Eisenstein Series for GL_2

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1 Setting from Tate's thesis

By a local field, we mean $F = \mathbb{R}, \mathbb{C}$, the archimedean local fields, or F is the fractional field of a discrete valuation ring with finite residue field, the non-archimedean local fields. In the later case, we set $\mathfrak{o}$ the integer ring, $\mathfrak{p}$ the maximal ideal, ϖ a uniformizer, $k \cong \mathbb{F}_q$ the residue field. F is either a finite extension of some p-adic field, the mixed characteristic case, or $F \cong \mathbb{F}_q(t)$. In all cases, we have a normalized norm $|u|dx = dux$ for all Haar measure on F. If $F = \mathbb{R}$, $|\cdot|$ is the usual absolute value. If $F = \mathbb{C}$, $|\cdot|$ is the square of the usual absolute value. If F is non-archimedean, $|u| = q^{-v(u)}$. We also define $\mathfrak{u}$ the subgroup of elements of trivial norm.

For a non-archimedean local field F and a nontrivial additive character ψ, we define $\mathfrak{c}(\psi) = \mathfrak{p}^{-d}$ the conductor of ψ as the maximal fractional ideal $\mathfrak{p}^{-d} \subset F$ on which ψ is trivial.

Proposition 1. For every nontrivial character ψ, $\psi_a(x) := \psi(ax)$ defines an isomorphism from F to $\widehat{F}$.

The self-dual Haar measure is given as follows:

1. $F = \mathbb{R}$, $\psi(x) = e^{2\pi i x}$, dx is the usual Lebesgue measure.

2. $F = \mathbb{C}$, $\psi(z) = e^{2\pi i(\mathrm{Tr}(z))}$, $dz = 2dxdy$.

3. F is non-archimedean. Let $\mathfrak{p}^{-d}$ be the conductor of ψ. Then dx is the Haar measure with $\mathrm{Vol}_+(\mathfrak{o}) = q^{-d/2}$.

In general, if dx is self-dual with respect to ψ, $|a|^{1/2}dx$ is the self-dual with respect to ψ_a.

It is obvious that $\dfrac{dx}{|x|}$ is a Haar measure of F^*. We choose $d^*\alpha = \dfrac{dx}{|x|}$ for $F = \mathbb{R}$ or $\mathbb{C}$, and $d^*\alpha = \dfrac{q}{q-1}\dfrac{dx}{|x|}$ for F non-archimedean. In the later case, $d^*\alpha$ is normalized so that $\mathrm{Vol}_\times(\mathfrak{u}) = q^{-d/2}$.

Let $f \in \mathcal{S}(F)$, χ be a quasi-character on $F^\times$ and we define the local integral

$$Z(s, \chi, f) := \int_{F^*} f(t)(\chi|\cdot|^s)(t)d*\alpha,$$

which converges absolutely when $\Re(s) >> 0$ and has meromorphic continuation on $\mathbb{C}$. Consider the Fourier transform

$$\widehat{f}(a) := \int_F f(x)\psi(-ax)dx.$$

The local theory in Tate's thesis says the ratio

$$\frac{Z(1-s, \chi^{-1}, \widehat{f})}{Z(s, \chi, f)}$$

is an meromorphic function on s that depends on χ and ψ, not on f. The ratio is denote by the gamma factor

$$\gamma(s, \chi, \psi) := \frac{Z(1-s, \chi^{-1}, \widehat{f})}{Z(s, \chi, f)}$$

The local L-function of $L(s, \chi)$ is defined as the normalized gcd of all possible $Z(s, \chi, f)$. In the nonarchimedean case, when χ is ramified, $L(s, \chi) = 1$; when χ is unramified, $L(s, \chi) = \dfrac{1}{1 - (\chi|\cdot|^s)(\varpi)}$. In the archimedean case, we define

$$\Gamma_\mathbb{R}(s) := \pi^{-s/2}\Gamma(s/2),\ \Gamma_\mathbb{C}(s) := 2(2\pi)^{-s}\Gamma(s) = \Gamma_\mathbb{R}(s)\Gamma_\mathbb{R}(s+1).$$

When $F = \mathbb{R}$, $L(s, \mathrm{sgn}^\epsilon|\cdot|^{s_0}) = \Gamma_\mathbb{R}(s + s_0 + \epsilon)$. When $F = \mathbb{C}$, $L(s, c_n|\cdot|^{s_0}) = \Gamma_\mathbb{C}(s + s_0 + |n|/2)$.

In all cases, we also define the ϵ-factor by

$$\epsilon(s, \chi, \psi) := \frac{L(s, \chi)}{L(1-s, \chi^{-1})}\gamma(s, \chi, \psi).$$

It is an exponential function.

Let K be a global field. We choose a nontrivial additive character $\psi = \prod_v \psi_v$ on $\mathbb{A}_K$, which is trivial on K. Let $da = \prod_v da_v$ be the corresponding self-dual Haar measure. This induces an isomorphism $\mathbb{A}_K \to \widehat{\mathbb{A}_K}$ and $K, A_K/K$ are dual to each other. The induced measure on A_K/K is the normalized Haar measure.

Let $\Phi \in \mathcal{S}(\mathbb{A}_K)$, χ be a quasi-character on $I_K/K^\times$ and define the global zeta integral

$$Z(s,\chi,\Psi) := \int_{I_K} \Phi(t)(\chi|\cdot|^s)(t)d^*t.$$

The global theory in Tate's thesis gives meromorphic continuations for global zeta integrals and the functional equation

$$Z(s,\chi,\Phi) = Z(1-s,\chi^{-1},\widehat{\Phi}).$$

2 Induced representations

Let $B(F) \subset \mathrm{GL}_2(F)$ be the subgroup of upper triangular matrices. The modular character δ_B is

$$\delta_B \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = |a/d|.$$

For some quasi-characters χ_1, χ_2 of $\mathbb{F}^\times$. We define

$$\mathcal{B}(\chi_1,\chi_2) := \mathrm{Ind}_{B(F)}^G(\chi),$$

where $\chi \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \chi_1(a)\chi_2(d)$. Let $|\chi_i| = |\cdot|^{s_i}$. Let $w_0 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $n(x) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. We have the Whittaker functional

$$f \mapsto \int_F f(w_0 n(x))dx = \lim_{N\to\infty} \int_{|x|\le q^N} f(w_0 n(x))dx,$$

which gives a Whittaker model for all induced representations.

We define the intertwining operator

$$M(f)(g) := \int_F f(w_0 n(x) g)dx.$$

The integral is absolutely convergent when $\Re(s_1 - s_2) > 0$, and defines an intertwining operator from $\mathcal{B}(\chi_1, \chi_2)$ to $\mathcal{B}(\chi_2, \chi_1)$. For general case, we consider the family $\mathcal{B}(\chi_1, \chi_2, s) := \mathcal{B}(\chi_1|\cdot|^s, \chi_2|\cdot|^{-s})$ and have the meromorphic continuation of intertwining operators as follows: When $\chi_1\chi_2^{-1}$ is ramified,

$$M(f)(g) := \lim_{N\to\infty} \int_{|x|\leq q^N} f(w_0 n(x) g) dx.$$

When $\chi_1\chi_2^{-1}$ is unramified,

$$M(f)(g) = \lim_{N\to\infty} \left(\int_{|x|\leq q^N} f\left(w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) dx + \mathrm{Vol}(\mathfrak{o}^\times) \frac{f(g)(\chi_1\chi_2^{-1}|\cdot|^{2s})(\varpi)^{N+1}}{1 - (\chi_1\chi_2^{-1}|\cdot|^{2s})(\varpi)} \right).$$

The only simple poles occur when $\chi_1\chi_2^{-1}|\cdot|^{2s}$ is the trivial character.

We define the Godement sections as follows: Let $\Phi \in \mathcal{S}(F^2)$.

$$f_{\Phi,\chi,s}(g) := (\chi_1|\cdot|^{s+1/2})(\det(g)) \int_{F^*} \Phi((0,t)g)(\chi_1\chi_2^{-1}|\cdot|^{2s+1})(t) d^*t.$$

Proposition 2.

$$f_{\Phi,\chi,s} \in \mathcal{B}(\chi_1, \chi_2, s).$$

Moreover, every element in $\mathcal{B}(\chi_1, \chi_2, s)$ has the form $f_{\Phi,\chi,s}$.

We consider the symplectic Fourier transform on $\mathcal{S}(F^2)$:

$$\begin{aligned}\widehat{\Phi}(u,v) &:= \int_F \int_F \Phi(x,y)\psi\left(\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \right) dxdy \\ &= \int_F \int_F \Phi(x,y)\psi(xv - yu) dxdy.\end{aligned}$$

If $\Phi = \Phi_1 \otimes \Phi_2$,

$$\widehat{\Phi} = \widehat{\Phi_2} \otimes \Phi_1^\vee.$$

Hence $\widehat{\widehat{\Phi}} = \Phi$.

Proposition 3.

$$M(\chi, s)(f_{\Phi,\chi,s}) = \gamma(1 - 2s, \chi_1^{-1}\chi_2, \psi) f_{\widehat{\Phi},\chi^c,-s}.$$

We consider the Weil representation of $\mathrm{GL}_2(F)$ on $\mathcal{S}(F^2)$ with respect to a given nontrivial character ψ. The representation $(r_\psi, \mathcal{S}(F^2))$ is given by

$$r_\psi \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \Phi(x,y) = |a|\Phi(ax, ay),\ r_\psi \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \Phi(x,y) = \psi(bxy)\Phi(x,y),$$

$$r_\psi(w_0)\Phi(x,y) = \int_F \int_F \Phi(u,v)\psi(uy+vx)dudv,\ r_\psi \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \Phi(x,y) = \Phi(ax,y).$$

Lemma 1. Let $\widetilde{\Phi}$ be the partial Fourier transform

$$\widetilde{\Phi}(x,y) = \int_F \Phi(x,u)\psi(uy)du.$$

Then

$$\widetilde{r_\psi(g)(\Phi)} = \rho(g)\widetilde{\Phi}.$$

We define the Whittaker function $W_{\Phi,\chi,s}$ by

$$W_{\Phi,\chi,s}(g) = (\chi_1|\cdot|^{s+1/2})(\det(g)) \int_{F^*} (r_\psi(g)\Phi)(t,t^{-1})(\chi_1\chi_2^{-1}|\cdot|^{2s})(t)d^*t.$$

Proposition 4.

$$\int_F f_{\widetilde{\Phi},\chi,s} \left(w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) \psi(-x)dx = W_{\Phi,\chi,s}(g).$$

In other words, $W_{\Phi,\chi,s}(g)$ is the Whittaker function corresponding to $f_{\widetilde{\Phi},\chi,s}$ under the Whittaker functional

$$f \mapsto \int_F f \left(w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) \psi(-x)dx.$$

3 Fourier expansion, constant term, and meromorphic continuations

Let χ_1, χ_2 be characters and $f \in \mathcal{B}(\chi_1,\chi_2,s)$. We define

$$E(f,s)(g) := \int_{\gamma \in B(K)\backslash \mathrm{GL}_2(K)} f(\gamma g).$$

The summation converges absolutely when $\Re(s) > 1/2$ and admits meromorphic continuation on $\mathbb{C}$.

In this GL_2 case, we have Fourier expansion. Assume s is in the range of convergence and consider the function

$$x \mapsto E\left(f, s, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right).$$

on $\mathbb{A}_F/F$. Define

$$C_\alpha(g, f) = \int_{\mathbb{A}/F} E\left(f, s, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \psi(-\alpha x) dx.$$

Then we have

$$E\left(f, s, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \sum_{\alpha \in F} c_\alpha(g, f) \psi(\alpha x).$$

In particular,

$$E(f, s, g) = \sum_{\alpha \in F} c_\alpha(g, f).$$

A set of representatives of $B(F)\backslash \mathrm{GL}_2(F)$ is $\{I_2\} \cup \left\{ w_0 \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \right\}_{\lambda \in F}$. Hence

$$\begin{aligned} c_\alpha(g, f) &= \int_{\mathbb{A}/F} f\left(\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)\right) \psi(-\alpha x) dx + \int_{\mathbb{A}} f\left(w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \psi(-\alpha x) dx \\ &= \int_{\mathbb{A}/F} f(g) \psi(-\alpha x) dx + \int_{\mathbb{A}} f\left(w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \psi(-\alpha x) dx. \end{aligned}$$

The first term is nonzero iff $\alpha = 0$ and in that case we get $f(g)$. To summarize, we have

$$c_\alpha(g, f) = \begin{cases} \int_{\mathbb{A}} f\left(w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \psi(-\alpha x) dx, & \alpha \neq 0 \\ f(g) + M(f)(g) & \alpha = 0 \end{cases}.$$

The Fourier coefficients can be formulated in terms of the Whittaker model. Let

$$W_f(g) = \int_{\mathbb{A}} f\left(w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \psi(\alpha x) dx.$$

Then $C_\alpha(g,f) = W_f\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} g\right)$ The image of a flat section under the intertwining operator is not a flat section. In terms of Godement sections we have an elegant functional equation. We consider the Eisenstein series along a Godement section $f_{\chi,\Phi,s} \in \mathcal{B}(\chi_1,\chi_2,s)$ and denote by $E_\chi(\Phi,s,g)$ the associated Eisenstein series.

Theorem 1. $E_\chi(\Phi,s,g)$, defined on $\mathrm{Res}(s) > 1/2$, has meromorphic continuation on $\mathbb{C}$ with the functional equation

$$E_\chi(\Phi,s,g) = E_{\chi^c}(g,\widehat{\Phi},-s).$$

The constant term is $f_{\chi,\Phi,s} + f_{\chi^c,\widehat{\Phi},-s}$, and $E_\chi(\Phi,s,g) - (f_{\chi,\Phi,s} + f_{\chi^c,\widehat{\Phi},-s})$ has analytic continuation.

Furthermore, the function

$$E_\chi(\Phi,s,g) - S(\Phi,s,g)$$

where

$$\begin{aligned} S(\Phi,s,g) = &-(\chi_1|\cdot|^{s+1/2})(\det(g))\Phi(0,0)\int_{|t|\le 1}(\chi_1\chi_2^{-1}|\cdot|^{2s+1})(t)d^*t \\ &+(\chi_2|\cdot|^{1/2-s})(\det(g))\widehat{\Phi}(0,0)\int_{|t|\le 1}(\chi_1^{-1}\chi_2|\cdot|^{1-2s})(t)d^*t, \end{aligned}$$

is entire and bounded in every strip with finite width when F is a number field ($E_\chi(\Phi,s,g)$ is a rational function in q^{-s} when F is a function field). When $\chi_1\chi_2^{-1}$ is ramified, $S(\Phi,s,g) = 0$ and the Eisenstein series is entire.

The proof involves a two dimensional version of Poisson summation formula and is similar to the meromorphic continuation for global L-function in the GL_1-case.

The Fourier expansion is valid for all s where the Eisenstein series doesn't have a pole.

4 Local newform theory

Let F be a non-archimedean local field, (π, V) be a generic irreducible representation of $\mathrm{GL}_2(F)$ with central character ω and conductor $c(\pi)$. For every $m \in \mathbb{Z}_{\ge 0}$, we consider the group

$$\Gamma_0(\mathfrak{p}^m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathfrak{o}) \;\middle|\; c \in \mathfrak{p}^m \right\} \subset \mathrm{GL}_2(\mathfrak{o}).$$

Definition 1. The subspace of vectors of level m is defined as

$$M(\mathfrak{p}^m, \pi) = \left\{ v \in V \;\middle|\; \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathfrak{p}^m),\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} v = \omega(d) v \right\}.$$

for $m > 0$, and the subspace of spherical vectors for $m = 0$.

Theorem 2.

$$\dim_{\mathbb{C}}(M(\mathfrak{p}^m, \pi)) = \begin{cases} m - c(\pi) + 1, & m \geq c(\pi), \\ 0, & m < c(\pi). \end{cases}$$

To be explicit, let ψ be a nontrivial character of conductor $c(\psi) = \mathfrak{o}$. Then $M(\mathfrak{p}^m, \pi)$ consists of vectors $v \in V$ with $\Psi(I_2, s, v)$ being a polynomial in q^{-s} of degree $\leq m - c(\pi)$.

From the theorem we have that $\dim_{\mathbb{C}}(M(\mathfrak{p}^{c(\pi)}, \pi)) = 1$ and $c(\pi)$ is the least possible m with $M(\mathfrak{p}^m, \pi)$ nonzero. Nonzero elements in the space are called new vectors.

Definition 2. New vectors in the Whittaker model and the Kirillov model are called Whittaker newforms and Kirillov newforms respectively. We also define the normalized Whittaker newform W^0 and the normalized Kirillov newform φ^0 as follows: When $c(\psi) = \mathfrak{o}$, $\Phi(I_2, s, \varphi_v)$ are constants. We define the normalized Kirillov newform as the Kirillov newform φ^0 with $\varphi^0(1) = 1$. As ψ varies, we have canonical isomorphisms between Kirillov models. We define the normalized Kirillov newform in each Kirillov model as the image under the isomorphism. The normalized Whittaker newform is the Whittaker newform corresponding to the normalized Kirillov newform.

Proposition 5. When $m \geq c(\pi)$, every vector in $\Gamma_0(\mathfrak{p}^m)$ has the form

$$P \begin{pmatrix} \pi^{-1} & 0 \\ 0 & 1 \end{pmatrix} v^0$$

for some polynomial P of degree $\leq m - c(\pi)$ and a fixed new vector v^0.

We assume that $c(\psi) = \mathfrak{o}$. Then the normalized Kirillov newform is the function φ^0 that satisfies $\varphi^0(ux) = \varphi^0(x)$ for all $u \in \mathfrak{u}$ and $\Psi(I_2, s, \varphi^0) = L(s, \pi)$. For the normalized Whittaker newforms, we still need to compute $w_0\varphi^0$ and use the functional equation. The normalized Kirillov newform for induced representations are given as follows: Let $\pi = \mathcal{B}(\chi_1, \chi_2)$. We have 3 cases:

1. χ_1, χ_2 are ramified. We have $\varphi^0 = 1_{\mathfrak{u}}$ and $w_0\varphi^0 = \epsilon(1/2, \pi, \psi)1_{\pi^{-c(\pi)}\mathfrak{u}}\omega_0$.

2. χ_1 is unramified while χ_2 is ramified. We have $\varphi^0 = 1_{\mathfrak{o}}\chi_1|\cdot|^{1/2}$.

$$w_0\varphi^0 = \epsilon(1/2, \chi_2, \psi)(\chi_1^{-1}|\cdot|^{1/2})(\pi)^{c(\chi_2)}1_{\pi^{-c(\chi_2)}\mathfrak{o}}\chi_2|\cdot|^{1/2}.$$

3. χ_1, χ_2 are unramified.

$$\varphi^0 = w_0\varphi^0 = 1_{\mathfrak{o}}|\cdot|^{1/2}\frac{\chi_1^{m+1} - \chi_2^{m+1}}{\chi_1(\pi) - \chi_2(\pi)}.$$

5 Classical Eisenstein series

Here we compute some Eisenstein series in terms of automorphic forms that recovers classical Eisenstein series. I don't know who wrote down the choice of local sections first. I learn the choice at ∞ from an unpublished note by Ming-Lun Hsieh and Shih-Yu Chen.

Let χ_1, χ_2 be two Dirichlet characters, algebraically identified with Hecke characters of $I_{\mathbb{Q}}/\,\mathbb{Q}^*$, with conductor

$$c(\chi_i) = \prod_p p^{c_{i,p}}.$$

. We consider the Eisenstein series associated with the normalized newform of $\mathcal{B}(\chi_1, \chi_2, s)$. Here we choose ψ so that $\psi_\infty(x) = e^{2\pi i x}$.

1. χ_1, χ_2 are unramified at p. Consider

$$\Phi(x, y) = 1_{\mathbb{Z}_p}(x)1_{\mathbb{Z}_p}(y).$$

We have $\widetilde{\Phi} = \Phi$. We have the Godement section

$$f_{\widetilde{\Phi},s}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \int_{\mathbb{Q}_p^*} 1_{\mathbb{Z}_p}(tc)1_{\mathbb{Z}_p}(td)(\chi_1\chi_2^{-1}|\cdot|^{2s+1})(t)d^*t$$
$$= \int_{\mathbb{Z}_p} (\chi_1\chi_2^{-1}|\cdot|^{2s+1})(t)d^*t = L_p(2s+1, \chi_1\chi_2^{-1})$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_p$. The corresponding element in the Whittaker model is

$$\begin{aligned} W_{\Phi,s}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} =& (\chi_1|\cdot|^{s+1/2})(a)\int_{\mathbb{Q}_p^*} 1_{\mathbb{Z}_p}(at)1_{\mathbb{Z}_p}(t^{-1})(\chi_1\chi_2^{-1}|\cdot|^{2s})(t)d^*t \\ =& 1_{\mathbb{Z}_p}(a)(\chi_1|\cdot|^{s+1/2})(a)\sum_{\ell=-v_p(a)}^{0}(\chi_1\chi_2^{-1}|\cdot|^{2s})(p^m) \\ =& 1_{\mathbb{Z}_p}(a)p^{-(s+1/2)v_p(a)}\sum_{\ell=0}^{v_p(a)}\chi_1(p)^{v_p(a)-\ell}\chi_2(p)^{\ell}p^{2s\ell}. \end{aligned}$$

2. χ_1 is unramified while χ_2 is ramified at p. Consider

$$\Phi(x,y) = \frac{1_{\mathbb{Z}_p}(x)1_{1+p^{c_{2,p}}\mathbb{Z}_p}(y)}{\mathrm{Vol}_\times(1+p^{c_{2,p}}\mathbb{Z}_p)}.$$

We have

$$\begin{aligned} \widetilde{\Phi}(x,y) =& 1_{\mathbb{Z}_p}(x)1_{p^{-c_{2,p}}\mathbb{Z}_p}(y)\frac{\mathrm{Vol}_+(1+p^{c_{2,p}}\mathbb{Z}_p)}{\psi}{}_p(y)\mathrm{Vol}_\times(1+p^{c_{2,p}}\mathbb{Z}_p) \\ =& 1_{\mathbb{Z}_p}(x)1_{p^{-c_{2,p}}\mathbb{Z}_p}(y)\psi_p(y)\zeta_p(1)^{-1}. \end{aligned}$$

The Godement section is

$$f_{\widetilde{\Phi},s}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \zeta_p(1)^{-1}\int_{\mathbb{Q}_p^*} 1_{\mathbb{Z}_p}(tc)1_{p^{-c_{2,p}}\mathbb{Z}_p}(td)\psi_p(td)(\chi_1\chi_2^{-1}|\cdot|^{2s+1})(t)d^*t$$
$$= 1_{p^{c_{2,p}}\mathbb{Z}_p}(c)1_{\mathfrak{u}_p}(d)(\chi_1|\cdot|^{2s+1})(p^{-c_{2,p}})\chi_2(d)\epsilon(0,\chi_2^{-1},\psi_p)$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_p$. The corresponding element in the Whittaker model is

$$\begin{aligned} W_{\Phi,s}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} =& (\chi_1|\cdot|^{s+1/2})(a)\int_{\mathbb{Q}_p^*}\frac{1_{\mathbb{Z}_p}(at)1_{1+p^{c_{2,p}}\mathbb{Z}_p}(t^{-1})}{\mathrm{Vol}_\times(1+p^{c_{2,p}}\mathbb{Z}_p)}(\chi_1\chi_2^{-1}|\cdot|^{2s})(t)d^*t \\ =& 1_{\mathbb{Z}_p}(a)p^{-(s+1/2)v_p(a)}\chi_1(a). \end{aligned}$$

3. χ_1 is ramified while χ_2 is unramified at p. Consider

$$\Phi(x,y) = 1_{\mathfrak{u}_p}(x)\chi_1(x)^{-1}1_{\mathbb{Z}_p}(y).$$

We have $\widetilde{\Phi} = \Phi$. The Godement section is

$$f_{\widetilde{\Phi},s}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi_1(ad-bc)\int_{\mathbb{Q}_p^*} 1_{\mathfrak{u}_p}(tc)\chi_1(tc)^{-1}1_{\mathbb{Z}_p}(td)(\chi_1\chi_2^{-1}|\cdot|^{2s+1})(t)d^*t$$
$$=1_{\mathfrak{u}_p}(c)\chi_1\left(\frac{ad-bc}{c}\right)$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_p$. The corresponding element in the Whittaker model is

$$\begin{aligned} W_{\Phi,s}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} =&(\chi_1|\cdot|^{s+1/2})(a)\int_{\mathbb{Q}_p^*} 1_{\mathfrak{u}_p}(at)\chi_1^{-1}(at)1_{\mathbb{Z}_p}(t^{-1})(\chi_1\chi_2^{-1}|\cdot|^{2s})(t)d^*t \\ =&1_{\mathbb{Z}_p}(a)p^{-(s+1/2)v_p(a)}(\chi_2(p)p^{2s})^{v_p(a)}. \end{aligned}$$

4. χ_1, χ_2 are ramified at p. Consider

$$\Phi(x,y) = \frac{1_{\mathfrak{u}_p}(x)\chi_1(x)^{-1}1_{1+p^{c_{2,p}}\mathbb{Z}_p}(y)}{\mathrm{Vol}_\times(1+p^{c_{2,p}}\mathbb{Z}_p)}.$$

We have

$$\widetilde{\Phi}(x,y) = 1_{\mathfrak{u}_p}(x)\chi_1(x)^{-1}1_{p^{-c_{2,p}}\mathbb{Z}_p}(y)\psi_p(y)\zeta_p(1)^{-1}.$$

The Godement section is

$$\begin{aligned} &f_{\widetilde{\Phi},s}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ =&\zeta_p(1)^{-1}\chi_1\left(\frac{ad-bc}{c}\right)\int_{\mathbb{Q}_p^*} 1_{\mathfrak{u}_p}(tc)1_{p^{-c_{2,p}}\mathbb{Z}_p}(td)\psi_p(td)(\chi_1\chi_2^{-1}|\cdot|^{2s+1})(t)d^*t \\ =&1_{p^{c_{2,p}}\mathfrak{u}_p}(c)1_{\mathfrak{u}_p}(d)\chi_1\left(\frac{ad-bc}{c}\right)(|\cdot|^{2s+1})(p^{-c_{2,p}})\chi_2(d)\epsilon(0,\chi_2^{-1},\psi_p) \end{aligned}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_p$. The corresponding element in the Whittaker model is

$$\begin{aligned}
&W_{\Phi,s}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \\
=&(\chi_1|\cdot|^{s+1/2})(a)\int_{\mathbb{Q}_p^*}\frac{1_{\mathfrak{u}_p}(at)\chi_1(at)^{-1}1_{1+p^{c_{2,p}}\mathbb{Z}_p}(t^{-1})}{\mathrm{Vol}_\times(1+p^{c_{2,p}}\mathbb{Z}_p)}(\chi_1\chi_2^{-1}|\cdot|^{2s})(t)d^*t \\
=&1_{\mathfrak{u}_p}(a).
\end{aligned}$$

5. $p=\infty$. Let $k\in\mathbb{N}$ such that $\chi_1\chi_2^{-1}(-1)=(-1)^k$ and define

$$\widetilde{\Phi}_k(x,y)=2^{-k}(x+iy)^k e^{-\pi(x^2+y^2)}.$$

Then we have

$$\begin{aligned}
f_{\widetilde{\Phi}_k,s}(\kappa_\theta)=&2^{-k}i^k e^{ik\theta}\int_{\mathbb{R}^*}t^k\,\mathrm{sgn}(t)^k|t|^{1+2s}d^*t \\
=&2^{-k}i^k e^{ik\theta}\Gamma_{\mathbb{R}}(2s+k+1).
\end{aligned}$$

The corresponding element in the Whittaker model is

$$\begin{aligned}
&W_{\Phi_k,s}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}=(\chi_1|\cdot|^{s+1/2})(a)\int_{\mathbb{R}}\int_{\mathbb{R}^*}\widetilde{\Phi}_k(ta,tx)(\chi_1\chi_2^{-1}|\cdot|)^{2s+1}(t)d^*tdx \\
=&(\chi_1|\cdot|)^{s+1/2}(a)2^{-k}\int_{\mathbb{R}}(a+ix)^k(a^2+x^2)^{-[s+\frac{k+1}{2}]}e^{-2\pi ix}dx.
\end{aligned}$$

The integral exists only when $\Re(s)>>0$, but it admits meromorphic continuation coming from the classical result of meromorphic continuation for the integral

$$\int_{-\infty}^{\infty}(x+iy)^{-\alpha}(x-iy)^{-\beta}e^{-2\pi ix}dx.$$

If $s=\dfrac{k-1}{2}$, we get

$$\begin{aligned}
&W_{\Phi_k,s}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}=(\chi_1|\cdot|^{k/2})(a)2^{-k}i^k\Gamma_{\mathbb{R}}(2k)\int_{\mathbb{R}}\frac{e^{-2\pi ix}}{(x+ia)^k}dx \\
=&1_{\mathbb{R}_{>0}}(a)a^{k/2}(-2\pi i)^{-k}\Gamma(k)(-2\pi i)e^{-2\pi a}\frac{(-2\pi i)^{k-1}}{(k-1)!}=1_{\mathbb{R}_{>0}}(a)a^{k/2}e^{-2\pi a}.
\end{aligned}$$

If $s = \dfrac{1-k}{2}$, we get

$$W_{\Phi_k,s}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = (\chi_1|\cdot|^{1-k/2})(a)2^{-k}\pi^{-1}i\int_{\mathbb{R}} \frac{e^{-2\pi ix}(a+ix)^{k-1}}{(x+ia)}dx$$

$$=1_{\mathbb{R}_{>0}}(a)a^{1-k/2}2^{-k}\pi^{-1}i(-2\pi i)e^{-2\pi a}(2a)^{k-1} = 1_{\mathbb{R}_{>0}}(a)a^{k/2}e^{-2\pi a}.$$

We consider the Fourier expansion on $\mathcal{H}$. Take $g = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}_\infty$. Then

$$W_{\chi,\Phi,s}\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right) = e^{2\pi i\alpha(x+iy)}(\alpha y)^{k/2}\prod_{p<\infty} W_{\chi,\Phi_p,s}\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}_\infty,$$

which is nonzero only if $\alpha \in \mathbb{N}$. The nth Fourier expansion is

$$y^{k/2}q^n\sum_{d|n}\chi_1(n/d)\chi_2(d)d^{k-1}$$

if $s = \dfrac{k-1}{2}$, and

$$y^{k/2}q^n\sum_{d|n}\chi_2(n/d)\chi_1(d)d^{k-1}$$

if $s = \dfrac{1-k}{2}$.

We also compute the constant term. We first note that $f_{\chi,\widetilde{\Phi}}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}_\infty = 0$ if $\chi_1 \neq 1$. Suppose $\chi_1 = 1$, $\chi_2(-1) = (-1)^k$. Let $\Phi'_p(t) = \Phi_p(0,t)$. Then

$$f_{\chi,\widetilde{\Phi}}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}_\infty = y^{s+1/2}\prod_p\int_{\mathbb{Q}_p^*}\widehat{\Phi'_p}(t)\chi_2^{-1}|\cdot|^{2s+1}d^*t$$

$$=y^{s+1/2}\prod_p Z_p(2s+1,\chi_2^{-1},\widehat{\Phi'_p}) = y^{s+1/2}\prod_p Z_p(-2s,\chi_2,\Phi'_p).$$

If $p < \infty$, by construction, $Z_p(-2s,\chi_2,\Phi'_p) = L(-2s,\chi_2)$. We consider $p = \infty$. Consider the functional equation

$$Z_\infty(-2s,\chi_2,\Phi'_\infty) = \frac{L(-2s,\chi_2)}{L(2s+1,\chi)}\epsilon^{-1}(-2s,\chi_2,\phi_\infty)Z_\infty(2s+1,\chi_2^{-1},\widehat{\Phi'_\infty}).$$

Suppose χ_2 is even, then $\epsilon = 1$. We get

$$Z_\infty(-2s,\chi_2,\Phi'_\infty) = 2^{-k}i^k\Gamma_{\mathbb{R}}(2s+k+1)\frac{\Gamma_{\mathbb{R}}(-2s)}{\Gamma_{\mathbb{R}}(2s+1)}$$

$$=2^{1+2s-k}i^k\pi^{2s}\Gamma_{\mathbb{R}}(2s+k+1)\Gamma(-2s)\cos(s\pi).$$

When $s = \dfrac{k-1}{2}$,

$$Z_\infty(-2s, \chi_2, \Phi'_\infty) = i^k \lim_{s \to \frac{k-1}{2}} \frac{\cos(\pi s)}{\sin(-2s\pi)} = \frac{i^k}{2\cos(k\pi/2))} = \frac{1}{2}.$$

When $s = \dfrac{1-k}{2}$,

$$Z_\infty(-2s, \chi_2, \Phi'_\infty) = 2^{2-2k} i^k \pi^{1-k} \Gamma_{\mathbb{R}}(2)\Gamma(k-1)\cos((1-k)\pi/2) = 0.$$

Suppose χ_2 is odd, then $\epsilon = i$. We get

$$Z_\infty(-2s, \chi_2, \Phi'_\infty) = 2^{-k} i^{k+1} \Gamma_{\mathbb{R}}(2s+k+1) \frac{\Gamma_{\mathbb{R}}(-2s+1)}{\Gamma_{\mathbb{R}}(2s+2)}$$
$$= 2^{1+2s-k} i^{k+1} \pi^{2s} \Gamma_{\mathbb{R}}(2s+k+1)\Gamma(-2s)\sin(s\pi).$$

When $s = \dfrac{k-1}{2}$,

$$Z_\infty(-2s, \chi_2, \Phi'_\infty) = i^{k-1} \lim_{s \to \frac{k-1}{2}} \frac{\sin(s\pi)}{\sin(2s\pi)} = \frac{i^{k-1}}{2\cos((k-1)\pi/2)} = \frac{1}{2}.$$

When $s = \dfrac{1-k}{2}$ and $k > 1$,

$$Z_\infty(-2s, \chi_2, \Phi'_\infty) = 2^{2-2k} i^{k+1} \pi^{1-k} \Gamma_{\mathbb{R}}(2)\Gamma(k-1)\sin((1-k)\pi/2) = 0.$$

We note that $\widehat{\Phi}$ is our choice of Φ when we interchange χ_1 and χ_2 and get the following conclusion: When $k > 1$, the constant term is

$$y^{k/2}\delta(\chi_1)\frac{L(1-k, \chi_2)}{2}$$

if $s = \dfrac{k-1}{2}$, and

$$y^{k/2}\delta(\chi_2)\frac{L(1-k, \chi_1)}{2}$$

if $s = \dfrac{1-k}{2}$. When $k = 1$ and $s = 0$, the constant term is

$$y^{k/2}\delta(\chi_1)\frac{L(1-k, \chi_2)}{2} + y^{k/2}\delta(\chi_2)\frac{L(1-k, \chi_1)}{2}.$$

These Eisenstein series we obtain are classical Eisenstein series $E_k^{\chi_1,\chi_2}(q)$.

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