# Finiteness of good moduli spaces of graded points

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**Abstract**

In this note we give a conceptual proof of the fact that, given a quasi-compact algebraic stack $\mathcal{X}$ with good moduli space $X$, the good moduli space of every quasi-compact connected component of $\mathrm{Grad}(\mathcal{X})$ is finite over $X$. Our main intermediate result states that if a stack $\mathcal{Y}$ is a $\Theta$-action retract onto a stack $\mathcal{Z}$ with good moduli space $Z$, then $Z$ is a categorical moduli space for $\mathcal{Y}$.

**1. Setup.** Let $\mathcal{X}$ be an algebraic stack, locally finitely presented over a quasi-separated and locally noetherian algebraic space $S$, with affine stabilizers and separated inertia. The stack of graded points of $\mathcal{X}$ is the mapping stack $\mathrm{Grad}(\mathcal{X}) = \mathrm{Map}(B\mathbb{G}_\mathrm{m}, \mathcal{X})$, which is algebraic and satisfies the same properties of $\mathcal{X}$ named above, see for example Halpern-Leistner [6]. There is an obvious map $\mathrm{tot}\colon \mathrm{Grad}(\mathcal{X}) \to \mathcal{X}$ called the *total point map*.

In this note, we work under the assumption that $\mathcal{X}$ is quasi-compact and has a good moduli space $p\colon \mathcal{X} \to X$ in the sense of Alper [1]. Let $\mathcal{X}_\alpha \subset \mathrm{Grad}(\mathcal{X})$ be a connected component and assume that $\mathcal{X}_\alpha$ is quasi-compact. It is shown in Ibáñez Núñez [7, Lemma 2.6.7] that $\mathcal{X}_\alpha$ has a good moduli space $p_\alpha\colon \mathcal{X}_\alpha \to X_\alpha$ and that the induced map $g_\alpha\colon X_\alpha \to X$ is affine. In this note we give a proof of the following fact.

**2. Theorem.** *The induced map $g_\alpha\colon X_\alpha \to X$ between good moduli spaces is finite.*

**3. Luna's result.** This result is known by Luna in the case of a stack of the form $\mathcal{X} = \mathrm{Spec}\, A/\mathrm{GL}_n$ over a field of characteristic 0, Luna [8, Théorème]. Therefore, if the base $S$ is the spectrum of a characteristic 0 field, then the theorem follows by taking étale slices as in Alper, Hall, and Rydh [2, Theorem 4.12].

**4. Stack of filtrations.** Our proof of Theorem 2 will use the stack of filtrations $\mathrm{Filt}(\mathcal{X})$, defined as the mapping stack $\mathrm{Filt}(\mathcal{X}) = \mathrm{Map}(\Theta, \mathcal{X})$, where $\Theta = \mathbb{A}^1/\mathbb{G}_\mathrm{m}$ is the quotient of $\mathbb{A}^1$ by the usual scaling action of $\mathbb{G}_m$, see Halpern-Leistner [6]. The associated graded map $\mathrm{gr}\colon \mathrm{Filt}(\mathcal{X}) \to \mathrm{Grad}(\mathcal{X})$, defined by precomposition along $0\colon B\mathbb{G}_\mathrm{m} \to \Theta$, is quasi-compact and induces a bijection on connected components [6, Lemma 1.3.8]. Let $\mathcal{X}_\alpha^+ \subset \mathrm{Filt}(\mathcal{X})$ be the connected component mapping to $\mathcal{X}_\alpha$.

Since $\mathcal{X}$ has a good moduli space, $\mathcal{X}$ is $\Theta$-*complete*, that is, the evaluation map $\mathrm{ev}\colon \mathrm{Filt}(\mathcal{X}) \to \mathcal{X}$, defined by precomposition along $1\colon \operatorname{Spec}\mathbb{Z} \to \Theta$, is universally closed, by Alper, Halpern-Leistner, and Heinloth [4, Theorem 5.4]. Since $\mathcal{X}_\alpha^+$ is quasi-compact, the restriction of the evaluation map $\mathrm{ev}_\alpha\colon \mathcal{X}_\alpha^+ \to \mathcal{X}$ is proper.

**5. Proposition.** *Let $p_\alpha^+\colon \mathcal{X}_\alpha^+ \to X_\alpha$ be the composition of the associated graded map $\mathrm{gr}_\alpha\colon \mathcal{X}_\alpha^+ \to \mathcal{X}_\alpha$ and the good moduli space $p_\alpha\colon \mathcal{X}_\alpha \to X_\alpha$. Then $p_\alpha^+\colon \mathcal{X}_\alpha^+ \to X_\alpha$ is a categorical moduli space, that is, any map $\mathcal{X}_\alpha^+ \to Y$, with $Y$ an algebraic space, factors uniquely through $p_\alpha^+$.*

*Proof.* Consider the maps $\mathbb{N} \to \mathbb{N}^2\colon b \mapsto (0,b)$ and $\mathbb{N}^2 \to \mathbb{N}\colon (a,b) \mapsto a+b$. After applying $\Theta_{(-)}$ as in Bu, Halpern-Leistner, Ibáñez Núñez and Kinjo [5, §5.1.2], we obtain maps $\Theta \to \Theta^2$ and $\Theta^2 \to \Theta$, respectively. After taking mapping stacks, we obtain maps $\mathrm{Filt}(\mathcal{X})^2 \to \mathrm{Filt}(\mathcal{X})$ and $\mathrm{Filt}(\mathcal{X}) \hookrightarrow \mathrm{Filt}(\mathcal{X})^2$, respectively. The second map is a closed and open immersion by [5, Theorem 5.1.4], and composed with the first one it gives the identity on $\mathrm{Filt}(\mathcal{X})$. Thus, the image of $\mathcal{X}_\alpha^+$ in $\mathrm{Filt}^2(\mathcal{X})$ under $\mathrm{Filt}(\mathcal{X}) \to \mathrm{Filt}^2(\mathcal{X})$ lies in the open and closed substack $\mathrm{Filt}(\mathcal{X}_\alpha^+) \subset \mathrm{Filt}(\mathrm{Filt}(\mathcal{X})) = \mathrm{Filt}^2(\mathcal{X})$. This realizes $\mathcal{X}_\alpha^+$ as a closed and open substack of $\mathrm{Filt}(\mathcal{X}_\alpha^+)$, witnessing the canonical $\Theta$-action on $\mathcal{X}_\alpha^+$.

We claim that the associated graded map $\mathrm{gr}\colon \mathrm{Filt}(\mathcal{X}_\alpha^+) \to \mathrm{Grad}(\mathcal{X}_\alpha^+)$ is identified with $\mathrm{gr}_\alpha\colon \mathcal{X}_\alpha^+ \to \mathcal{X}_\alpha$ when restricted to $\mathcal{X}_\alpha^+$. For this, consider the map $\mathbb{Z}\times\mathbb{N} \to \mathbb{Z}\colon (a,b) \mapsto a+b$, which gives a morphism $\mathrm{B}\mathbb{G}_\mathrm{m} \times \Theta \to \mathrm{B}\mathbb{G}_\mathrm{m}$. We obtain commuting squares

$$\begin{tikzcd} \mathrm{B}\mathbb{G}_\mathrm{m}\times\Theta \arrow[r] \arrow[d] & \mathrm{B}\mathbb{G}_\mathrm{m} \arrow[d] \\ \Theta^2 \arrow[r] & \Theta \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} \mathrm{Grad}(\mathrm{Filt}(\mathcal{X})) & \mathrm{Grad}(\mathcal{X}) \arrow[l, hook'] \\ \mathrm{Filt}(\mathrm{Filt}(\mathcal{X})) \arrow[u, "\mathrm{gr}"] & \mathrm{Filt}(\mathcal{X}), \arrow[l, hook'] \arrow[u, "\mathrm{gr}"'] \end{tikzcd}$$

where the horizontal arrows in the right square are open and closed immersions. The claim follows.

Now take a map $f\colon \mathcal{X}_\alpha^+ \to Y$ with $Y$ an algebraic space. Applying Filt and Grad, we obtain a diagram

$$\begin{tikzcd} & \mathcal{X}_\alpha^+ \arrow[r, "f"] & Y \\ \mathcal{X}_\alpha^+ \arrow[ur, dash, "\mathrm{id}"] \arrow[r, hook] \arrow[d, "\mathrm{gr}_\alpha"'] & \mathrm{Filt}(\mathcal{X}_\alpha^+) \arrow[r] \arrow[u, "\mathrm{ev}"'] \arrow[d, "\mathrm{gr}"'] & \mathrm{Filt}(Y) \arrow[u, "\sim"'] \arrow[d, "\sim"] \\ \mathcal{X}_\alpha \arrow[r, hook] & \mathrm{Grad}(\mathcal{X}_\alpha^+) \arrow[r] & \mathrm{Grad}(Y). \end{tikzcd}$$

This shows that $f$ factors through $\mathrm{gr}_\alpha$. This factorization is unique because $\mathrm{gr}_\alpha$ has a section. The result follows because $\mathcal{X}_\alpha \to X_\alpha$ is a categorical moduli space [3, Theorem 3.12]. □

**6.** We remark that the proof of Proposition 5 only uses that $\mathcal{X}_\alpha$ has a good moduli space. It is not necessary to assume that $\mathcal{X}$ has a good moduli space. Thus, we have the following stronger result.

**7. Proposition.** *Let $\mathcal{Y} \to \mathcal{Z}$ be a $\Theta$-action retract in the sense of [5, §5.1.10], where $\mathcal{Y}$ and $\mathcal{Z}$ are algebraic stacks satisfying the basic assumptions [5, §1.1.11]. Suppose that $\mathcal{Z}$ has a good moduli space $\mathcal{Z} \to Z$. Then the composition $\mathcal{Y} \to \mathcal{Z} \to Z$ is a categorical moduli space.*

**8.** The categorical moduli space $\mathcal{Y} \to Z$ is an example of *non-reductive moduli space* in the sense of forthcoming work of David Rydh.

The next ingredient in the proof of Theorem 2 is the following useful property of $p_\alpha^+$.

**9. Proposition.** *We have a canonical isomorphism $p_{\alpha,*}^+ \mathcal{O}_{\mathcal{X}_\alpha^+} = \mathcal{O}_{X_\alpha}$. Here, $p_{\alpha,*}^+$ denotes the underived pushforward.*

*Proof.* Let $R = \mathrm{Spec}_{\mathcal{O}_{X_\alpha}}(p_{\alpha,*}^+ \mathcal{O}_{\mathcal{X}_\alpha^+})$ and denote $r\colon R \to X_\alpha$ the obvious map. The map $\mathcal{X}_\alpha^+ \to R$ factors uniquely through $p_\alpha^+$ by Proposition 5, giving a map $u\colon X_\alpha \to R$. We have a commutative diagram

$$\begin{tikzcd}
 & X_\alpha \arrow[dr, "u"] \arrow[dd, "\mathrm{id}"] & \\
\mathcal{X}_\alpha^+ \arrow[ur, "p_\alpha^+"] \arrow[rr] \arrow[dr, "p_\alpha^+"'] & & R \arrow[dl, "r"] \\
 & X_\alpha &
\end{tikzcd}$$

Applying affinization over $X_\alpha$, we obtain a new commutative diagram

$$\begin{tikzcd}
 & X_\alpha \arrow[dr, "u"] \arrow[dd, "\mathrm{id}"] & \\
R \arrow[ur, "r"] \arrow[rr, "\mathrm{id}"] \arrow[dr, "r"'] & & R \arrow[dl, "r"] \\
 & X_\alpha &
\end{tikzcd}$$

that shows that $u$ and $r$ are inverses of each other. The result follows. □

**10.** Proposition 9 is still valid if we replace $p_\alpha^+$ by any categorical good moduli space, since this is the only property of $p_\alpha^+$ that we used in the proof.

**11. Proof of Theorem 2.** First, we note the commutativity of the diagram

$$\begin{tikzcd}
 & \mathcal{X}_\alpha^+ \arrow[dl, "\mathrm{gr}_\alpha"'] \arrow[dr, "\mathrm{ev}_\alpha"] & \\
\mathcal{X}_\alpha \arrow[d, "p_\alpha"'] \arrow[drr, "p \circ \mathrm{tot}_\alpha"] & & \mathcal{X} \arrow[d, "p"] \\
X_\alpha \arrow[rr, "g_\alpha"'] & & X.
\end{tikzcd}$$

Indeed, commutativity of the lower part of the diagram is immediate from the definition of $g_\alpha$, while the upper part of the diagram commutes by [7, Lemma 2.2.2]. We know from [7, Lemma 2.6.7] that $g_\alpha$ is affine, so we need to show that $g_{\alpha,*}(\mathcal{O}_{X_\alpha})$ is coherent. By Proposition 9, we have

$$g_{\alpha,*}(\mathcal{O}_{X_\alpha}) = g_{\alpha,*}p^+_{\alpha,*}(\mathcal{O}_{\mathfrak{X}^+_\alpha}) = g_{\alpha,*}\mathrm{gr}_{\alpha,*}p_{\alpha,*}(\mathcal{O}_{\mathfrak{X}^+_\alpha}) = p_*\mathrm{ev}_{\alpha,*}(\mathcal{O}_{\mathfrak{X}^+_\alpha}).$$

Now $\mathrm{ev}_\alpha$ is proper, by §4, and thus $\mathrm{ev}_{\alpha,*}(\mathcal{O}_{\mathfrak{X}^+_\alpha})$ is coherent, while $p_*$ preserves coherence by Alper [1, Theorem 4.16, (x)]. We conclude.

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## References

[1] Alper, J., 'Good moduli spaces for Artin stacks', *Annales de l'Institut Fourier*, 63/6 (2013), 2349–2402. doi: 10.5802/aif.2833.

[2] Alper, J., Hall, J., and Rydh, D., 'A Luna étale slice theorem for algebraic stacks', *Annals of Mathematics*, 191/3 (2020), 675. doi: 10.4007/annals.2020.191.3.1.

[3] Alper, J., Hall, J., and Rydh, D., *The étale local structure of algebraic stacks*, Preprint, arXiv:1912.06162v3, 2021. doi: https://doi.org/10.48550/arXiv.1912.06162, arXiv: 1912.06162v3 [math.AG].

[4] Alper, J., Halpern-Leistner, D., and Heinloth, J., 'Existence of moduli spaces for algebraic stacks', *Inventiones Mathematicae* (Aug. 2023). doi: 10.1007/s00222-023-01214-4.

[5] Bu, C., Halpern-Leistner, D., Ibáñez Núñez, A., and Kinjo, T., 'Intrinsic Donaldson–Thomas theory. I. Component lattices of stacks', version 1 (19 Feb. 2025), arXiv: 2502.13892v1.

[6] Halpern-Leistner, D., *On the structure of instability in moduli theory*, Preprint, arXiv:1411.0627v5, 2022.

[7] Ibáñez Núñez, A., *Refined Harder-Narasimhan filtrations in moduli theory*, Preprint. arXiv:2311.18050, 2023.

[8] Luna, D., 'Adhérence d'orbite et invariants', *Inventiones Mathematicae*, 29/3 (Oct. 1975), 231–238. doi: 10.1007/bf01389851.

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