{Re}\,\lambda, (\omega^\vee)_P^{P_\alpha} \rangle$

$$\begin{aligned} \mathfrak{a}_0 &= \mathfrak{a}_0^G \oplus \mathfrak{a}_G \\ &= \mathfrak{a}_0^{P_\alpha \ni \alpha} \oplus \mathfrak{a}_{P_\alpha} \\ &= \mathfrak{a}_0^P \oplus \underbrace{\mathfrak{a}_P^{\ni \alpha}}_{\dim = 1 + \dim \mathfrak{a}_{P_\alpha}} \end{aligned}$$

$$\mathfrak{a}_P = \overbrace{\mathfrak{a}_P^{P_\alpha}}^{\text{1-dim'l, spanned by } \alpha} \oplus \mathfrak{a}_{P_\alpha}, \qquad \mathfrak{a}_P^{P_\alpha} = \mathfrak{a}_0^{P_\alpha} \cap \mathfrak{a}_P$$

$\Rightarrow (\varpi^\vee)_P^{P_\alpha}$ is prop'l to $\alpha^\vee$

In fact, $\langle \alpha, (\varpi^\vee)_P^{P_\alpha} \rangle = \langle \alpha_0^{P_\alpha}, \varpi^\vee \rangle = \langle \alpha, \varpi^\vee \rangle > 0$
& $\langle \alpha, \alpha^\vee \rangle > 0$

$\Rightarrow (\varpi^\vee)_P^{P_\alpha} = c\,\alpha^\vee$ for $c > 0$ (same for $(\omega^\vee)_Q^{P_\alpha}$ and β)

$$\begin{aligned} \Rightarrow \langle (\mathrm{Re}\,\lambda)_0^{P_\alpha}, \varpi^\vee \rangle &= \langle \mathrm{Re}\,\lambda, (\varpi^\vee)_P^{P_\alpha} \rangle \\ &< 0 \\ &\leq \langle \mathrm{Re}\,\mu, (\omega^\vee)_Q^{P_\alpha} \rangle = \langle (\mathrm{Re}\,\mu)_0^{P_\alpha}, \varpi^\vee \rangle \end{aligned}$$

$\square$

The main part of the ASLE solved by the Eisenstein series is obtained by turning the following result into linear eq'ns

<u>THM</u> Fix $\varphi \in \mathcal{A}_P$, $\lambda \in X_P$ s.t. $\langle \mathrm{Re}\,\lambda + \rho_P, \alpha^\vee \rangle \gg 0$ (depends on φ) for all $\alpha \in \Delta_P$. Let $\psi = E(\varphi, \lambda)$. Then,

$$\mathcal{L}(\psi) = \mathcal{L}(\varphi_\lambda)$$

This follows from a "global Bernstein-Zelevinsky geometric Lemma" that computes

replace sum over $\frac{G(\mathbb{Q})}{P(\mathbb{Q})}$ to $\frac{Q(\mathbb{Q})}{Q_\omega(\mathbb{Q})}$

$$C_{G,Q}\,\psi = \sum_{\omega \in {}_Q W_P} E^Q\big(M(\omega,\lambda)\big(C_{P,P_\omega}\varphi, \omega\lambda\big)\big)$$

This can be rewritten to control

$$C^{\mathrm{cusp}}_{G,Q}(\psi) - C^{\mathrm{cusp}}_{P,Q}(\varphi_\lambda)$$

(Need to introduce a lot of notation to be more precise - see pages 21-25 of [BL])

$\Rightarrow$ We can introduce diff'l op's to kill all non-leading cuspidal exponents & turn the above fact into linear equations.

There's another "type" of equ's in the ASLE but these just ensure that the ASLE is l.f.t.

We don't need them for uniqueness.

$\lambda \leftrightarrow s$
$X_p \cong \mathbb{C}$

§ Case $G = SL(2)$ $\left(P = \begin{pmatrix} a & * \\ & a^{-1} \end{pmatrix}\right)$

No need to work adelically. Over $\mathbb{R}$ is fine.

We have $X = \Gamma \backslash \mathcal{H}$, $\Gamma = SL_2(\mathbb{Z})$, $\mu = \frac{dx\,dy}{y^2}$

Let $y: \mathbb{H} \to \mathbb{R}_{>0}$ be $y(z) = \mathrm{Im}(z) \Rightarrow y \in \mathcal{A}_P$

We have $C_c^\infty(K\backslash G/K) \circlearrowright L^1_{loc}(\mathbb{H})$ via convolution, denote the action as $f \mapsto \delta(h) f$

We have $\delta(h)\, y^{s+\frac{1}{2}} = \hat{h}(s)\, y^{s+\frac{1}{2}}$ (some $\hat{h}(s)$ entire)

FACT For all $s \in \mathbb{C}$, $\exists h \in C_c^\infty(K\backslash G/K)$ s.t. $\hat{h}(s) \neq 0$

We have $(C_{G,P} f)(\overset{y>0}{y}) = \int_{\mathbb{Z}\backslash\mathbb{R}} f(x+iy)\,dx = \int_0^1 f(x+iy)\,dx$

Our Eis series is $E(z;s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} y(\gamma z)^{s+\frac{1}{2}}$

This converges abs. for $\mathrm{Re}(s) > \frac{1}{2}$ & $\in \mathcal{F}_{umg}(X)$

FACT (1) $\delta(h) E(\cdot\,; s) = \hat{h}(s) E(\cdot\,; s)$

(2) $\exists$ some holom. fct $m(s)$ s.t.

"equiv" to our THM $\mathcal{L}(\psi) = \mathcal{L}(\varphi_x)$

$$CE(y;s) = y^{s+\frac{1}{2}} + m(s)\, y^{-s+\frac{1}{2}} \quad (y>0)$$

(3) $E(\cdot\,;s) \perp_X f$ for any cusp form $f \in X$

Our THM $\mathcal{L}(\psi) = \mathcal{L}_p(\varphi_\lambda)$ here means

$$\mathcal{L}_p(\varphi_\lambda) = \mathcal{L}_p(y^{s+\frac{1}{2}}) = \{s\}$$

b/c the action is $(T_a f)(y) = a^{-1/2} f(ay)$

& $y^{s+\frac{1}{2}}$ has eigenvalue a^s under T_a

$\Rightarrow$ $CE(y;s) - y^{s+\frac{1}{2}}$ should not have any leading cuspidal exponents, i.e.

$$(T_a - a^{-s})(CE(y;s) - y^{s+\frac{1}{2}}) \equiv 0, \ \forall s$$

<u>THM</u> $E(z,s)$ is the unique solution in $\mathcal{F}_{umg}(X)$ for $\mathrm{Re}(s) > 1/2$ of ALSE

$$\delta(h)\psi = \hat{h}(s)\psi, \quad \forall h \in C_c^\infty(K\backslash G/K)$$

$$(T_a - a^{-s})(C\psi(y) - y^{s+\frac{1}{2}}) \equiv 0, \quad \forall a > 0$$

$$(\psi, f)_X = 0, \quad \forall \text{ cusp forms } f \text{ on } X$$

<u>THM</u> This ASLE is l.f.t.