Notes on Deligne-Ribet's p-adic L-function

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October 31st, 2022

1 Strict Ray Class Group

Let K be a totally real field of degree r over $\mathbb{Q}$. We assume $K \neq \mathbb{Q}$. Let $\mathcal{O} = \mathcal{O}_K$ be the ring of integers and $\mathcal{D}$ be the different of $\mathcal{O}$.

Let $N : K \to \mathbb{Q}$ be the norm map. Given any $\mathbb{Q}$-algebra R, it extends to a map $N : K \otimes R \to R$. In particular, for the finite adèles $R = \hat{\mathbb{Q}} = \mathbb{A}_{\mathbb{Q}}^{(\infty)}$, we have $N : \hat{K} = \mathbb{A}_K^{(\infty)} \to \hat{\mathbb{Q}}$.

We will describe the strict ray class group of K in terms of (fractional) ideals of K. Let I_0 be the monoid of (fractional) ideals of K. It contains the submonoid A_0 of integral ideals of K.

Definition. Let $\mathfrak{a}, \mathfrak{b} \in I_0$ and $\mathfrak{f} \in A_0$. We say $\mathfrak{a}$ is equivalent to $\mathfrak{b}$ modulo $\mathfrak{f}$, denoted $\mathfrak{a} \sim_{\mathfrak{f}} \mathfrak{b}$, if $\mathfrak{a}\mathfrak{b}^{-1} = (\alpha)$ for some

$$\alpha \in 1 + \mathfrak{f}\mathfrak{b}^{-1} \ , \ \alpha \gg 0$$

where this second part means that α is totally positive in K. This equivalence relation preserves integrality

Definition. Let $\mathbb{A}_{\mathfrak{f}} = A_0/{\sim_{\mathfrak{f}}} \supset \mathbb{A}_{\mathfrak{f}}^{\times} = G_{\mathfrak{f}} =$ strict ray class group of K of conductor $\mathfrak{f}$. It is contained in $I_{\mathfrak{f}} = I_0/\sim_{\mathfrak{f}}$.

By taking different $\mathfrak{f}$'s, we obtain an inverse system and we can take the inverse limits to get

$$G = \varprojlim_{\mathfrak{f}} G \subset A = \varprojlim_{\mathfrak{f}} A_{\mathfrak{f}} \subset I = \varprojlim_{\mathfrak{f}} I_{\mathfrak{f}}$$

In the case $K = \mathbb{Q}$, we would recover $\hat{\mathbb{Z}}^{\times} \subset \hat{\mathbb{Z}} \subset \hat{\mathbb{Q}}$. Most of the rest of the talk could be phrased in term of the (absolute) strict ray class group G of K but the original paper uses I, and so will we.

Proposition 1. *The norm map $N : I_0 \to \mathbb{N}$ (different norm map than above, obviously) extends to $N : I \to \hat{\mathbb{Q}}$ s.t. $N(\mathfrak{a}\mathfrak{b}) = N(\mathfrak{a})N(\mathfrak{b})$. We have $N(A) \subset \hat{\mathbb{Z}}$ and $N(G) \subset \hat{\mathbb{Z}}^{\times}$.*

We want to compare I and $\hat{K}$. Let $\alpha \in \hat{K}$. For any $\mathfrak{f} \in A_0$, choose any $\alpha_{\mathfrak{f}} \in K$ such that

$$\alpha_f \gg 0 \text{ and } \alpha \cong \alpha_f \mod \mathfrak{f}$$

such that $(\alpha_{\mathfrak{f}})_{\mathfrak{f}}$ is a "compatible sequence" with respect to the the transition maps $K/\mathfrak{f}' \to K/\mathfrak{f}$ whenever $\mathfrak{f}' \subset \mathfrak{f}$. I think that to interpret this condition, we need to think of $K/\mathfrak{f}$ as an $\mathcal{O}$-module, but not quite sure. It's not terribly important. The idea should be clear.

Definition. Let $i : \hat{K} \to I$ as $\alpha \mapsto ((\alpha_{\mathfrak{f}}))_{\mathfrak{f}}$, where $(\alpha_{\mathfrak{f}}) \in I_{\mathfrak{f}}$ is the (fractional) ideal generated by α_f. One can show that this is well-defined and $N((i(\alpha)) = N(\alpha)$.

Remark 1. If $\mathfrak{a} = (\alpha)$ is the fractional ideal obtained by looking at the $\mathfrak{p}$-valuation of α for every prime $\mathfrak{p}$ of K, we do not have $N(\mathfrak{a}) = N(\alpha)$ in general.

We will care about the product $(\mathfrak{a}\cdot\alpha) = \mathfrak{a}\cdot\alpha := \mathfrak{a}\cdot i(\alpha) \in I$ for $\mathfrak{a} \in I$, $\alpha \in \hat{K}$.

Remark 2. Since 0 is not totally positive, $i(0)$ just corresponds to a compatible sequence of principal ideals generated by positive elements of each $\mathfrak{f}$. In particular, it is not 0. So very confusingly, $\alpha \cdot 0$ is not 0.

Proposition 2. *Given $x \in \hat{K}^\times$, $(x)\cdot x^{-1} \in G$. In fact, the map $j : \hat{K}^\times \to G$ as $x \mapsto (x)\cdot x^{-1}$ is just the Artin map.*

2 L-function and Analytic Properties

Let $\epsilon : I \to \mathbb{C}$ be a Schwartz function (i.e. locally constant and compact support).

Definition. $L(s,\epsilon) = \sum_{\mathfrak{a}\in I_0} \epsilon(\mathfrak{a}) N\mathfrak{a}^{-s}$. In a lot of cases, we can restrict our attention to ϵ's supported on A_0 so this infinite sum would only run over integral ideals.

Remark 3.

1. ϵ is not necessarily a character of I.
2. $L(s,-)$ is linear in ϵ
3. The above converges for $\mathrm{Re}(s)$ large enough. By writing it as a linear combination of partial zeta function, we can directly obtain its meromorphic continuation on $\mathbb{C}$.

We briefly talk about its functional equation. To do so, we need the notion of parity.

Definition. Let σ be a $\mathbb{R}$-place of K. Let $c_\sigma \in G$ be complex conjugation at that place (i.e. an "Archimedean Frobenius"). Let $\Sigma^\pm$ be the subgroup of G generated by the c_σ's. It has order 2^r where recall, $r = [K:\mathbb{Q}]$.

We say that ϵ has parity $(a_\sigma)_\sigma$, for some sequence of integers $a_\sigma = 0, 1$, if

$$\epsilon(\sigma g) = (-1)^{a_\sigma}\epsilon(g), \quad \forall g \in G, \sigma$$

We say that ϵ is even (resp. odd) if $a_\sigma = 0$ (resp. 1) for all σ.

Remark 4. An arbitrary ϵ does not have to have any parity in general. However, we can always decompose it according to the eigenspace of the natural action of $\Sigma^\pm$ on the space of Schwartz functions on I. In other words, we can write

$$\epsilon = \sum_S \epsilon_S$$

as the sum runs over all possible parity S (there are 2^r of them) and ϵ_S has parity S. In particular, for $S = (0)_\sigma$, we have

$$\epsilon^+(x) = \epsilon_{(0)}(x) = \frac{1}{2^{-r}} \sum_{c\in\Sigma^\pm} \epsilon(cx)$$

and for $S = (1)_\sigma$, we have

$$\epsilon^-(x) = \epsilon_{(1)}(x) = \frac{1}{2^{-r}} \sum_{c\in\Sigma^\pm} N(c)\epsilon(cx)$$

We can think of these the "even" and "odd" part of ϵ. I think these formulae are easy to generalize to obtain ϵ_S for a general parity S. Note that $N(c_\sigma) = -1$.

We also need the notion of the "Fourier transform" $T\epsilon : I \to \mathbb{C}$ of ϵ but we won't give the actual expression of this new Schwartz function (cf. beginning of Section 3 of Deligne-Ribet's paper).

Finally, let $\Gamma_\mathbb{R}(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)$.

Theorem 1. *Let $\epsilon : I \to \mathbb{C}$ with some parity $(a_\sigma)_\sigma$. Let $A = \sum a_\sigma$ and $\gamma(s) = \prod_\sigma \Gamma_\mathbb{R}(s + a_\sigma$. Let $\Lambda(s,\epsilon) = \gamma(s)L(s,\epsilon)$. Then,*

$$\Lambda(s,\epsilon) = i^A \Lambda(1-s, T\epsilon)$$

Moreover, Lambda(s,ϵ) is holomorphic on $\mathbb{C}$, except if ϵ is even. In that case, we have at worst simple poles at $s = 0$ and 1.

By looking at the poles of $\gamma(s)$ coming from the simple poles of $\Gamma(s)$ at non-positive integers, we get the following corollary about trivial zeros of this L-function.

Corollary 1. *Let $\epsilon : I \to \mathbb{C}$ be of some parity. Then,*

1. *If ϵ is not even, $L(1-k,\epsilon) = 0$ for all even $k \geq 1$.*
2. *If ϵ is not odd, $L(1-k,\epsilon) = 0$ for all odd $k \geq 1$.*

Therefore, for a general Schwartz function $\epsilon : I \to \mathbb{C}$, $L(1-k,\epsilon) = L(1-k,\epsilon^{\pm})$ (depending on whether k is even or odd).

Remark 5. This theorem is false for $K = \mathbb{Q}$ and $k = 1$.

3 Cusps and q-Expansions

We fix a polarization ideal $\mathfrak{c}$ of I_0, $\alpha \in \hat{K}^\times$ and $\mathfrak{f} \in A_0$. Take $\mathfrak{a} = (\alpha) \in I_0$ and $\mathfrak{b} = \mathfrak{a}\mathfrak{c}^{-1}$. Then, we have

$$\mathfrak{a}^{-1} \otimes \mathbb{Q} = \mathcal{O} \otimes \mathbb{Q}$$

in $K \otimes \mathbb{Q}$ and this gives a canonical isomorphism $j_{can} : \mathfrak{a}^{-1} \otimes R \xrightarrow{\sim} \mathcal{O} \otimes R$ for any $\mathbb{Q}$-algebra R.

Furthermore, let $\epsilon : \mathfrak{f}^{-1}\mathcal{O}/\mathcal{O} \to \mathfrak{f}^{-1}\mathfrak{a}^{-1}/\mathfrak{a}^{-1}$ be the isomorphism obtained from

$$\mathfrak{f}^{-1}\mathcal{O}/\mathcal{O} \to \mathfrak{f}^{-1}\hat{\mathcal{O}}/\hat{\mathcal{O}}^{-1} \xrightarrow{\times\alpha^{-1}} \mathfrak{f}^{-1}\hat{\mathfrak{a}}^{-1}/\hat{\mathfrak{a}}^{-1} \to \mathfrak{f}^{-1}\mathfrak{a}^{-1}/\mathfrak{a}^{-1}$$

where the hat's $\hat{}$ denote adelic closure. This information $(\mathfrak{a},\mathfrak{b},\epsilon,j)$ (all determined by α) is enough to construct an Hilbert-Blumenthal abelian variety $X_{\mathfrak{a},\mathfrak{b}}$ over R with a $\Gamma_\infty(\mathfrak{f})$-structure (coming from ϵ) and a basis vector $\omega_{can} = \omega(j_{can})$ of $\underline{\omega}_{X_{\mathfrak{a},\mathfrak{b}}}$. This is called a Tate variety, generalizing the "usual" Tate curve.

Given an Hilbert modular form $F \in M_k(\Gamma_\infty(\mathfrak{f}), R)$, its q-expansion at the cusp α is its evaluation at $X_{\mathfrak{a},\mathfrak{b}}$. Denote it by F_α. To obtain a p-adic q-expansion, we need a p-adic version of this Tate variety, namely a $\Gamma_\infty(p^{-\infty}\mathfrak{f})$-structure.

To do this, we take the isomorphism $\epsilon_n : p^{-n}\mathfrak{f}^{-1}\mathcal{O}/\mathcal{O} \to p^{-n}\mathfrak{f}^{-1}\mathfrak{a}^{-1}/\mathfrak{a}^{-1}$ obtained by multiplication by α^-1 as above (but with a division by p^n). Then, a similar argument (which we don't write out), the sequence $i(\epsilon) = (\epsilon_n)_n$ yields a $\Gamma_\infty(p^{-\infty}\mathfrak{f})$-structure of $X_{\mathfrak{a},\mathfrak{b}}$.

Note that ϵ_0 is just the same as ϵ above and $i(\epsilon)$ as a whole gives us another basic vector ω of $\underline{\omega}_{X_{\mathfrak{a},\mathfrak{b}}}$. Indeed, tensoring ϵ_n with $\mathfrak{f}$ over $\mathcal{O}$, we get

$$p^{-n}\mathcal{O}/\mathcal{O} \xrightarrow{\sim} p^{-n}\mathfrak{a}^{-1}/\mathfrak{a}^{-1} \Rightarrow \mathbb{Z}_p \otimes \mathcal{O} \xrightarrow{\sim} \mathbb{Z}_p \otimes \mathfrak{a}^{-1}$$

and inverting the latter gives us an isomorphism $j : \mathfrak{a}^{-1} \otimes R \to \mathcal{O} \otimes R$ for any p-adic ring R. Note that j is simply α_p times j_{can}. So $\omega := \omega(j) = \alpha_p \omega_{can}$.

Suppose R is a flat $\mathbb{Z}_p$-algebra and let $E = R \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Let $\pi_k : M_k(\Gamma_\infty(\mathfrak{f}), E) \to V(\Gamma_\infty(\mathfrak{f}), E)$ be the "p-adification" of classical Hilbert modular forms. The p-adic q-expansion of F at α is the evaluation of $\pi_k F$ at $(X_{\mathfrak{a},\mathfrak{b}}, i(\epsilon))$, which is the same as the evaluation of F at $(X_{\mathfrak{a},\mathfrak{b}}, \epsilon, j)$. Note that we use j here instead of j_{can}.

Then, the relation between ω and ω_{can} together with the definition of "weight k" give us the following.

Proposition 3. *Let $F \in M_k(\Gamma_\infty(\mathfrak{f}), E)$. Then, the p-adic q-expansion of F at α is given by*

$$N\alpha_p^{-k} F_\alpha$$

Theorem 2 (q-expansion principle)**.** *Let $F \in V(\Gamma_\infty(\mathfrak{f}), E$ and $\alpha \in \hat{K}^\times$. If the coefficients fo the p-adic q-expansion of F at α are all in R, then*

$$F \in^V (\Gamma_\infty(\mathfrak{f}), R)$$

This follows from the irreducibility of the Igusa tower over the ordinary locus of the Hilbert moduli space.

Corollary 2. *Suppose that $F \in V(\Gamma_\infty(\mathfrak{f}), E)$ has a p-adic q-expansion at α given by*

$$F_\alpha = c_0 + \sum_{\mu \in \mathfrak{ab}, \mu \gg 0} c_\mu q^\mu$$

such that $c_\mu \in R$ for all μ (but not necessarily c_0). Let $\beta \in \hat{K}^\times$ be another cusp and let d_0, d_μ be the coefficients of the q-expansion of F at β.

Then, $d_0 - c_0 \in R$ and all $d_\mu \in R$.

Proof. Apply the q-expansion principal to $F - c_0$, where we see c_0 as a p-adic modular form induced from the constant weight 0 modular form c_0. ☐

4 Eisenstein Series

Let β be any fractional ideal of K. We construct an Eisenstein series $G_{k,\epsilon}$ in $M_k(\Gamma_\infty(\mathfrak{f}), \mathbb{C})$ where the polarization module is $\mathfrak{c} = \beta^{-1}$.

Definition. Given a Schwartz function $\epsilon : I \to \mathbb{C}$, we define its *modification* with respect to β to be $\tilde{\epsilon} : I \to \mathbb{C}$

$$\tilde{\epsilon}(x) := \begin{cases} 0, & \text{if } x \not\subset \beta \\ \epsilon(x^{-1} \cdot 0), & \text{if } x \subset \beta \end{cases}$$

where $x^{-1} \cdot 0 \in I$ is the element defined at the end of Section 1 in Remark 2.

For $c \in G$, let $\tilde{\epsilon}_c := (\tilde{\epsilon})_c$ be its twist by c.

Theorem 3. *Let $k \geq 1$, $\epsilon : I \to \mathbb{C}$ as above such that ϵ has parity $(-1)^k$, i.e. has the same parity as k. Suppose that ϵ is supporte on A and defined modulo $\mathfrak{f}$. Then, there exists an Eisenstein series*

$$G_{k,\epsilon} \in M_k(\Gamma_\infty(\mathfrak{f}), \mathbb{C})$$

whose q-expansion at the cusp $\alpha \in \hat{K}^\times$ is

1. *If $k > 1$:*

$$N(\mathfrak{a})^k \left[2^{-r} L(1-k, \epsilon_c) + \sum_{\mu \gg 0} \left(\sum_{x \subset \beta \mathfrak{a}^2} \epsilon(\mu x^{-1}) N(\mu x^{-1})^{k-1} \right) q^\mu \right]$$

2. *If $k = 1$:*

$$N(\mathfrak{a}) \left[2^{-r} L(0, \epsilon_c + \tilde{\epsilon}_c) + \sum_{\mu \gg 0} \left(\sum_{x \subset \beta \mathfrak{a}^2} \epsilon(\mu x^{-1}) \right) q^\mu \right]$$

where $\mathfrak{a} = (\alpha)$ *and* $c = \mathfrak{a} \cdot \alpha^{-1}$.

A rationality result of Siegel tells us that if ϵ actually takes values in $\mathbb{Q}$, then $L(1-k,\epsilon) \in \mathbb{Q}$ for all integers k. By linearity in the second variable, the same is true once changing $\mathbb{Q}$ for $\mathbb{Q}_p$ or $\hat{\mathbb{Q}}$. Shortly, we will assume that ϵ "almost" takes $\mathbb{Z}_p$-values (or actually $\hat{\mathbb{Z}}$-values), i.e. that the expression involving ϵ appearing in the non-constant terms of the q-expansion above take $\hat{\mathbb{Z}}$-values. Then, after taking care of the small difference between p-adic q-expansion and the usual one, we will conclude that differences between constant terms, i.e. difference between certain L-values, are also $\hat{\mathbb{Z}}$-rational.

5 Deligne-Ribet p-adic L-function

Let's make the final remark of the last section more precise. Let $\alpha \in \hat{K}^\times$ and $c = (\alpha) \cdot \alpha^{-1} = \mathfrak{a} \cdot \alpha^{-1} \in G$. Then, we have

$$N(c) = N(\mathfrak{a}) \cdot N(\alpha)^{-1}$$

Note that this is not equal to 1 in general (but it is if $\alpha \in K$ and $\alpha \gg 0$).

Therefore, after fixing an isomorphism $\overline{\mathbb{Q}}_p \cong \overline{\mathbb{Q}}$, inducing an isomorphism $\hat{\overline{\mathbb{Q}}}_p = \mathbb{C}_p \cong \mathbb{C}$, we can see our Eisenstein series above over $\mathbb{C}_p$ and takes its p-adic q-expansion at α. The formulas from our theorem above remain the same, except the factor $N(\mathfrak{a})$ is replaced by $N(\mathfrak{c})$. This motivates the following definition.

Definition. Fix $k \geq 1$, $c \in G$. Assume $\epsilon : I \to \hat{\mathbb{Q}}$ is a Schwartz function. Let

$$\Delta_c(1-k,\epsilon) := L(1-k,\epsilon) - N\mathfrak{c}^k L(1-k,\epsilon_c)$$

By our comments above, this is simply the difference between the constant terms on the p-adic q-expansion of $G_{k,\epsilon}$ at the cusps 1 and α (where $c = (\alpha) \cdot \alpha^{-1}$), seen as an element of $\hat{\mathbb{Q}}$ by looking at all primes p at once.

We define the distribution $\mu_{c,k}$ on I with values in $\hat{\mathbb{Q}}$ as $\epsilon \mapsto \lambda_c(1-k,\epsilon)$. If $k=1$, we simply write $\mu_c := \mu_{c,1}$.

Theorem 4 (Main theorem)**.** *For each* $c \in G$, μ_c *is actually a measure on* I *with values in* $\hat{\mathbb{Z}}$. *Moreover,*

$$\int \epsilon \cdot N^{k-1} d\mu_c = \Delta_c(1-k,\epsilon) \ ,$$

namely $\mu_{c,k} = N^{k-1}\mu_c$, *where this norm map is* $N : I \to \hat{\mathbb{Q}}$.

To prove it, we need to use the *generalized Kummer congruences.*

Theorem 5. *Let* $(\epsilon_k)_k$ *be a sequence of* $\hat{\mathbb{Q}}$*-valued Schwartz functions on* I *with* $\epsilon_k = 0$ *for all but finitely many k's. Assume that* ϵ_k *has parity* $(-1)^k$.

Let $\varphi = \sum_{k\geq 1} \epsilon_k N^{k-1} : I \to \hat{\mathbb{Q}}$. *Let* $\beta \in I_0$ *be any ideal of* K *and assume*

$$\sum_{x\subset\beta, x\in I_0} \varphi(\mu x^{-1}) \in \hat{\mathbb{Z}},\ \forall \mu \gg 0 \text{ in } K$$

Then, $\Delta_c(0,\epsilon_1+\tilde{\epsilon}) + \sum_{k\geq 2} \Delta_c(1-k,\epsilon_k) \in 2^r\hat{\mathbb{Z}}$, *where* $\tilde{\epsilon}_1$ *is the modification of* ϵ_1 *with respect to* β.

Proof. After some consideration, we can reduce to the case where $\beta = \mathcal{O}$, all ϵ_k are supported on A and are all defined modulo the same conductor $\mathfrak{f}$. Therefore, we can construct Eisenstein series $F_k := G_{k,\epsilon_k}$.

Let $S := \sum_{k\geq 1} F_k$. If we compose all ϵ_k with the projection $\hat{\mathbb{Q}} \to \mathbb{Q}_p$, we can think of S as a p-adic Hilbert modular form over $\mathbb{Q}_p$. The assumption about φ therefore simply means that all non-constant coefficients of its p-adic q-expansion at 1 are in $\mathbb{Z}_p$.

Then, the q-expansion principle yields that the difference of its constant term at 1 and at $\alpha \in \hat{K}^\times$ is in $\mathbb{Z}_p$. By considering all primes p at once, this exactly says

$$\left(2^{-r}L(0,\epsilon+\tilde{\epsilon}) + \sum_{k\geq 2} 2^{-r}L(1-k,\epsilon)\right) - \left(2^{-r}Nc^{-1}L(0,\epsilon_c+\tilde{\epsilon}_c) + \sum_{k\geq 2} 2^{-r}Nc^{-k}L(1-k,\epsilon_c)\right) \in \hat{\mathbb{Z}}$$

where $c = (\alpha)\cdot\alpha^{-1}$ as usual. After multiplying by 2^r, we get the desired conclusion. □

We can now prove the main theorem above.

Proof. We first need to explained that if ϵ is $\hat{\mathbb{Z}}$-valued, then so is its integral against μ_c. Using the trivial zeroes of certain $L(s,\epsilon_S)$ for most parity S at $s=0$ (i.e. for $k=1$ in Corollary 1 of Section 2), we know that

$$\Lambda_c(0,\epsilon) = \Lambda_c(0,\epsilon^-)$$

Then, using the theory of "exceptional" fields, Deligne-Ribet show explicitly that $\Lambda_c(0,\epsilon^-) \in \hat{\mathbb{Z}}$ using the fact that ϵ^- is odd.

Now, we have to prove that $N^{k-1}\mu_c = \mu_{c,k}$ as distributions. This is achieved by integrating any $\hat{\mathbb{Z}}$-integral ϵ against both sides and show that both sides are congruent modulo m for all integers m.

Let $\epsilon^\dagger$ be ϵ^+ or ϵ^- depending on whether $k\geq 2$ is even or odd. Then again, trivial vanishing yields

$$\Lambda_c(1-k,\epsilon) = \Lambda_c(1-k,\epsilon^+)$$

and $\int \epsilon N^{k-1}\mu_c = \int \epsilon^\dagger N^{k-1}\mu_c$ (apply vanishing to $\Lambda_c(0,\epsilon N^{k-1})$ by noting that $\epsilon^\dagger N^{k-1}$ is always odd). Therefore, there is no loss in generality by assuming that ϵ has parity $(-1)^k$.

Now fix $m\geq 0$ and choose any locally constant function $\eta : A \to \hat{\mathbb{Z}}$ that is odd and congruent to N modulo m. Then, by applying our generalized Kummer congruence with $\epsilon_1 = \epsilon N^{k-1}$ and $\epsilon_k = -\epsilon$ (all other $\epsilon_{k'} = 0$), we obtain

$$\Lambda_c(1-k,\epsilon) \equiv \Lambda_c(0,\epsilon\eta^{k-1}) + \Lambda_c(0,\widetilde{\epsilon\eta^{k-1}}) \mod m$$

and we need to get rid of this last term. But this is not actually hard because $\widetilde{\epsilon\eta^{k-1}}$ is supported on elements of I of the form $x^{-1}\cdot 0$ and these have norm zero. So we have

$$\widetilde{\epsilon\eta^{k-1}}(x^{-1}\cdot 0) = 0 \in m\hat{\mathbb{Z}} \Rightarrow \widetilde{\epsilon\eta^{k-1}} \equiv 0 \mod m$$

so $\Lambda_c(1-k,\epsilon) \equiv \Lambda_c(0,\epsilon\eta^{k-1})$ modulo m, as desired. □

This measure μ_c is not quite what we want. Instead, we fix some prime p and project $\hat{\mathbb{Z}}$ to $\mathbb{Z}_p$ to see μ_c as an element of the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[G]]$. Then, define the "pseudo-measure"

$$\lambda := \frac{1}{1-c}\mu_c \in \mathrm{Frac}(\Lambda)$$

This is well-defined and independent of $c\in G$ since $\mu_{c'} = (1-c')\lambda$. The advantage here is that for any character $\epsilon : G \to \mathbb{C}^\times$, we have

$$\int \epsilon N^k d\lambda = L(1-k,\epsilon)$$

and this yields p-adic variation of the usual L-function, which we call the Deligne-Ribet p-adic L-function.