REFINED HARDER–NARASIMHAN FILTRATIONS IN MODULI THEORY

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To my parents

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STATEMENT OF ORIGINALITY

This thesis contains no material that has already been accepted, or is concurrently being submitted, for any degree, diploma, certificate or other qualification at the University of Oxford or elsewhere. To the best of my knowledge and belief, the work contained in this thesis is original and my own, unless indicated otherwise. Most of the content in the first five chapters of this thesis has appeared on arXiv [45].

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ABSTRACT

We define canonical refinements of Harder–Narasimhan filtrations and stratifications in moduli theory. More precisely, we construct a canonical stratification for every noetherian algebraic stack $\mathcal{X}$ with affine diagonal that admits a good moduli space and is endowed with a norm on graded points. The strata live in a newly defined *stack of sequential filtrations* $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ of $\mathcal{X}$. Therefore, the stratification gives a canonical sequential filtration, the *iterated balanced filtration*, for each point of $\mathcal{X}$. In the presence of suitable Θ-stratifications, the iterated balanced filtration provides canonical refinements of Harder–Narasimhan filtrations. Our construction extends Kirwan's refined stratifications beyond the Geometric Invariant Theory case, and the iterated Haiden–Katzarkov–Kontsevich–Pandit (HKKP) filtration beyond the case of linear moduli problems, as well as giving a precise link between the two theories.

We introduce the machinery of *chains of stacks* as a tool to compute the iterated balanced filtration in combinatorial frameworks. We use it to give, in the case of quotient stacks by diagonalisable algebraic groups, an explicit description of the iterated balanced filtration in terms of convex geometry.

When the stack $\mathcal{X}$ parametrises objects in an abelian category, we show that the iterated balanced filtration agrees with the iterated HKKP filtration for modular lattices. A key ingredient in the proof is a new characterisation of the HKKP filtration for lattices as the minimiser of a certain norm function on the set of paracomplemented filtrations.

Examples where our theory can be applied include moduli of principal bundles on a curve, moduli of objects at the heart of a Bridgeland stability condition and moduli of K-semistable Fano varieties. We conjecture that the iterated balanced filtration describes the asymptotics of the Kempf–Ness flow in Geometric Invariant Theory, extending a theorem of Haiden, Katzarkov, Kontsevich and Pandit in the quiver case. This is part of a larger project aiming to describe the asymptotics of natural flows in moduli theory.

CONTENTS

Acknowledgements **iii**

Statement of originality **v**

Abstract **vii**

Contents **viii**

1 Introduction **1**

1.1 Overview . . . 1
1.2 Normed good moduli stacks . . . 5
1.3 Sequential filtrations and stratifications . . . 6
1.4 The balancing stratification . . . 9
1.5 Relation to convex geometry . . . 11
1.6 Comparison with the iterated HKKP filtration . . . 13
1.7 Conjecture on asymptotics of flows . . . 15
1.8 Notation and conventions . . . 18

2 Preliminaries **19**

2.1 Good moduli spaces and local structure theorems . . . 19
2.2 Stacks of rational filtrations and graded points . . . 22
2.3 Normed stacks . . . 31
2.4 Linear forms on stacks . . . 34
2.5 Θ-stratifications . . . 36
2.6 Θ-stratifications for stacks proper over a normed good moduli stack . 42
2.7 Formal fans and the degeneration fan . . . 50

3 Sequential stratifications and the iterated balanced filtration **57**

3.1 The balanced filtration . . . 57

3.2 Stacks of sequential filtrations . . . 62
3.3 Sequential stratifications . . . 67
3.4 Central rank and $B\mathbb{G}_m^n$-actions . . . 70
3.5 The balancing stratification and the iterated balanced filtration . . . 72
3.6 Examples . . . 79

4 Chains of stacks **85**
4.1 Chains . . . 85
4.2 The balancing chain . . . 87
4.3 The torsor chain . . . 89

5 The iterated balanced filtration and convex geometry **97**
5.1 Polarised states . . . 98
5.2 From states to good moduli stacks . . . 104
5.3 The refined Harder–Narasimhan filtration of a polarised state . . . 113

6 Modular lattices **117**
6.1 Generalities about lattices . . . 117
6.2 Background on the HKKP filtration and statement of main result . . 119
6.3 Distributive artinian lattices and directed acyclic graphs . . . 121
6.4 Maximal distributive sublattices of artinian lattices . . . 125
6.5 The degeneration fan of an artinian lattice . . . 127
6.6 Normed lattices and the HKKP filtration . . . 140
6.7 The HKKP chain . . . 149

7 Comparison with the iterated HKKP filtration **151**
7.1 Agreement of degeneration fans . . . 151
7.2 The case of nilpotent quiver representations . . . 155
7.3 Lamps and linearly lit good moduli stacks . . . 157
7.4 Examples of linearly lit stacks . . . 163
7.5 Main comparison result . . . 170

Bibliography **179**

CHAPTER 1

INTRODUCTION

1.1 OVERVIEW

There is a fascinating relation in moduli theory between the algebraic notion of stability and the existence of solutions to certain differential equations. To illustrate this, consider a vector bundle E on a smooth projective curve C. By the Narasimhan–Seshadri theorem, there exists a hermitian metric on E whose associated connection satisfies the Yang–Mills equations if and only if E is polystable.

One naturally wonders if this correspondence can be extended in some way to non-polystable objects. There is a space M_E of hermitian metrics on E, and the Yang–Mills equations define a flow, the hermitian Yang–Mills flow, on M_E. If E is polystable, then this flow converges to a minimum of the Yang–Mills functional from any initial metric.

Question: Can the asymptotic behaviour of the Hermitian-Yang-Mills flow on M_E be detected from stability-like properties of E?

A complete answer to this question has been given by Haiden–Katzarkov–Kontsevich–Pandit [33, 34]. Let $\mathbb{R}^\infty$ denote the partially ordered set of eventually-zero sequences of real numbers, ordered lexicographically.

THEOREM 1.1.1 (Haiden–Katzarkov–Kontsevich–Pandit)**.** *There exists a canonical filtration*

$$0 = E_0 \subsetneqq E_1 \subsetneqq \cdots \subsetneqq E_n = E, \tag{1.1}$$

labelled by elements

$$c_1 > \cdots > c_n, \tag{1.2}$$

where each $c_i = (c_{i0}, c_{i1}, \ldots) \in \mathbb{R}^\infty$*, such that, in the ith piece of the filtration, the hermitian Yang–Mills flow of E from any initial metric grows asymptotically by a factor of*

$$e^{-c_{i0}t} t^{-c_{i1}} (\log t)^{-c_{i2}} (\log \log t)^{-c_{i3}} \cdots, \quad t \gg 0, \tag{1.3}$$

up to bounded terms.

The filtration of E consisting of those E_i where $c_{i,0}$ jumps, that is, where $c_{i,0} > c_{i+1,0}$, is precisely the Harder–Narasimhan filtration of E [40]. Thus we call the chain (1.1), together with the labels (1.2), the *refined Harder–Narasimhan filtration* of E. It is an $\mathbb{R}^\infty$-filtration, in the sense that the labels (1.2) are also part of the data of the filtration.

From (1.3), we see that the Harder-Narasimhan filtration of E controls the exponential growth rate of the hermitian Yang-Mills flow, while the refinement controls the polynomial, logarithmic and iterated logarithmic growth rate.

The relation between stability and special metrics is not exclusive to the moduli of vector bundles on a curve. For example, consider the following moduli problems:

- Moduli of G-bundles on a smooth projective complex curve C, for G a connected reductive group.
- Moduli of orbits for a linear action of a reductive group G on a projective variety X over $\mathbb{C}$.
- Moduli of Fano varieties.

In each of the three situations, there is a Hitchin–Kobayashi type correspondence between polystable objects and solutions to a differential equation:

- A principal G-bundle P on C admits a Hermitian–Einstein metric if and only if P is polystable (Narasimhan–Seshadri and Ramanathan).
- A G-orbit of a point $x \in X(\mathbb{C})$ contains a zero of the moment map if and only if x is polystable (Kempf–Ness theorem).
- A smooth Fano variety Y admits a Kähler–Einstein metric if and only if Y is K-polystable (Yau–Tian–Donaldson conjecture, now a theorem).

In each of the three examples, there is an analogue M of the space of metrics on a given object. For a G-bundle P, $M = M_P$ is the space of reductions of structure group of P to a fixed maximal compact subgroup $K \subset G$. For a point $x \in X(\mathbb{C})$, we take $M = M_x$ to be the quotient G/K. For a smooth Fano variety Y, $M = M_Y$ is the space of Kähler metrics on Y. In each example, there is a natural flow on M (the hermitian Yang–Mills flow, the negative gradient flow for the Kempf–Ness potential and the Calabi flow, respectively) that converges precisely if the object is polystable.

Question: In each of the three moduli problems above, is there a reasonable notion of refined Harder–Narasimhan filtration controlling the asymptotics of the natural flow on the space M of metrics?

The difficulty is that we are dealing with *non-linear moduli problems*, that is, moduli of objects that do not come from an abelian category. The definition of the refined Harder–Narasimhan filtration of a vector bundle E by Haiden–Katzarkov–Kontsevich–Pandit uses as input solely the partially ordered set of subbundles of E, which is a *modular lattice*, together with the data of the rank and the degree of every subbundle. However, for non-linear moduli problems such a lattice is not available, and thus the methods of Haiden–Katzarkov–Kontsevich–Pandit do not apply. Even more concerning is the fact that, without a lattice, it seems unclear what the data of an $\mathbb{R}^\infty$-filtration of an object in a nonlinear moduli problem should be.

Even though modular lattices are exclusive to linear moduli problems, the one type of algebraic structure that is present in all the moduli problems mentioned so far is that of an *algebraic stack*. There is an algebraic stack $\mathrm{Bun}(C)$ parametrising vector bundles on C, and a stack $\mathrm{Bun}_G(C)$ parametrising principal G-bundles on C. The quotient stack X/G is naturally associated to the action of G on X. The case of Fano varieties is much more complicated, but an algebraic stack $\mathcal{X}^{Kss}$ parametrising K-semistable Fano varieties has recently been constructed and proven to enjoy nice properties by Xu and collaborators [5, 13, 14, 76]. It is then natural to rephrase our question as follows.

Question: Can the refined Harder–Narasimhan filtration of a vector bundle E be computed intrinsically in terms of the geometry of the algebraic stack $\mathrm{Bun}(C)$?

One of the main contributions of this thesis is a positive answer to this question. This naturally yield a version of the refined Harder–Narasimhan filtration for objects in our three nonlinear moduli problems.

In moduli theory, the assignment of canonical filtrations is intimately related to stratifications of moduli stacks. In the case of $\mathrm{Bun}(C)$, the assignment of the Harder–Narasimhan filtration to every vector bundle defines a stratification of $\mathrm{Bun}(C)$ by *Harder–Narasimhan type*. The analogue of this stratification for quotient stacks X/G was defined by Kirwan [55]. It is called the HKKN stratification of X/G after Hesselink–Kempf–Kirwan–Ness. It is related to the assignment, for every point x of X, of Kempf's maximally destabilising one-parameter subgroup [53]. Indeed, for the moduli stack X/G, the correct notion of filtration of a point x is a one-parameter subgroup $\lambda\colon \mathbb{G}_{m,\mathbb{C}} \to G$ such that the limit $\lim_{t\to 0} \lambda(t)x$ exists in X, considered up to certain equivalence relation. This is not a surprise from the point of view of algebraic stacks, since such a one-parameter subgroup λ corresponds to a map $\mathbb{A}^1_{\mathbb{C}}/\mathbb{G}_{m,\mathbb{C}} \to X/G$ sending 1 to x, and a filtration of E labelled by integers also corresponds to a map $\mathbb{A}^1_{\mathbb{C}}/\mathbb{G}_{m,\mathbb{C}} \to \mathrm{Bun}(C)$ sending 1 to E. Here, $\mathbb{A}^1_{\mathbb{C}}/\mathbb{G}_{m,\mathbb{C}}$ is the quotient of the multiplicative group $\mathbb{G}_{m,\mathbb{C}}$ by the natural scaling action of $\mathbb{G}_{m,\mathbb{C}}$. This stack is so important in moduli theory that it has its own name: $\Theta_{\mathbb{C}} = \mathbb{A}^1_{\mathbb{C}}/\mathbb{G}_{m,\mathbb{C}}$. The stratification of $\mathrm{Bun}(C)$ by Harder–Narasimhan type and the HKKN stratification of X/G have been abstracted to general algebraic stacks by Halpern-Leistner [36] in what is called Θ-stratifications.

It is natural to ask whether the assignment of the refined Harder–Narasimhan filtration for every vector bundle E defines a stratification of $\mathrm{Bun}(C)$ refining the stratification by Harder–Narasimhan type, famously studied by Atiyah–Bott [10]. This is unclear from the work of Haiden–Katzarkov–Kontsevich–Pandit. However, canonical refinements of the HKKN stratification of X/G and of the stratification by Harder–Narasimhan type of $\mathrm{Bun}(C)$ had already been constructed by Kirwan [57, 58], with the study of cohomology rings of moduli spaces as a motivation. Kirwan's refined stratification of X/G are defined via an iteration of blow-ups and HKKN stratifications. It has the caveat that it does not give a notion of $\mathbb{R}^\infty$-filtration for every point x of X. Another issue is that Kirwan's construction cannot be applied to $\mathcal{X}^{\mathrm{Kss}}$, since this stack is not a semistable locus in the GIT sense.

The results of this thesis link Kirwan's refined stratifications and Haiden–Katzarkov–Kontsevich–Pandit's refined Harder–Narasimhan filtrations of vector bundles on C, as well as extending both to moduli problems where they were not previously defined.

Theorem 1.1.2. *Let $\mathcal{X}$ be one of the algebraic stacks $\mathrm{Bun}(C)$, $\mathrm{Bun}_G(C)$, X/G or $\mathcal{X}^{Kss}$. For every point $x \in \mathcal{X}(\mathbb{C})$ of $\mathcal{X}$ there is a naturally-defined set $\mathbb{Q}^\infty\text{-}\mathrm{Filt}(\mathcal{X}, x)$ of* $\mathbb{Q}^\infty$-filtrations *of x, and a canonical element $\lambda_{rHN}(x) \in \mathbb{Q}^\infty\text{-}\mathrm{Filt}(\mathcal{X}, x)$, called the* refined Harder–Narasimhan filtration *of x, defined intrinsically in terms of the stack $\mathcal{X}$. Moreover, the assignment of $\lambda_{rHN}(x)$ for every x defines a stratification of $\mathcal{X}$ into locally closed substacks by type of refined Harder–Narasimhan filtration.*

In the case of X/G and $\mathrm{Bun}(C)$, the stratification by type of refined Harder–Narasimhan filtration agrees with Kirwan's refined stratification.

In the case of $\mathrm{Bun}(C)$ and a vector bundle $E \in \mathrm{Bun}(C)(\mathbb{C})$, the set $\mathbb{Q}^\infty\text{-}\mathrm{Filt}(\mathrm{Bun}(C), E)$ agrees with the set of $\mathbb{Q}^\infty$-filtrations of E in the usual sense, and $\lambda_{rHN}(x)$ agrees with the refined Harder–Narasimhan filtration of E as defined by Haiden–Katzarkov–Kontsevich–Pandit.

This theorem follows from Theorems 1.6.1 and 3.5.2 and from Section 3.6. It is more natural and simpler to replace $\mathbb{R}^\infty$ by $\mathbb{Q}^\infty$ in the general theory.

Our definition of the refined Harder–Narasimhan filtration involves as main ingredients blow-ups of stacks and Θ-stratifications [36], as well as the construction of a *stack of sequential filtrations* $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ for an algebraic stack $\mathcal{X}$ (Definition 3.2.2).

Our construction is valid for very general algebraic stacks endowed with suitable stability data (Definition 3.5.10).

Despite the abstract nature of its definition, the refined Harder–Narasimhan filtration is often computable in terms of combinatorial data, which in this thesis come in two flavours: convex geometry and modular lattices. In Chapter 5, we establish a convex geometric algorithm to compute the filtration in the case of quotient stacks by the action of a diagonalisable algebraic group (Corollary 5.2.18). The proof involves the theory of *chains of stacks*, developed in Chapter 4. In Chapter 7, we use these techniques to show that the refined Harder–Narasimhan filtration agrees with the one defined by Haiden–Katzarkov–Kontsevich–Pandit when the latter, which is defined in the setting of modular lattices, makes sense (Theorem 1.6.1). To that aim, in Chapter 6 we establish a new characterisation of the HKKP filtration as the minimiser of a norm function in the set of paracomplemented filtrations of the lattice (Theorem 6.6.26).

Our motivation for this work is the expectation that the refined Harder–Narasimhan filtration describes the asymptotics of natural flows in moduli theory, like the Yang–Mills flow for principal bundles, the Calabi flow for varieties and the gradient flow of the Kempf–Ness potential in Geometric Invariant Theory. We provide a conjectural statement for this expectation in the case of GIT for affine spaces (Conjecture 1.7.1).

1.2 NORMED GOOD MODULI STACKS

The stratification of $\mathrm{Bun}(C)$ by Harder–Narasimhan type is a Θ-stratification. In particular, every stratum $\mathcal{S}$ has an $\mathbb{A}^1$-retraction $\mathcal{S} \to \mathcal{Z}$ onto what is called its centre $\mathcal{Z}$. In this case, $\mathcal{Z}$ enjoys the property of admitting a good moduli space $\mathcal{Z} \to Z$, that is, a map to an algebraic space Z that best approximates the stack $\mathcal{Z}$ (we recall the precise definition from Alper [2] in Definition 2.1.1). Having a good moduli space is a strong condition on the stack that implies many desirable properties.

We may hope that algebraic stacks admitting good moduli spaces have canonical stratifications and, in a suitable sense, canonical filtrations for every point. These could then be pulled back along the retractions $\mathcal{S} \to \mathcal{Z}$ from each centre $\mathcal{Z}$ and produce, in the case of $\mathrm{Bun}(C)$, the sought-after stratification by HKKP type, as well as recovering the iterated HKKP filtration of every point. This is close to being true. What is missing is stability type data on $\mathcal{Z}$ on which this stratification depends. The correct notion for our purposes turns out to be a *norm on graded points* of $\mathcal{Z}$, a concept from the Beyond GIT programme of Halpern-Leistner [36].

A *graded point* of a stack $\mathcal{X}$ is a map $g\colon B\mathbb{G}_{m,k} \to \mathcal{X}$, where $B\mathbb{G}_{m,k}$ is the classifying stack of the multiplicative group over a field k. Graded points on $\mathcal{X}$ can also be seen as ordinary points of the mapping stack $\mathrm{Grad}(\mathcal{X}) = \underline{\mathrm{Hom}}(B\mathbb{G}_m, \mathcal{X})$. A norm on graded points of $\mathcal{X}$ is, roughly speaking, the data of a positive real number $\|g\|$ for every graded point $g\colon B\mathbb{G}_{m,k} \to \mathcal{X}$, and this data is required to satisfy some local constancy and nondegeneracy conditions (Definition 2.3.3). For $\mathrm{Bun}(C)$, a natural norm is induced by the rank of vector bundles. In this case, a graded point $g\colon B\mathbb{G}_{m,\mathbb{C}} \to \mathrm{Bun}(C)$ corresponds to a vector bundle E together with a direct sum decomposition $E = \bigoplus_{c\in\mathbb{Z}} E_c$, and its norm is then defined by the formula $\|g\|^2 = \sum_{c\in\mathbb{Z}} c^2 \,\mathrm{rk}(E_c)$. This restricts to a norm on graded points on each of the centres $\mathcal{Z}$ of the Harder–Narasimhan stratification.

Many other moduli stacks $\mathcal{X}$ are also naturally endowed with a Θ-stratification, related to the assignment, for every point x of $\mathcal{X}$, of a Harder–Narasimhan filtration of x in a generalised sense. It is often the case that each centre $\mathcal{Z}$ of the stratification has a good moduli space and is naturally endowed with a norm on graded points (see Section 3.6 for examples). Therefore, in order to define refined Harder–Narasimhan filtrations in moduli theory in great generality, it is enough to deal with algebraic stacks admitting a good moduli space and endowed with a norm on graded points. We call these *normed good moduli stacks* for simplicity. Our aim thus becomes to stratify and produce canonical filtrations for normed good moduli stacks.

1.3 SEQUENTIAL FILTRATIONS AND STRATIFICATIONS

The first obstacle we encounter is the very meaning of filtration in this generality. The iterated HKKP filtration of a vector bundle E consists of both a chain

$$0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_n = E$$

of subbundles and a chain

$$c_1 > c_2 > \cdots > c_n$$

of labels $c_i \in \mathbb{Q}^\infty$. Here, $\mathbb{Q}^\infty$ is the set of eventually zero sequences of rational numbers, ordered lexicographically. In this sense the iterated HKKP filtration is a *sequential filtration*, or $\mathbb{Q}^\infty$*-filtration*, of E.

We first look at the well-studied case of $\mathbb{Z}$-filtrations, that is, when the labels c_i are integers. These are closely related to the quotient stack $\mathbb{A}^1/\mathbb{G}_m$ of the affine line $\mathbb{A}^1$ by the scaling action of the multiplicative group, also denoted $\Theta = \mathbb{A}^1/\mathbb{G}_m$. Indeed, the mapping stack $\mathrm{Filt}(\mathrm{Bun}(C)) = \underline{\mathrm{Hom}}(\mathbb{A}^1/\mathbb{G}_m, \mathrm{Bun}(C))$ parametrises vector bundles endowed with a $\mathbb{Z}$-filtration (see Heinloth [41, Lemma 1.10] and Alper–Halpern-Leistner–Heinloth [8, Corollary 7.13]). Therefore it makes sense to define a $\mathbb{Z}$-filtration of a k-point x in a general stack $\mathcal{X}$ to be a map $f\colon \Theta_k \to \mathcal{X}$ together with an isomorphism $x \sim f(1)$, and to define the stack of filtrations on $\mathcal{X}$ to be the mapping stack $\mathrm{Filt}(\mathcal{X}) = \underline{\mathrm{Hom}}(\Theta, \mathcal{X})$. This idea lies at the heart of Halpern-Leistner's approach to moduli theory beyond GIT [36]. There is an *associated graded map* $\mathrm{gr}\colon \mathrm{Filt}(\mathcal{X}) \to \mathrm{Grad}(\mathcal{X})$ and a forgetful map $\mathrm{ev}_1\colon \mathrm{Filt}(\mathcal{X}) \to \mathcal{X}$. For our purposes, it is better to consider $\mathbb{Q}$-filtrations, and we work instead with the stacks of $\mathbb{Q}$-filtrations $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ and of $\mathbb{Q}$-gradings $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$, that we construct by formally localising with respect to the natural action of the monoid $(\mathbb{Z}_{>0}, \cdot, 1)$ on $\mathrm{Filt}(\mathcal{X})$ and $\mathrm{Grad}(\mathcal{X})$ (Definition 2.2.7). Rational filtrations of a point $x \in \mathcal{X}(k)$ are defined in the same way and they form a set $\mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, x)$.

Our first goal is to construct an algebraic stack $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ that, in the case $\mathcal{X} = \mathrm{Bun}(C)$, parametrises vector bundles E endowed with a sequential filtration. To this aim, we observe that giving a $\mathbb{Q}^\infty$-filtration on a vector bundle E on C is equivalent to first giving a $\mathbb{Q}$-filtration $F_\bullet$ of E, then a $\mathbb{Q}$-filtration of the associated graded object $\mathrm{gr}\, F_\bullet$, and so on, until the process finishes in finitely many steps. For general stacks, we formalise this idea by defining the stack $\mathrm{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X})$ of $\mathbb{Q}^n_{\mathrm{lex}}$-filtrations inductively as a fibre product

$$\mathrm{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X}) = \mathrm{Filt}_{\mathbb{Q}^{n-1}_{\mathrm{lex}}}(\mathcal{X}) \times_{\mathrm{gr}, \mathrm{Grad}_{\mathbb{Q}^{n-1}}(\mathcal{X}), \mathrm{ev}_1} \mathrm{Filt}_{\mathbb{Q}}\left(\mathrm{Grad}_{\mathbb{Q}^{n-1}}(\mathcal{X})\right).$$

Here, $\mathbb{Q}^n_{\mathrm{lex}}$ is $\mathbb{Q}^n$ endowed with the lexicographic order, and $\mathrm{Grad}_{\mathbb{Q}^l}(\mathcal{X})$ is defined inductively as $\mathrm{Grad}_{\mathbb{Q}^l}(\mathcal{X}) = \mathrm{Grad}_{\mathbb{Q}}\left(\mathrm{Grad}^{l-1}_{\mathbb{Q}}(\mathcal{X})\right)$. The associated graded map $\mathrm{gr}\colon \mathrm{Filt}_{\mathbb{Q}^l_{\mathrm{lex}}}(\mathcal{X}) \to \mathrm{Grad}_{\mathbb{Q}^l}(\mathcal{X})$ is also defined by induction. Then we set:

DEFINITION 1.3.1 (Definition 3.2.2)**.** The stack $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ of *sequential filtrations*, or *$\mathbb{Q}^\infty$-filtrations*, of $\mathcal{X}$ is the colimit of the stacks $\mathrm{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X})$ when n tends to ∞.

Under reasonable conditions on $\mathcal{X}$, the stack of $\mathbb{Q}^\infty$-filtrations $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ is algebraic (Proposition 3.2.3). There is also a map $\mathrm{ev}_1\colon \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \to \mathcal{X}$ corresponding to "forgetting the filtration", so it now makes sense to define a *$\mathbb{Q}^\infty$-filtration* of a field-valued point $x\colon \mathrm{Spec}\, k \to \mathcal{X}$ to be a k-point $\lambda \in \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})(k)$ together with an isomorphism $x \sim \mathrm{ev}_1(\lambda)$. We denote $\mathbb{Q}^\infty\text{-}\mathrm{Filt}(\mathcal{X}, x)$ the set of $\mathbb{Q}^\infty$-filtrations of x (Definition 3.2.13).

The relation between stratifications of $\mathcal{X}$ and sequential filtrations is encapsulated in the definition of a *sequential stratification*.

DEFINITION 1.3.2 (Definition 3.3.1)**.** A *sequential stratification* of an algebraic stack $\mathcal{X}$ is a family $(\mathcal{S}_\alpha)_{\alpha\in\Gamma}$ of locally closed substacks of $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$, indexed by a partially ordered set Γ, such that

1. each composition $\mathcal{S}_c \to \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \to \mathcal{X}$ is a locally closed immersion,
2. the topological spaces $|\mathcal{S}_c|$ are pairwise disjoint and cover $|\mathcal{X}|$, and
3. for every $c \in \Gamma$, the union $\bigcup_{c'\leq c} |\mathcal{S}_{c'}|$ is open in $\mathcal{X}$.

Thus the strata $\mathcal{S}_c$ are locally closed substacks of $\mathcal{X}$ together with a choice of lift to $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$. Therefore a sequential stratification provides each point x of $\mathcal{X}$ with a choice of sequential filtration of x.

The definition of a sequential stratification is inspired in Halpern-Leistner's definition of a Θ-stratification of a stack $\mathcal{X}$ [36] (recalled in Definition 2.5.1), which roughly speaking is a partition of $\mathcal{X}$ into locally closed substacks $\mathcal{S}_c$ of $\mathcal{X}$ that are also open substacks of $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$. Very importantly, each stratum $\mathcal{S}_c$ retracts onto what is called its *centre* $\mathcal{Z}_c$, which is an open substack of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$. Establishing existence and nice properties of certain Θ-stratifications is a crucial ingredient of our construction of the balancing stratification.

If $\mathcal{X}$ is a normed good moduli stack and $f: \mathcal{Y} \to \mathcal{X}$ is a representable polarised projective morphism it is a result of Halpern-Leistner [36, Theorem 5.6.1] that $\mathcal{Y}$ has a natural Θ-stratification induced by the norm on $\mathcal{X}$ and the polarisation. This generalises Kirwan's construction of the instability stratification in GIT [55]. Our first result establishes a property of this stratification that will be fundamental for our purposes:

THEOREM 1.3.3 (Theorem 2.6.4)**.** *Let $\mathcal{X}$ be a normed noetherian good moduli stack with affine diagonal. Consider a representable projective morphism $f: \mathcal{Y} \to \mathcal{X}$ and the Θ-stratification $(\mathcal{S}_c)_{c\in\mathbb{Q}_{\geq 0}}$ of $\mathcal{Y}$ induced by an f-ample line bundle $\mathcal{L}$ and the norm on graded points. Then the centre $\mathcal{Z}_c$ of every stratum $\mathcal{S}_c$ has a good moduli space.*

For the proof, we need to consider the more general case when f is proper and the line bundle $\mathcal{L}$ is replaced by the weaker notion of a *linear form on graded points*, and then use the concepts of Θ-monotonicity and S-monotonicity developed in [36] to check Θ-reductivity and S-completeness of $\mathcal{Z}_c$, which implies the existence of a good moduli space [8].

1.4 THE BALANCING STRATIFICATION

Our main construction (Theorem 3.5.2 and Definition 3.5.3) produces a canonical sequential stratification for every normed noetherian good moduli stack $\mathcal{X}$ with affine diagonal. We call it the *balancing stratification*, since the adjective *balanced* was used both by Kirwan and Haiden–Katzarkov–Kontsevich–Pandit to describe the filtrations they studied. The balancing stratification produces a canonical sequential filtration, the *iterated balanced filtration*, for every point of the stack $\mathcal{X}$.

Recall that a point p in a stack $\mathcal{X}$ with good moduli space $\pi\colon \mathcal{X} \to X$ is said to be *polystable* if it is closed in the fibre of π containing p. Every fibre of π contains a unique polystable point. If $x \in \mathcal{X}(k)$ is a non-polystable geometric point, it follows from a result of Kempf [53] that there is a filtration $\lambda\colon \Theta_k \to \mathcal{X}$ of x such that $\lambda(0) = y$ is polystable. The question arises whether one can choose a canonical such polystable degeneration λ. For this problem, it is useful to endow $\mathcal{X}$ with a norm on graded points. After replacing $\mathcal{X}$ with a π-saturated open substack containing x, we may assume that y lies in the closed substack $\mathcal{X}^{\max}$ of points with maximal stabiliser dimension, defined by Edidin–Rydh [27]. Kempf defines a natural number $\langle \lambda, \mathcal{X}^{\max} \rangle \in \mathbb{N}$ (Definition 3.1.1) that can be thought of as measuring the velocity at which λ converges to y. The definition extends to rational filtrations λ, giving a rational number $\langle \lambda, \mathcal{X}^{\max} \rangle$. It turns out that there is a unique rational filtration $\lambda_{\mathrm{b}}(x) \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, x)$ such that $\langle \lambda, \mathcal{X}^{\max} \rangle \geq 1$ and $\|\lambda\|$ is minimal among rational filtrations with this property (Theorem 3.1.3). We call $\lambda_{\mathrm{b}}(x)$ the *balanced filtration* of x. We can think of the balanced filtration as degenerating x to its associated polystable point with optimal velocity and minimal cost.

If we lift x to the blow-up $\mathcal{B} = \mathrm{Bl}_{\mathcal{X}^{\max}} \mathcal{X}$, then the balanced filtration of x coincides with the filtration of x associated to the natural Θ-stratification $(\mathcal{S}_c)_{c \in \mathbb{Q}_{\geq 0}}$ on $\mathcal{B}$ from Theorem 1.3.3. If $\mathcal{E} \subset \mathcal{B}$ is the exceptional divisor, then we have a stratification of $\mathcal{X}$ by type of balanced filtration where the strata are $\mathcal{X}^{\max}$ and the $\mathcal{S}_c \setminus \mathcal{E}$. This is the first approximation to the iterated balanced filtration, that we get by iterating this procedure from the centres of the strata $\mathcal{S}_c$.

The balancing stratification is indexed by a totally ordered set $\mathbf{\Gamma}$ defined explicitly (Definition 3.5.1). It consists of sequences $((d_0, c_0), \ldots, (d_n, c_n))$ with $d_0, d_1, \ldots, d_n \in \mathbb{N}$, $c_0, \ldots, c_{n-1} \in \mathbb{Q}_{>0}$, $c_n = \infty$, and satisfying some other conditions, and the poset structure is given by lexicographic order. The balancing stratification is uniquely characterised by the following theorem. See Theorem 2.6.4 for a more precise formulation.

Theorem 1.4.1 (Existence and characterisation of the balancing stratification)**.** *There is a unique way of assigning, to every normed noetherian good moduli stack $\mathcal{X}$ with affine diagonal, a sequential stratification $(\mathcal{S}_\alpha^{\mathcal{X}})_{\alpha\in\boldsymbol{\Gamma}}$ of $\mathcal{X}$, called the* balancing stratification *of $\mathcal{X}$, in such a way that for every such $\mathcal{X}$ the following holds. Let $(\mathcal{S}_c)_{c\in\mathbb{Q}_{\geq 0}}$ be the Θ-stratification of the blow-up $\mathcal{B} = \mathrm{Bl}_{\mathcal{X}^{\max}}\mathcal{X}$ induced by the natural ample line bundle on $\mathcal{B}$ and the norm on graded points, let $\mathcal{Z}_c$ be the centres of the stratification, which have good moduli spaces by Theorem 1.3.3, let $\mathcal{E}\subset\mathcal{B}$ be the exceptional divisor, let d be the maximal stabiliser dimension of $\mathcal{X}$, let $\pi\colon\mathcal{X}\to X$ be the good moduli space of $\mathcal{X}$, and let $\mathcal{U} = \mathcal{X}\setminus\pi^{-1}\pi(\mathcal{X}^{\max})$. Then*

1. *the highest stratum is $\mathcal{S}^{\mathcal{X}}_{(d,\infty)} = \mathcal{X}^{\max}$, embedded in $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ via the trivial filtration map $\mathcal{X}\to\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$;*
2. *for every $c\in\mathbb{Q}_{>0}$, and every $\alpha\in\boldsymbol{\Gamma}$ with $\mathcal{S}_\alpha^{\mathcal{Z}_c}\neq\varnothing$, we have*
$$\mathcal{S}^{\mathcal{X}}_{((d,c),\alpha)} = \left(\mathcal{S}_c\times_{\mathcal{Z}_c}\mathcal{S}_\alpha^{\mathcal{Z}_c}\right)\setminus\mathcal{E}$$
with its natural structure of locally closed substack of $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$; and
3. *for every $\alpha\in\boldsymbol{\Gamma}$ with $\mathcal{S}_\alpha^{\mathcal{U}}\neq\varnothing$, we have $\mathcal{S}_\alpha^{\mathcal{X}} = \mathcal{S}_\alpha^{\mathcal{U}}$, as a substack of $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$.*

In the statement we are implicitly using that the $\mathcal{Z}_c$ have good moduli spaces, which is Theorem 1.3.3 above. Let us explain how to realise $\mathcal{S}^{\mathcal{X}}_{((d,c),\alpha)}$ inside $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$. For $c\in\mathbb{Q}_{>c}$, we can pull back each nonempty $\mathcal{S}_\alpha^{\mathcal{Z}_c}$ along the associated graded map $\mathcal{S}_c\to\mathcal{Z}_c$ to get a stack $\mathcal{V}_{c,\alpha} = \mathcal{S}_c\times_{\mathcal{Z}_c}\mathcal{S}_\alpha^{\mathcal{Z}_c}$. The centre $\mathcal{Z}_c$ is an open substack of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{B})$, so $\mathcal{S}_\alpha^{\mathcal{Z}_c}$ lives in $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathrm{Grad}_{\mathbb{Q}}(\mathcal{B}))$. The iterative definition of $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{B})$ gives a cartesian square

$$\begin{array}{ccc}
\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{B}) & \longrightarrow & \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathrm{Grad}_{\mathbb{Q}}(\mathcal{B})) \\
\downarrow & \ulcorner & \downarrow \\
\mathrm{Filt}_{\mathbb{Q}}(\mathcal{B}) & \xrightarrow{\mathrm{gr}} & \mathrm{Grad}_{\mathbb{Q}}(\mathcal{B})
\end{array}$$

and hence $\mathcal{V}_{c,\alpha}$ lives in $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{B})$. After subtracting the exceptional divisor $\mathcal{E}$, we get a locally closed substack $\mathcal{S}^{\mathcal{X}}_{((d,c),\alpha)} := \mathcal{V}_{c,\alpha}\setminus\mathcal{E}$ of $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$.

In order to prove Theorem 1.4.1, we introduce the concept of the *central rank* of a stack. The central rank of $\mathcal{X}$, denoted $z(\mathcal{X})$, is the largest natural number n such that $B\mathbb{G}_m^n$ acts on $\mathcal{X}$ in a nondegenerate way (Definition 3.4.1). The condition can be thought of as requiring that every stabiliser of $\mathcal{X}$ contains a copy of $\mathbb{G}_m^n$ in its centre. The maximal dimension of a stabiliser of $\mathcal{X}$ is denoted $d(\mathcal{X})$. The proof is by induction on $N(\mathcal{X}) := d(\mathcal{X}) - z(\mathcal{X})$.

The main observation is that, whenever we have a blow-up $f\colon\mathcal{Y} = \mathrm{Bl}_{\mathcal{R}}\mathcal{X}\to\mathcal{X}$ of $\mathcal{X}$ along some closed substack $\mathcal{R}$, if the Θ-stratification of $\mathcal{Y}$ is $(\mathcal{S}_c)_{c\in\mathbb{Q}_{\geq 0}}$, then for every unstable stratum $\mathcal{S}_c$, its centre $\mathcal{Z}_c$ has bigger central rank than $\mathcal{X}$: $z(\mathcal{Z}_c) >$

$z(\mathcal{X})$ (Lemma 3.4.7). We also have $d(\mathcal{Z}_c) \leq d(\mathcal{X})$ by representability of f, so that $N(\mathcal{Z}_c) < N(\mathcal{X})$.

The balancing stratification defines a canonical sequential filtration for every point:

Definition 1.4.2 (Definition 3.5.8)**.** Let $\mathcal{X}$ be a normed noetherian good moduli stack with affine diagonal. The *iterated balanced filtration* of a field-valued point $x \in \mathcal{X}(k)$ is the element $\lambda_{\mathrm{ib}}(x) \in \mathbb{Q}^\infty\text{-}\operatorname{Filt}(\mathcal{X}, x)$ determined by the balancing stratification of $\mathcal{X}$.

The iterated balanced filtration is defined over k even if k is not perfect. This has to do with the assumption that $\mathcal{X}$ has a good moduli space instead of just an adequate moduli space, Alper [3], which is a more general notion in positive characteristic.

For a stack $\mathcal{Y}$ endowed with a Θ-stratification $(\mathcal{S}_c)$ such that all the centres $\mathcal{Z}_c$ of the strata are normed good moduli stacks, the balancing stratification of each $\mathcal{Z}_c$ can be pulled back to the $\mathcal{S}_c$ to define a sequential stratification of $\mathcal{Y}$. This produces a *refined Harder–Narasimhan filtration* for every point of $\mathcal{Y}$ (Definition 3.5.10).

We show that the balancing stratification has nice functorial properties.

Proposition 1.4.3 (Proposition 3.5.5)**.** *Let $f\colon \mathcal{X} \to \mathcal{X}'$ be a morphism between noetherian normed good moduli stacks with affine diagonal. If f is either a closed immersion or a base change from a map $X \to X'$ between the good moduli spaces of $\mathcal{X}$ and $\mathcal{X}'$, then for all $\alpha \in \mathbf{\Gamma}$, the stratum $\mathcal{S}_\alpha^{\mathcal{X}}$ equals the pullback $\mathcal{S}_\alpha^{\mathcal{X}'} \times_{\mathcal{X}',f} \mathcal{X}$ of $\mathcal{S}_\alpha^{\mathcal{X}'}$ along f, with its natural structure of locally closed substack of* $\operatorname{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$.

1.5 RELATION TO CONVEX GEOMETRY

Despite its seemingly convoluted definition involving several blow-ups and Θ-stratifications, the iterated balanced filtration has a particularly simple description for a point x in a quotient stack of the form $\operatorname{Spec} A/G$ with G a diagonalisable algebraic group over a field k (for example a split torus $\mathbb{G}_{m,k}^n$) and A a finite type k-algebra. For simplicity, in this introduction we will assume that $\operatorname{Spec} A$ is the total space of a vector space $V = k^l$, that G acts on V via the characters $\chi_1, \dots, \chi_l \in \Gamma_{\mathbb{Z}}(G) = \operatorname{Hom}(G, \mathbb{G}_{m,k})$ and that we are interested in computing the iterated balanced filtration of the point $x = (1, \dots, 1)$ in the quotient stack V/G. We denote $N = \Gamma^{\mathbb{Q}}(G)$ the set of rational cocharacters of G. The norm on graded points of V/G corresponds to a rational inner product $(-,-)$ on $N = N_0$, and we identify N and its dual via this inner product.

We now describe a sequence of elements $\lambda_0, \ldots, \lambda_n \in N$ in terms of the *state* $\Xi_0 = \{\chi_1, \ldots, \chi_l\}$ of x. Let F_0 be the smallest face containing 0 of $\mathrm{cone}(\Xi_0)$, the convex cone generated by Ξ_0 inside N. Then we let λ_0 be the unique element of the orthogonal complement $N_1 := F_0^{\perp}$ such that $(\lambda_0, \alpha) \geq 1$ for all $\alpha \in \Xi_0 \setminus F_0$ and $\|\lambda_0\|$ is minimal. This is a convex optimisation problem.

To compute $\lambda_1, \cdots, \lambda_n$, we proceed as follows. We let

$$\Xi_1 = \{p_1(\alpha) \mid \alpha \in \Xi_0, \ (\lambda_0, \alpha) = 1\} \subset N_1,$$

where $p_1 \colon N \to N_1$ is the orthogonal projection. It will always be the case that $\lambda_0 \in \mathrm{cone}\,(\Xi_1) \subset N_1$ (Theorem 5.1.18), and thus there is a smallest face F_1 of $\mathrm{cone}\,(\Xi_1)$ containing λ_0. Let N_2 be the orthogonal complement of F_1 inside N_1. Then we let λ_1 be the unique element in N_2 such that $(\lambda_1, \alpha) \geq 1$ for all $\alpha \in \Xi_1 \setminus F_1$ and $\|\lambda_1\|$ is minimal. We define $\Xi_2 = \{p_2(\alpha) \mid \alpha \in \Xi_1, \ (\lambda_1, \alpha) = 1\} \subset N_2$, where $p_2 \colon N_1 \to N_2$ is the orthogonal projection. Repeating this process, we get $\lambda_2, \ldots, \lambda_n$. The algorithm terminates when we get to $\Xi_{n+1} \subset F_{n+1}$.

A $\mathbb{Q}^\infty$-filtration of x in V/G is determined by a finite sequence of elements of N (Remark 3.2.16), and under this correspondence we have:

Theorem 1.5.1 (Theorem 5.2.16 and Corollary 5.2.18)**.** *The iterated balanced filtration of x in V/G is given by the sequence $\lambda_0, \ldots, \lambda_n$ described above.*

For the proof, we use the machinery of *chains of stacks*. A chain of stacks is the data of a sequence of pointed k-stacks $(\mathcal{X}_n, x_n)$, together with a $\mathbb{Q}$-filtration λ_n of each x_n and link maps $(\mathcal{X}_{n+1}, x_{n+1}) \to \big(\mathrm{Grad}(\mathcal{X}_n), \mathrm{gr}\,\lambda_n\big)$ (Definition 4.1.1). Associated to every chain there is a $\mathbb{Q}^\infty$-filtration of the point x_0 in $\mathcal{X}_0$ (Definition 4.1.3). For a pointed normed good moduli stack $(\mathcal{X}, x)$, we give two different constructions of chains computing sequential filtrations of $(\mathcal{X}, x)$, the *balancing chain* (Construction 4.2.1) and the *torsor chain* (Construction 4.3.1). The former is closely related to the balancing stratification, while the latter tends to be closer to combinatorial structures, such as states or lattices. We show that both chains compute the iterated balanced filtration of $(\mathcal{X}, x)$ (Proposition 4.2.6 and Theorem 4.3.4).

In order to relate the convex-geometrical picture of states to chains of stacks, we define a category of normed semistable polarised states (Definition 5.1.1) which is combinatorial in nature. We then define an analogue of the notion of chain in this category and define a canonical *balancing chain* for every object. There is a functor from normed semistable polarised states to pointed normed good moduli stacks (Definitions 5.2.1 and 5.2.9), and we show that it sends the balancing chain of a state to the torsor chain of the corresponding stack (Theorem 5.2.16).

1.6 COMPARISON WITH THE ITERATED HKKP FILTRATION

The theory of chains of stacks will be used to establish, in Chapter 7, a correspondence between the iterated balanced filtration and the iterated HKKP filtration for normed artinian lattices as defined by Haiden–Katzarkov–Kontsevich–Pandit [33]. In order to prove such a correspondence, we establish first a new characterisation of the HKKP filtration. Recall that a *lattice* L is a partially ordered set such that every pair of elements $a, b \in L$ has a supremum $a \vee b$ and an infimum $a \wedge b$. It is *modular* if $(x \vee a) \wedge b = (x \wedge b) \vee a$ for all $x, a, b \in L$ with $a \leq b$, and it is artinian if it is nonempty, modular and of finite length. A (rational) norm X on L is the data of a positive rational number $X([a, b])$ for every nontrivial interval $[a, b]$ in L, subject to some compatibility conditions (Definition 6.6.11). For an artinian lattice with minimal element 0 and maximal element 1, a $\mathbb{Q}$-filtration (resp. $\mathbb{R}$-filtration) F of L is a chain of elements $0 = a_0 < a_1 < \cdots < a_n = 1$ in L together with a chain of rational numbers (resp. real numbers) $c_1 > \cdots > c_n$. We write $F_{\geq c} = \bigvee_{c_i \geq c} a_i$ for $c \in \mathbb{R}$, and we say that F is *paracomplemented* if the interval $[F_{\geq c+1}, F_c]$ is complemented (the lattice analogue of semisimple) for all $c \in \mathbb{R}$. Haiden–Katzarkov–Kontsevich–Pandit defined a canonical $\mathbb{R}$-filtration F (the HKKP filtration) for every normed artinian lattice L, the HKKP filtration. They characterise the HKKP filtration F of L as the unique paracomplemented $\mathbb{R}$-filtration making another lattice $\Lambda(F)$ semistable. Their proof of the existence and uniqueness of the HKKP filtration is complicated, and requires minimising a certain *mass* function $m\,(\Lambda(F))$ on the set of paracomplemented $\mathbb{R}$-filtrations.

In Chapter 6 we characterise the HKKP filtration of a normed artinian lattice L as the unique minimiser of the much simpler norm function

$$\|F\|^2 := \sum_{c \in \mathbb{R}} c^2 X([F_{>c}, F_{\geq c}])$$

on the set of paracomplemented $\mathbb{R}$-filtrations. In this way, we get a simpler proof of the existence and uniqueness of the HKKP filtration, and we can deduce that the HKKP filtration is actually a $\mathbb{Q}$-filtration if the norm X is rational. The main idea is to consider maximal distributive sublattices D of L (where distributivity refers to the operations $\wedge$ and $\vee$). Every such D is isomorphic to the lattice of subrepresentations of a representation of an acyclic quiver Q with dimension vector consisting only of 1s, and every two filtrations of L factor through some D. These observations allow us to reduce to the distributive case, where the characterisation can be established directly. Maximal distributive sublattices of L can be viewed as the lattice analogue of maximal tori of reductive groups, and with this analogy our arguments resemble Kempf's proof of the existence of maximally destabilising one-parameter subgroups [53]. Our motivation to consider the norm function is that it is much closer to the theory of instability on algebraic stacks, and thus allows to make a precise link between the HKKP filtration and the balanced filtration. However, we hope that our approach to the HKKP filtration for lattices also clarifies and simplifies the original work of Haiden–Katzarkov–Kontsevich–Pandit. The construction of the HKKP filtration can be iterated, giving rise to the iterated HKKP filtration (Definition 6.6.27).

We can now state our main comparison result, the proof of which occupies the majority of Chapter 7.

Theorem 1.6.1 (special case of Corollary 7.5.11)**.** *Let k be an algebraically closed field and let $\mathcal{A}$ be a locally noetherian k-linear Grothendieck abelian category. Suppose that the moduli stack of objects $\mathcal{M}_{\mathcal{A}}$ is an algebraic stack locally of finite type over k. Let $\mathcal{X}$ be a quasi-compact open substack of $\mathcal{M}_{\mathcal{A}}$ admitting a good moduli space $\pi\colon \mathcal{X} \to X$, endowed with a linear norm on graded points. For any k-point $x \in \mathcal{X}(k)$, there is a canonically defined normed artinian lattice L_x and a canonical bijection $\mathbb{Q}^\infty\text{-}\mathrm{Filt}(L_x) \cong \mathbb{Q}^\infty\text{-}\mathrm{Filt}(\mathcal{X}, x)$ under which the iterated HKKP filtration of L_x and the iterated balanced filtration of $(\mathcal{X}, x)$ agree.*

Here, the set $\mathbb{Q}^\infty\text{-}\mathrm{Filt}(L)$ of sequential filtrations of L is defined as in the case of vector bundles above. The norm being linear means it is compatible with the underlying abelian category in a precise sense. Examples of this setup include moduli stacks of Bridgeland semistable objects, moduli stacks of semistable vector bundles on a curve C, and moduli stacks of quiver representations (Section 3.6). For abelian categories $\mathcal{A}$ satisfying suitable finiteness conditions, the algebraicity of $\mathcal{M}_{\mathcal{A}}$ has been established in Fernandez Herrero–Lennen–Makarova [29]. In that context, choices of K-theory classes of the category $\mathcal{A}$ produce open substacks $\mathcal{X} \subset \mathcal{M}_{\mathcal{A}}$ having a good moduli space and a norm on graded points [29, Theorem 4.17], so Theorem 1.6.1 applies.

If a k-point x of $\mathcal{X}$ corresponds to an object M of $\mathcal{A}$, then L_x is the lattice of subobjects $N \subset M$ such that the associated graded $N \oplus M/N$ is also a point of $\mathcal{X}$. In practice, $\mathcal{X}$ is the set of semistable objects for some stability condition on $\mathcal{A}$, and L_x is the lattice of semistable subobjects of M (of the same phase), but we do not need to mention the specific stability condition to formulate the theorem. We would however like to have a definition of the lattice L_x formulated purely in the language of algebraic stacks and that is canonical and independent of the presentation of $\mathcal{X}$ as a substack of $\mathcal{M}_{\mathcal{A}}$ for some abelian category $\mathcal{A}$. However, the structure of algebraic stack of $\mathcal{X}$ is not enough for this purpose. To explain what else is needed, recall that a map $g\colon B\mathbb{G}_{m,k} \to \mathcal{X}$ corresponds to an object M of $\mathcal{A}$ together with a grading $M = \oplus_{n\in\mathbb{Z}} M_n$. The graded points g such that $M_n = 0$ for all $n < 0$ form a closed and open substack $\mathrm{Grad}(\mathcal{X})_{\geq 0}$ of $\mathrm{Grad}(\mathcal{X})$ that we call the *lamp*. The lamp is precisely the extra data needed to define the lattice L_x intrinsically from the stack $\mathcal{X}$. From the lamp, we can recover the natural poset structure on $\mathbb{Z}\text{-}\mathrm{Filt}(\mathcal{X}, x)$ and define L_x as a specific subposet. It turns out that L_x is an artinian lattice, and since the norm on graded points of $\mathcal{X}$ is linear it endows L_x with a norm in the lattice-theoretic sense. Moreover, there are natural bijections between sets of filtrations (labelled by $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{Q}^\infty$) of $(\mathcal{X}, x)$ in the stacky sense and filtrations of L_x in the lattice-theoretic sense. Since lamps give us a way to speak about L_x intrinsically, we can formulate a more general version of Theorem 1.6.1 for good moduli stacks $\mathcal{X}$ endowed with well-behaved lamps, what we call linearly lit good moduli stacks (Definition 7.3.7). We provide a criterion (Theorem 7.4.4) for a lamp to define a linearly lit good moduli stack that we hope can be applied to all versions of constructions of stacks of objects in linear categories.

The first step in the proof of Theorem 1.6.1 is deformation theoretic. We prove that for any closed k-point y of $\mathcal{X}$ corresponding to an object E in $\mathcal{A}$, there is a (non-canonical) pointed closed immersion $\iota : (\pi^{-1}\pi(y), y) \hookrightarrow \big(\mathrm{Ext}^1(E,E)/\mathrm{Aut}(E), 0\big)$. Since $\mathrm{Ext}^1(E,E)/\mathrm{Aut}(E)$ is isomorphic to a stack of representations for some quiver Q, every point x in $\pi^{-1}\pi(y)$ corresponds to some representation N of Q. As expected, the lattice L_x associated to x is isomorphic to the poset of subrepresentations of N. Since the iterated balanced filtration is unaltered under closed immersions, this reduces us to the case of a nilpotent quiver representation. It can be shown explicitly that in that case Kempf's number $\langle \lambda, \mathcal{X}^{\max}\rangle$, for $\lambda \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, x)$, can be interpreted lattice theoretically as the *complementedness* of the filtration F^λ of L_x corresponding to λ (Proposition 5.2.14). From the equality of norms $\|F_\lambda\| = \|\lambda\|$ we deduce that the balanced filtration of $(\mathcal{X}, x)$ equals the HKKP filtration of L_x. However, showing that the full iterated balanced filtration of $(\mathcal{X}, x)$ equals the iterated HKKP filtration of L_x is more complicated, and it requires the results on chains of stacks from Chapter 4.

1.7 CONJECTURE ON ASYMPTOTICS OF FLOWS

In view of Theorem 1.6.1 and the results of Haiden–Katzarkov–Kontsevich–Pandit on asymptotics of flows in the case of quiver representations and vector bundles on a smooth projective complex algebraic curve [33, 34], we expect the iterated balanced filtration to play a role in describing asymptotics of natural flows in moduli theory. We now make this expectation precise in the case of Geometric Invariant Theory on affine spaces.

We begin by recalling the Kempf–Ness potential in a more general framework. Let G be a connected reductive algebraic group over $\mathbb{C}$, endowed with a norm on cocharacters, that is, the data of a Weyl-invariant rational inner product on the set $\Gamma^{\mathbb{Z}}(T)$ of cocharacters of a maximal torus T of G (Definition 2.3.8). Let K be a maximal compact subgroup of G. We denote $\mathfrak{g}$ the Lie algebra of G and $\mathfrak{k}$ the Lie algebra of K. We consider a smooth projective-over-affine scheme X over $\mathbb{C}$, endowed with an action of G. An ample line bundle $\mathcal{L}$ on X/G (that is, an ample line bundle L on X with a G-equivariant structure) defines an open semistable locus $(X/G)^{\mathrm{ss}} = X^{\mathrm{ss}}/G$, which is the open stratum of a Θ-stratification of X/G (Theorem 2.6.4). The line bundle $\mathcal{L}$ restricts to a line bundle $\overline{\mathcal{L}}$ on the differentiable stack "of metrics" X/K, and we endow $\overline{\mathcal{L}}$ with a hermitian norm $\|-\|$ (that is, we endow L with a K-invariant hermitian norm). For a point $x \in X(\mathbb{C})$, fixing a choice of nonzero lift x^* of x to the total space of $\overline{\mathcal{L}}$, the *Kempf–Ness potential* is defined to be

$$p_x : G/K \to \mathbb{R} : Kg \mapsto \log\|x^*\| - \log\|gx^*\|.$$

Here, G/K denotes the quotient by the action of K on G on the left. Note that the Kempf–Ness potential is independent of the choice of lift x^* of x. Kempf–Ness type theorems state that, under some conditions, the following equivalences hold:

1. x is semistable if and only if p_x is bounded below.
2. x is polystable if and only if p_x attains a minimum.

This is the case, for example, if either X is projective [31, 52, 55] or X is affine and some additional conditions are satisfied [43, 54, 63]. In these cases, if x is polystable, then the negative gradient flow of p_x converges to a minimum from any starting point. In the strictly semistable case, we are interested in understanding the asymptotic behaviour of this flow.

In order to define the gradient flow, we need a Riemannian metric on G/K. This comes from the norm on cocharacters of G, which gives a K-invariant euclidean inner product on $\mathfrak{k}$. By Hadamard's theorem, the map

$$\varphi : \mathfrak{k} \to G/K : v \mapsto K \exp\left(\frac{1}{2i} v\right)$$

is a diffeomorphism (see [31, Appendix A]). The isomorphism $d_0\varphi : \mathfrak{k} \to T_e(G/K)$ induces an inner product on $T_e(G/K)$, and we extend it to a Riemannian metric on G/K using the action of G on G/K by right translations (see [31, Appendix C] for details). This allows us to define the gradient vector field ∇p_x on G/K.

The semistable locus X^{ss}/G is a good moduli stack, endowed with a norm on graded points coming from the norm on cocharacters of G. If $x \in X(\mathbb{C})$ is a semistable point, then the iterated balanced filtration $\lambda_{\mathrm{ib}}(x)$ of x is defined, and by Remark 3.2.16 it is identified with an equivalence class of sequences of commuting rational one-parameter subgroups $\lambda_1, \dots, \lambda_n$. We may choose representatives λ_j that are compatible with K, in the sense that for some power $\lambda_j^l \colon \mathbb{C}^\times \to G$ that is integral (and hence for any such power), the inclusion $\lambda_j^l(S^1) \subset K$ holds. The λ_j induce well-defined linear maps on Lie algebras $\mathrm{Lie}(\lambda_j) \colon \mathbb{C} \to \mathfrak{g}$, giving elements $\nu_j = i\,\mathrm{Lie}(\lambda_j)(1) \in \mathfrak{k}$.

We now shift our attention to a particular case of the above setting. We will take $X = V$ to be a finite dimensional G-representation. Let $\alpha \colon G \to \mathbb{G}_{m,\mathbb{C}}$ be a character, and take the linearisation $\mathcal{L} = \mathcal{O}_{X/G}(\alpha) = (X/G \to B\mathbb{G}_{m,\mathbb{C}})^*\alpha$, where we regard α as an element of $\mathrm{Pic}(BG)$. Choose a K-invariant hermitian metric $\|-\|$ on V. The total space of $\mathcal{L}$ is $(V \times \mathbb{C})/G$, where the action is $g(x,c) = (gx, \alpha(g)c)$. The norm $\|-\|$ defines a hermitian norm $\|-\|_{\overline{\mathcal{L}}}$ on $\overline{\mathcal{L}}$ by the formula

$$\|(v,c)\|_{\overline{\mathcal{L}}} = e^{-\|v\|^2}\,|c|\,.$$

The associated Kempf–Ness potential for a point $x \in V$ is

$$p_x(Kg) = \|gx\|^2 - \log|\alpha(g)| - \|x\|^2.$$

Suppose that $x \in V$ is semistable and let $\nu_1, \dots, \nu_n \in \mathfrak{k}$ represent the iterated balanced filtration of x as above. We denote $\log = \varphi^{-1} \colon G/K \to \mathfrak{k}$ the inverse of the map φ defined above. With this setup, we conjecture:

Conjecture 1.7.1. *Let $h \colon (0,\infty) \to G/K$ be a flow line for $-\nabla p_x$. Then the expression*

$$\log h(t) + \log(t)\nu_1 + \log\log(t)\nu_2 + \cdots + \underbrace{\log\cdots\log}_{n}(t)\nu_n$$

in $\mathfrak{k}$ is bounded for $t \gg 0$.

In the case where $V = \bigoplus_{a \in Q_1} \mathrm{Hom}(\mathbb{C}^{d_{s(a)}}, \mathbb{C}^{d_{t(a)}})$ is the representation space of a quiver Q with dimension vector d and $G = \prod_{i \in Q_0} \mathrm{GL}_{d_i,\mathbb{C}}$ with the standard action, the conjecture is true for specific choices of hermitian norm on V and norm on cocharacters of G by Theorem 1.6.1 and [33, Theorem 5.11]. See Example 5.2.19 for a simple example where the conjecture is checked beyond the quiver case. We hope to return to Conjecture 1.7.1 in future work.

Our expectation is that the iterated balanced filtration describes also the asymptotics of natural gradient flows in other moduli problems. Examples include the

Calabi flow for a K-semistable Fano variety, the Yang–Mills flow for semistable G-bundles on a smooth projective curve and the gradient flow for the Kempf–Ness potential in more general GIT situations. In all these examples, there is an underlying normed good moduli stack of semistable objects, so the iterated balanced filtration is defined (Section 3.6).

1.8 NOTATION AND CONVENTIONS

The set of nonnegative integers, or natural numbers, is denoted $\mathbb{N} = \mathbb{Z}_{\geq 0}$. We will also denote $\mathbb{N}^* = \mathbb{Z}_{>0}$.

We follow the definitions and conventions of [72] regarding algebraic stacks and algebraic spaces. We denote $\mathbf{St}_{\mathrm{fppf}}$ the 2-category of stacks for the fppf site of schemes. For an algebraic stack $\mathcal{X}$, we denote $|\mathcal{X}|$ its topological space and $\pi_0(\mathcal{X})$ the set of connected components of $|\mathcal{X}|$. By a *geometric point* of $\mathcal{X}$ we mean a morphism $\operatorname{Spec} k \to \mathcal{X}$ where k is an algebraically closed field. If $x\colon T \to \mathcal{X}$ a T-point, with T a scheme, we denote $\operatorname{Aut}(x)$ the automorphism group of x as a group algebraic space over T. The multiplicative group over $\mathbb{Z}$ is denoted $\mathbb{G}_m = \operatorname{Spec} \mathbb{Z}[t, t^{-1}]$ and, for a scheme T, we use the notation $\mathbb{G}_{m,T} = \mathbb{G}_m \times T$.

For a flat and finitely presented group scheme G over a base S and an algebraic space X over S endowed with a G-action, we use the notation X/G for the quotient stack, omitting the customary brackets. The *classifying stack* BG *of* G is the quotient $BG := S/G$, where S is endowed with the trivial G-action.

For a field k and an algebraic group G over k, we denote $\Gamma_{\mathbb{Z}}(G), \Gamma^{\mathbb{Z}}(G), \Gamma_{\mathbb{Q}}(G)$ and $\Gamma^{\mathbb{Q}}(G)$ the sets of characters, cocharacters, rational characters and rational cocharacters of G, respectively. If $\lambda \in \Gamma^{\mathbb{Z}}(G)$ is a cocharacter, and G acts on a scheme X over k, then we denote $X^{\lambda,0}$ the fixed point locus of the induced $\mathbb{G}_m$-action on X and $X^{\lambda,+}$ the *attractor*, defined functorially on k-schemes T by the formula $\operatorname{Hom}(T, X^{\lambda,+}) = \operatorname{Hom}^{\mathbb{G}_{m,k}}(\mathbb{A}^1_T, X)$, where $\operatorname{Hom}^{\mathbb{G}_{m,k}}$ denotes $\mathbb{G}_{m,k}$-equivariant maps and $\mathbb{A}^1_T$ is endowed with the usual scaling action [26]. For the particular case of the conjugation action of G on itself, we denote $L(\lambda) = G^{\lambda,0}$ and $P(\lambda) = G^{\lambda,+}$. If G is reductive, then $P(\lambda)$ is a parabolic subgroup with Levi factor $L(\lambda)$. If $g \in G(k)$, we denote $\lambda^g = g\lambda g^{-1}$.

If $\mathcal{F}$ is a vector bundle on an algebraic stack $\mathcal{X}$, the total space of $\mathcal{F}$ is $\mathbb{A}(\mathcal{F}) := \operatorname{Spec}_{\mathcal{X}} \operatorname{Sym}_{\mathcal{O}_{\mathcal{X}}} \mathcal{F}^\vee$ and the associated projective bundle is $\mathbb{P}(\mathcal{F}) := \operatorname{Proj}_{\mathcal{X}} \operatorname{Sym}_{\mathcal{O}_{\mathcal{X}}} \mathcal{F}^\vee$.

CHAPTER 2

PRELIMINARIES

In this chapter we start by recalling, mainly following [36], the kind of stability structures on algebraic stacks that we will use in the rest of this thesis and how they give rise to stratifications. The two main concepts are that of a *norm on graded points* (Definition 2.3.3) and of a *linear form on graded points* (Definition 2.4.1), and these give rise to Θ-*stratifications* (Definition 2.5.1). While in [36] the stacks $\mathrm{Grad}(\mathcal{X})$ of graded points and $\mathrm{Filt}(\mathcal{X})$ of filtrations of an algebraic stack $\mathcal{X}$ are used, in this work we will need a generalisation of these, what we call the stack of *rational graded points* $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ and the stack of *rational filtrations* $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$, that we define in Section 2.2.

The main result in this chapter is Theorem 2.6.4, where we prove that, for a representable projective morphism $f\colon \mathcal{Y} \to \mathcal{X}$ into a stack $\mathcal{X}$ with a good moduli space and a norm on graded points, the stack $\mathcal{Y}$ carries a natural Θ-stratification $(\mathcal{S}_c)_{c\in\mathbb{Q}_{\geq 0}}$ whose centres $(\mathcal{Z}_c)_{c\in\mathbb{Q}_{\geq 0}}$ have good moduli spaces. This is an improvement of Halpern-Leistner's result [36, Theorem 5.5.10] that $\mathcal{Y}$ has a weak Θ-stratification whose semistable locus $\mathcal{Z}_0$ has a good moduli space. Existence of good moduli spaces for all centres $\mathcal{Z}_c$ will be fundamental in our construction of the balancing stratification (Theorem 3.5.2).

We conclude by recalling the combinatorial notion of *formal fan* from [36], and how it can be used to encode natural extra structure on the set of filtrations of a point in a stack. This will be needed for Chapters 6 and 7.

2.1 GOOD MODULI SPACES AND LOCAL STRUCTURE THEOREMS

We start by recalling the definition of good moduli space from [2, Definition 4.1], with the slightly modified conventions of [7, 1.7.3, 1.7.4].

Definition 2.1.1 (Good moduli space)**.** A morphism $\pi\colon \mathcal{X} \to X$ from an algebraic stack $\mathcal{X}$ to an algebraic space X is said to be a *good moduli space* if

1. the map π is quasi-compact and quasi-separated;
2. the map $\mathcal{O}_X \to \pi_* \mathcal{O}_{\mathcal{X}}$ is an isomorphism; and
3. the pushforward functor π_* on quasi-coherent sheaves is exact, and the same is true after any base change $X' \to X$, where X' is an algebraic space.

If X is quasi-separated, then the third condition can be simplified to π_* being exact, the statement for any base change being then automatic by [2, Proposition 3.10, (vii)]. We will often use the term *good moduli stack* meaning an algebraic stack $\mathcal{X}$ that admits a good moduli space $\pi: \mathcal{X} \to X$.

Remark **2.1.2.** Using good moduli spaces, we can recover the concept of linear reductivity. Indeed, an affine algebraic group G over a field k is linearly reductive precisely when the map $BG \to \operatorname{Spec} k$ is a good moduli space.

A good moduli space $\pi: \mathcal{X} \to X$ enjoys many special properties, for example:

1. Any map $\mathcal{X} \to Y$ with Y an algebraic space factors uniquely through π [7, Theorem 3.12]. In particular, the good moduli space π is uniquely determined by $\mathcal{X}$.
2. Any base change of π along a morphism $X' \to X$ with X' an algebraic space is a good moduli space [2, Proposition 4.7, (i)].
3. If $h: \mathcal{X}' \to \mathcal{X}$ is an affine morphism, then $\mathcal{X}'$ has a good moduli space $\mathcal{X}' \to X'$ and the induced map $X' \to X$ is affine with $X' = \operatorname{Spec}_X \pi_* h_* \mathcal{O}_{\mathcal{X}'}$ [2, Lemma 4.14]. In particular, if h is a closed immersion, then so is $X' \to X$.
4. For every point $p \in |X|$, the fibre $\pi^{-1}(p)$ has a unique closed point q, and the dimension of the stabiliser of q is bigger than that of any other point of $\pi^{-1}(p)$ [2, Proposition 9.1]. Moreover, the stabiliser of q is linearly reductive [2, Proposition 12.14]. The points $q \in |\mathcal{X}|$ that are closed in the fibre of π containing q are said to be *polystable*. More generally, a field-valued point $x: \operatorname{Spec} k \to \mathcal{X}$ is said to be *polystable* if x is closed in the fibre $\pi^{-1}\pi(x)$ or, equivalently, if the point of $|\mathcal{X}|$ underlying x is polystable.

Stacks with good moduli spaces are étale locally quotient stacks. More precisely:

Theorem 2.1.3 (Local structure [7])**.** *Let $\mathcal{X}$ be an algebraic stack and $\pi: \mathcal{X} \to X$ a good moduli space. Assume that $\mathcal{X}$ is of finite presentation over a quasi-compact and quasi-separated algebraic space B and that $\mathcal{X}$ has affine diagonal.*

Then there is a natural number n, an affine scheme $\operatorname{Spec} A$ *endowed with an action of* GL_n*,* *and a cartesian square*

$$\begin{array}{ccc} (\operatorname{Spec} A)/\mathrm{GL}_n & \longrightarrow & \mathcal{X} \\ \big\downarrow & \lrcorner & \big\downarrow \pi \\ \operatorname{Spec}(A^{\mathrm{GL}_n}) & \xrightarrow{h} & X \end{array}$$

with h an affine Nisnevich cover (in particular, étale). Here, A^{GL_n} denotes the ring of invariants. Moreover, $X \to B$ is of finite presentation and X has affine diagonal.

The theorem is [7, Theorem 6.1], together with the argument at the end of the proof of [7, Theorem 5.3] to guarantee that h can be taken to be affine. To see that X has affine diagonal, just take good moduli spaces for the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$, which is affine, to obtain the diagonal of X.

From Theorem 2.1.3, it follows that stacks whose good moduli space is a point are necessarily quotient stacks.

Corollary 2.1.4. *Let $\mathcal{X}$ be an algebraic stack of finite presentation over a field k and assume that $\pi\colon \mathcal{X} \to \operatorname{Spec} k$ is a good moduli space with affine diagonal. Then $\mathcal{X} \cong (\operatorname{Spec} A)/\mathrm{GL}_n$, where A is a k-algebra of finite type and $\operatorname{Spec} A$ is endowed with a GL_n-action.*

Over an algebraically closed field, there is a stronger result.

Corollary 2.1.5 (of [6, Theorem 4.12])**.** *Let $\mathcal{X}$ be an algebraic stack of finite presentation over an algebraically closed field k and suppose that $\pi\colon \mathcal{X} \to \operatorname{Spec} k$ is a good moduli space with affine diagonal. Let $x \in \mathcal{X}(k)$ be the unique closed k-point of $\mathcal{X}$ and let G be the stabiliser of x. Then $\mathcal{X} \cong (\operatorname{Spec} A)/G$, where A is a finite type k-algebra and $\operatorname{Spec} A$ is endowed with an action of G.*

Although it will not be used in this thesis, we finish this section with a relative version of the Luna étale slice theorem, recently proved in joint work of the author with Mark Andrea de Cataldo and Andres Fernandez Herrero [23, Theorem 2.1].

Theorem 2.1.6. *Let k be an algebraically closed field, and let $f\colon \mathcal{X} \to \mathcal{Y}$ be a smooth morphism between smooth algebraic stacks over k. Let $x \in \mathcal{X}(k)$ be a closed k-point mapping to a closed k-point y of $\mathcal{Y}$. Suppose that the stabilisers G_x and G_y of x and y are linearly reductive and that G_y is smooth. Denote N_x and N_y the normal spaces of $\mathcal{X}$ at x and of $\mathcal{Y}$ at y, endowed with respective actions of G_x and G_y. Then there is a commutative diagram*

$$\begin{array}{ccccc} (N_x/G_x, 0) & \longleftarrow & (\mathcal{U}, u) & \longrightarrow & (\mathcal{X}, x) \\ \big\downarrow & & \big\downarrow & & \big\downarrow f \\ (N_y/G_y, 0) & \longleftarrow & (\mathcal{V}, v) & \longrightarrow & (\mathcal{Y}, y) \end{array}$$

of pointed stacks where the horizontal arrows are étale and induce isomorphisms of stabiliser groups at the respective points. Here, the left vertical map is the one naturally induced by f.

We recall the definition of the normal space N_x. The point x defines a closed immersion $i\colon BG_x \to \mathcal{X}$, its residual gerbe, with ideal sheaf $\mathcal{I} \leq \mathcal{O}_{\mathcal{X}}$. The *normal space* N_x is the representation of G_x corresponding to the coherent sheaf $i^*(\mathcal{I})^\vee = (\mathcal{I}/\mathcal{I}^2)^\vee$ on BG_x. We will also use the normal space in Section 7.4.1.

2.2 STACKS OF RATIONAL FILTRATIONS AND GRADED POINTS

In [36], Halpern-Leistner defines the stacks $\mathrm{Grad}(\mathcal{X})$ of graded points and $\mathrm{Filt}(\mathcal{X})$ of filtrations of an algebraic stack $\mathcal{X}$ as mapping stacks. If $\mathcal{X}$ parametrises objects in an abelian category, then $\mathrm{Grad}(\mathcal{X})$ parametrises objects endowed with a $\mathbb{Z}$-grading and $\mathrm{Filt}(\mathcal{X})$ parametrises objects endowed with a $\mathbb{Z}$-filtration [8, Proposition 7.12 and Corollary 7.13], but $\mathrm{Grad}(\mathcal{X})$ and $\mathrm{Filt}(\mathcal{X})$ can be defined for very general $\mathcal{X}$. In this section we revisit the construction of $\mathrm{Grad}(\mathcal{X})$ and $\mathrm{Filt}(\mathcal{X})$, and extend it to consider rational filtrations and gradings.

Following [36], we define the stack Θ over $\mathrm{Spec}(\mathbb{Z})$ to be the quotient stack $\Theta = \mathbb{A}^1_{\mathbb{Z}}/\mathbb{G}_{m,\mathbb{Z}}$, the action of $\mathbb{G}_{m,\mathbb{Z}}$ on $\mathbb{A}^1_{\mathbb{Z}}$ being the usual scaling action. For an algebraic space S we denote $\Theta_S = \Theta \times S$.

Recall that for $\mathcal{Y}$ and $\mathcal{Z}$ two stacks over a base space S, an object of the *mapping stack* $\underline{\mathrm{Hom}}_S(\mathcal{Y},\mathcal{Z})$ over a scheme T is a map $T \to S$ together with a morphism $T \times_S \mathcal{Y} \to \mathcal{Z}$ over S. In the case $S = \mathrm{Spec}(\mathbb{Z})$, we omit the subindex from the notation. The following definition is in [36, Section 1.1], except that we do not work relative to a base algebraic stack.

DEFINITION 2.2.1 (Stacks of filtrations and graded points)**.** Let $\mathcal{X}$ be an algebraic stack and let n be a positive integer. We define the stack $\mathrm{Grad}^n(\mathcal{X})$ of $\mathbb{Z}^n$*-graded points* of $\mathcal{X}$ to be the mapping stack $\mathrm{Grad}^n(\mathcal{X}) := \underline{\mathrm{Hom}}(B\mathbb{G}^n_{m,\mathbb{Z}}, \mathcal{X})$. Similarly, we define the stack $\mathrm{Filt}^n(\mathcal{X})$ of $\mathbb{Z}^n$*-filtrations* of $\mathcal{X}$ to be $\mathrm{Filt}^n(\mathcal{X}) := \underline{\mathrm{Hom}}(\Theta^n, \mathcal{X})$.

We will simply denote $\mathrm{Filt}(\mathcal{X}) = \mathrm{Filt}^1(\mathcal{X})$ and $\mathrm{Grad}(\mathcal{X}) = \mathrm{Grad}^1(\mathcal{X})$.

LEMMA 2.2.2 (Independence of base for mapping stacks)**.** *Let $\mathcal{Y}$ be an algebraic stack such that $\mathcal{Y} \to \mathrm{Spec}\,\mathbb{Z}$ is a good moduli space, and let $\mathcal{X}$ be an algebraic stack defined over an algebraic space B. Then there is a canonical isomorphism*

$$\underline{\mathrm{Hom}}(\mathcal{Y},\mathcal{X}) \cong \underline{\mathrm{Hom}}_B(\mathcal{Y}_B,\mathcal{X})$$

of mapping stacks. In particular, there is a canonical map $\underline{\mathrm{Hom}}(\mathcal{Y},\mathcal{X}) \to B$.

Proof. For a scheme T, an object of the groupoid $\underline{\mathrm{Hom}}_B(\mathcal{Y}_B, \mathcal{X})(T)$ is a pair (a, b) with $b\colon T \to B$ and $a\colon T \times_B \mathcal{Y}_B = T \times \mathcal{Y} \to \mathcal{X}$ a morphism over B. Since $T \times \mathcal{Y} \to T$ is a good moduli space, for any given $a\colon T \times \mathcal{Y} \to \mathcal{X}$, an object of $\underline{\mathrm{Hom}}(\mathcal{Y}, \mathcal{X})(T)$, the composition $T \times \mathcal{Y} \to \mathcal{X} \to B$ factors uniquely through $T \times \mathcal{Y} \to T$, giving a unique $b\colon T \to B$ such that (a, b) is an object of $\underline{\mathrm{Hom}}_B(\mathcal{Y}_B, \mathcal{X})(T)$. □

Applying the Lemma for $\mathcal{Y} = \Theta^n$ or $\mathcal{Y} = B\mathbb{G}_m^n$, we see that $\mathrm{Filt}^n(\mathcal{X})$ and $\mathrm{Grad}^n(\mathcal{X})$ are independent of the base algebraic space considered.

To guarantee that $\mathrm{Filt}^n(\mathcal{X})$ and $\mathrm{Grad}^n(\mathcal{X})$ are well-behaved, we consider the following assumption on an algebraic stack $\mathcal{X}$ defined over an algebraic space B.

Assumption **2.2.3.** The algebraic space B is quasi-separated and locally noetherian, and the map $\mathcal{X} \to B$ is locally finitely presented and has affine diagonal.

Example **2.2.4.** Suppose that $\mathcal{X}$ is a noetherian algebraic stack with affine diagonal and $\pi\colon \mathcal{X} \to X$ is a good moduli space. Then X is also noetherian by [2, Theorem 4.16], and π is of finite type by [6, Theorem A.1]. The diagonal of X is affine since it is obtained from the diagonal of $\mathcal{X}$ by taking good moduli spaces. In particular, $\mathcal{X}$ satisfies Assumption 2.2.3 with $B = X$.

Under Assumption 2.2.3, the stacks $\mathrm{Filt}^n(\mathcal{X})$ and $\mathrm{Grad}^n(\mathcal{X})$ are algebraic and also satisfy Assumption 2.2.3 [7, Theorem 6.22]. Note that by [7, Remark 6.16] it is not necessary to assume that B is excellent in order to apply [7, Theorem 6.22], since the stacks $B\mathbb{G}^n_{m,B}$ and Θ^n_B satisfy condition (N) in [7]. See also [39, Theorem 5.1.1] for a related algebraicity result.

There are several maps relating $\mathrm{Grad}(\mathcal{X})$, $\mathrm{Filt}(\mathcal{X})$ and $\mathcal{X}$:

1. The "evaluation at 1" map $\mathrm{ev}_1\colon \mathrm{Filt}(\mathcal{X}) \to \mathcal{X}$, defined by precomposition along $\{1\} \to \Theta$. It is representable and separated [36, Proposition 1.1.13].
2. The "associated graded" map $\mathrm{gr}\colon \mathrm{Filt}(\mathcal{X}) \to \mathrm{Grad}(\mathcal{X})$, defined by precomposition along $B\mathbb{G}_m = \{0\}/\mathbb{G}_m \to \Theta$.
3. The "forgetful" map $u\colon \mathrm{Grad}(\mathcal{X}) \to \mathcal{X}$, defined by precomposition along $\mathrm{Spec}(\mathbb{Z}) \to B\mathbb{G}_m$.
4. The "evaluation at 0" map $\mathrm{ev}_0\colon \mathrm{Filt}(\mathcal{X}) \to \mathcal{X}$, which is the composition $\mathrm{ev}_0 = u \circ \mathrm{gr}$.
5. The "split filtration" map $\sigma\colon \mathrm{Grad}(\mathcal{X}) \to \mathrm{Filt}(\mathcal{X})$, defined by precomposition along the canonical representable morphism $\Theta \to B\mathbb{G}_m$.
6. The "trivial grading" map $\mathcal{X} \to \mathrm{Grad}(\mathcal{X})$, given by precomposition along $B\mathbb{G}_m \to \mathrm{Spec}\,\mathbb{Z}$. It is an open and closed immersion.

7. The "trivial filtration" map $\mathcal{X} \to \mathrm{Filt}(\mathcal{X})$, defined by precomposing along $\Theta \to \operatorname{Spec} \mathbb{Z}$. It is an open and closed immersion.

Remark **2.2.5.** The fact that the "trivial grading" and "trivial filtration" maps are closed and open immersions follows from [36, Proposition 1.3.9] by the argument in [36, Proposition 1.3.11].

Remark **2.2.6.** Sometimes, the assumption that $\mathcal{X} \to B$ has affine diagonal can be relaxed to $\mathcal{X}$ having affine stabilisers and being quasi-separated over B. This is the case for the construction and algebraicity of $\mathrm{Filt}^n_{\mathbb{Q}}(\mathcal{X})$, $\mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X})$, $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ and $\mathrm{Grad}_{\mathbb{Q}^\infty}(\mathcal{X})$ (Definition 2.2.7 and Definition 3.2.2). Representability of $\mathrm{ev}_1 \colon \mathrm{Filt}(\mathcal{X}) \to \mathcal{X}$ follows under the additional hypothesis that $\mathcal{X}$ has separated inertia.

The assumption that B is locally noetherian guarantees that the topological spaces of the algebraic stacks considered are locally connected, and hence their connected components are open.

The monoid $(\mathbb{N}^*, \cdot, 1)$ acts[1] on the stacks $\mathrm{Filt}(\mathcal{X})$ and $\mathrm{Grad}(\mathcal{X})$. A natural number $n > 0$ acts on $\mathrm{Filt}(\mathcal{X})$ by the map $\mathrm{Filt}(\mathcal{X}) \xrightarrow{(\bullet)^n} \mathrm{Filt}(\mathcal{X})$ given by precomposition along the nth power map $\Theta_T \to \Theta_T$, and similarly in the case of $\mathrm{Grad}(\mathcal{X})$. Now denote $\mathcal{Y}$ one of the stacks $\mathrm{Grad}(\mathcal{X})$ or $\mathrm{Filt}(\mathcal{X})$ with its $(\mathbb{N}^*, \cdot, 1)$-action. We define a diagram $D_{\mathcal{Y}} \colon (\mathbb{N}^*, |) \to \mathbf{St}_{\mathrm{fppf}}$, i.e. a pseudofunctor, on the 2-category $\mathbf{St}_{\mathrm{fppf}}$ of stacks for the fppf site of schemes. The index category is the filtered poset $(\mathbb{N}^*, |)$ of positive integers with the divisibility order, and $D_{\mathcal{Y}}$ is defined by setting $D_{\mathcal{Y}}(n) = \mathcal{Y}$ for all n, and $D_{\mathcal{Y}}(n|m)$ to be the "rising to the $\frac{m}{n}$th power" map $\mathcal{Y} \xrightarrow{(\bullet)^{m/n}} \mathcal{Y}$ defined above.

Definition 2.2.7 (Stacks of rational filtrations and rational graded points)**.** The stacks $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ of *rational filtrations* and $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ of *rational graded points* are the colimits

$$\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) := \operatorname{colim} D_{\mathrm{Filt}(\mathcal{X})} \quad \text{and} \quad \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}) := \operatorname{colim} D_{\mathrm{Grad}(\mathcal{X})}$$

in the cocomplete 2-category $\mathbf{St}_{\mathrm{fppf}}$.

Remark **2.2.8.** There are also maps $\mathrm{ev}_1 \colon \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) \to \mathcal{X}$, $\mathrm{gr} \colon \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) \to \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$, etcetera, relating the stacks $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$, $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ and $\mathcal{X}$, just because the version of these maps for Filt and Grad are compatible with the colimits defining $\mathrm{Filt}_{\mathbb{Q}}$ and $\mathrm{Grad}_{\mathbb{Q}}$.

[1] Formally, an action of $(\mathbb{N}^*, \cdot, 1)$ on a stack $\mathcal{Y}$ is a pseudofunctor $B(\mathbb{N}^*, \cdot, 1) \to \mathbf{St}_{\mathrm{fppf}}$ sending the unique object of $B(\mathbb{N}^*, \cdot, 1)$ to $\mathcal{Y}$. Here, we are denoting $B(\mathbb{N}^*, \cdot, 1)$ the category with one object and endomorphism monoid equal to $(\mathbb{N}^*, \cdot, 1)$, and $\mathbf{St}_{\mathrm{fppf}}$ is the 2-category of stacks on the category of schemes with the fppf topology.

PROPOSITION 2.2.9. *Let $\mathcal{X}$ be an algebraic stack over an algebraic space B, satisfying Assumption 2.2.3. Then* $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ *and* $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ *are algebraic and satisfy Assumption 2.2.3.*

Proof. By [36, Proposition 1.3.11], the "rising to the nth power" map $\mathrm{Filt}(\mathcal{X}) \to \mathrm{Filt}(\mathcal{X})$ is a closed and open immersion for $n > 0$. The same argument shows that the analogue map $\mathrm{Grad}(\mathcal{X}) \to \mathrm{Grad}(\mathcal{X})$ is a closed and open immersion too. Thus the algebraicity result follows from Lemma 2.2.10. Since $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ and $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ are increasing unions of closed and open substacks isomorphic to $\mathrm{Filt}(\mathcal{X})$ and $\mathrm{Grad}(\mathcal{X})$ respectively, they also satisfy Assumption 2.2.3. □

LEMMA 2.2.10. *Let I be a filtered poset, seen as a category, and let $D: I \to \mathbf{St}_{\mathrm{fppf}}$ be a pseudofunctor such that for all arrows $s \to t$ in I the induced $D(s) \to D(t)$ is representable by open immersions. Let $\mathcal{Y} = \operatorname{colim} D$ in the 2-category $\mathbf{St}_{\mathrm{fppf}}$. If $D(s)$ is algebraic for all objects s of D, then so is $\mathcal{Y}$.*

Proof. In this proof, we consider the site of *affine* schemes with the fppf topology. This does not change the 2-category $\mathbf{St}_{\mathrm{fppf}}$ of stacks, but it will be useful to consider only quasi-compact test schemes.

The stack $\mathcal{Y}$ is the stackification of the colimit $\mathcal{Y}^{\mathrm{pre}} = \operatorname{colim} D$ in the 2-category of prestacks (meaning presheaves of groupoids). A morphism $T \to \mathcal{Y}^{\mathrm{pre}}$ is a pair (s, f), with s an object of I and $f: T \to D(s)$ a map. A 2-morphism $(s, f) \to (s', f')$ is a pair (t, r) with $t \in I$ such that there are arrows $t/s: s \to t$ and $t/s': s' \to t$, and $r: D(t/s) \circ f \to D(t/s') \circ f'$ a 2-morphism. From this description, and using the facts that (1) I is filtered and (2) every object in the site considered, i.e. every affine scheme, is quasi-compact, it follows that $\mathcal{Y}^{\mathrm{pre}}$ is already a stack, so $\mathcal{Y} = \mathcal{Y}^{\mathrm{pre}}$. Moreover, each of the maps $D(i) \to \mathcal{Y}$ is an open immersion. Indeed, if $(s, f): T \to \mathcal{Y}$ is a map, with T affine, then $D(i) \times_{\mathcal{Y}} T = D(i) \times_{D(s')} T$ if $i, s \leq s'$, which is open in T. Thus $\bigsqcup_{s \in I} D(s) \to \mathcal{Y}$ is a smooth representable surjection, so $\mathcal{Y}$ is algebraic. □

Remark **2.2.11** (Functor of points of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ and $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$). From the proof of Lemma 2.2.10 we get a simple description of points in $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ and $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$. Namely, if T is a quasi-compact scheme, then a T-point of $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ will be denoted as $\frac{1}{n}\lambda$, where λ is a T-point of $\mathrm{Filt}(\mathcal{X})$ and n is a positive integer. An isomorphism between T-points $\frac{1}{n}\lambda$ and $\frac{1}{n'}(\lambda')$ of $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ is an isomorphism between $n'\lambda$ and $n(\lambda')$ in $\mathrm{Filt}(\mathcal{X})$. Here we are using additive notation for the "rising to the nth power" maps. A similar description applies to $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$.

Remark **2.2.12.** Since $(\mathbb{N}, \cdot, 1)$ also acts on $\mathrm{Filt}^n(\mathcal{X})$ and $\mathrm{Grad}^n(\mathcal{X})$ for all positive integers n, we can form the stacks $\mathrm{Filt}^n_{\mathbb{Q}}(\mathcal{X})$ and $\mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X})$ in a similar fashion. For the same reasons, they are algebraic and satisfy Assumption 2.2.3.

Example **2.2.13.** Let k be a field, let X be a separated scheme of finite type over k, endowed with an action of a smooth affine algebraic group G over k that admits a k-split maximal torus T. We denote $W = N_G(T)/Z_G(T)$ the Weyl group. Then we have natural isomorphisms

$$\mathrm{Grad}^n(X/G) = \bigsqcup_{\lambda \in \mathrm{Hom}(\mathbb{G}^n_{m,k}, T)/W} X^{\lambda,0}/L(\lambda)$$

and

$$\mathrm{Filt}^n(X/G) = \bigsqcup_{\lambda \in \mathrm{Hom}(\mathbb{G}^n_{m,k}, T)/W} X^{\lambda,+}/P(\lambda)$$

by [36, Theorem 1.4.8]. Here $L(\lambda)$, $P(\lambda)$, $X^{\lambda,0}$ and $X^{\lambda,+}$ are the centraliser, associated parabolic subgroup, fixed point locus and attracting locus of λ, see Section 1.8 for the precise definitions. The same description holds for $\mathrm{Grad}^n_{\mathbb{Q}}(X/G)$ and $\mathrm{Filt}^n_{\mathbb{Q}}(X/G)$ when replacing $\mathrm{Hom}(\mathbb{G}^n_{m,k}, T)/W$ by $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Hom}(\mathbb{G}^n_{m,k}, T)/W$. This can be seen by identifying the "rising to the nth power" maps with the "scaling by n" in $\mathrm{Hom}(\mathbb{G}^n_{m,k}, T)$. We will use this throughout.

Over a general base B, this description still holds for a quotient stack of the form X/GL_N with X an algebraic space that is quasi-separated and locally finitely presented over B [36, Theorem 1.4.7].

The formation of $\mathrm{Filt}^n_{\mathbb{Q}}(\mathcal{X})$ and $\mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X})$ is well-behaved with respect to base change from a target algebraic space.

Proposition 2.2.14. *Let $\mathcal{X} \to B$ and $\mathcal{X}' \to B'$ satisfy Assumption 2.2.3 and let*

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ \downarrow & \lrcorner & \downarrow \\ X' & \longrightarrow & X \end{array}$$

be a cartesian square with X and X' algebraic spaces. Then

$$\mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X}') \cong \mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X}) \times_{\mathcal{X}} \mathcal{X}' \cong \mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X}) \times_X X'$$

and

$$\mathrm{Filt}^n_{\mathbb{Q}}(\mathcal{X}') \cong \mathrm{Filt}^n_{\mathbb{Q}}(\mathcal{X}) \times_{\mathrm{ev}_1, \mathcal{X}} \mathcal{X}' \cong \mathrm{Filt}^n_{\mathbb{Q}}(\mathcal{X}) \times_X X'$$

for all n. The same holds for Filt^n *and* Grad^n.

Proof. The case of $\mathrm{Filt}^n(\mathcal{X}')$ and $\mathrm{Grad}^n(\mathcal{X}')$ is [36, Corollary 1.3.17]. The result follows for $\mathrm{Filt}^n_{\mathbb{Q}}(\mathcal{X}')$ and $\mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X}')$ after covering these by copies of $\mathrm{Filt}^n(\mathcal{X}')$ and $\mathrm{Grad}^n(\mathcal{X}')$. □

Proposition 2.2.15. *Let $\mathcal{X}$ be an algebraic stack defined over an algebraic space B, satisfying Assumption 2.2.3, and let $\mathcal{X}' \to \mathcal{X}$ be a closed immersion. Then*

$$\mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X}') \cong \mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X}) \times_{\mathcal{X}} \mathcal{X}'$$

and

$$\mathrm{Filt}^n_{\mathbb{Q}}(\mathcal{X}') \cong \mathrm{Filt}^n_{\mathbb{Q}}(\mathcal{X}) \times_{\mathrm{ev}_1, \mathcal{X}} \mathcal{X}'$$

for all n. The same holds for Filt^n and Grad^n.

Proof. As above, it is enough to see the fact for Filt^n and Grad^n, which is [36, Proposition 1.3.1]. □

It will be useful for the sequel the fact that Grad preserves properness.

Proposition 2.2.16. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a representable proper finitely presented morphism of algebraic stacks over a base algebraic space B satisfying Assumption 2.2.3. Then*

$$\mathrm{Grad}(f)\colon \mathrm{Grad}(\mathcal{X}) \to \mathrm{Grad}(\mathcal{Y})$$

and

$$\mathrm{Grad}_{\mathbb{Q}}(f)\colon \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}) \to \mathrm{Grad}_{\mathbb{Q}}(\mathcal{Y})$$

are representable and proper.

Proof. (Halpern-Leistner) It is enough to prove the statement for $\mathrm{Grad}(f)$. Let T be a scheme and $T \to \mathrm{Grad}(\mathcal{Y})$ a map, corresponding to $B\mathbb{G}_{m,T} \to \mathcal{Y}$. Form a cartesian square

$$\begin{array}{ccc} \mathcal{Z} & \longrightarrow & B\mathbb{G}_{m,T} \\ \downarrow & \ulcorner & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{Y}. \end{array}$$

The 1-category of representable algebraic stacks over $B\mathbb{G}_{m,T}$ is equivalent to the category of algebraic spaces over T endowed with a $\mathbb{G}_{m,T}$-action, and the equivalence is given by pullback along $T \to B\mathbb{G}_{m,T}$. Therefore $\mathcal{Z} = Z/\mathbb{G}_{m,T}$ for a T-algebraic space Z acted on by $\mathbb{G}_{m,T}$. Forming now the fibre product

$$\begin{array}{ccc} \mathcal{U} & \longrightarrow & T \\ \downarrow & \ulcorner & \downarrow \\ \mathrm{Grad}(\mathcal{X}) & \longrightarrow & \mathrm{Grad}(\mathcal{Y}) \end{array}$$

and given a T-scheme S, a map $S \to \mathcal{U}$ over T is a section of $\mathcal{Z} \to B\mathbb{G}_{m,T}$ over $B\mathbb{G}_{m,S} \to B\mathbb{G}_{m,T}$, which is in turn a $\mathbb{G}_{m,T}$-equivariant map $S \to Z$. Therefore $\mathcal{U} = Z^{\mathbb{G}_{m,T}}$, the fixed points of Z, as a stack over T. Since $Z \to T$ is finitely presented, we have by [36, Proposition 1.4.1] that the map $Z^{\mathbb{G}_{m,T}} \to Z$ is a closed immersion and also, by hypothesis, that $Z \to T$ is proper. Thus $\mathcal{U} \to T$ is proper. □

We recall the definition of $\mathbb{Z}$-*flag spaces* from [36, Definition 1.1.15] and introduce the natural counterpart of $\mathbb{Q}$-*flag spaces*.

Definition 2.2.17 (Flag spaces)**.** Let $\mathcal{X}$ be an algebraic stack over an algebraic space B satisfying Assumption 2.2.3, and let $x\colon T \to \mathcal{X}$ be a scheme-valued point. We define the $\mathbb{Z}$-flag space $\mathrm{Flag}(\mathcal{X}, x)$ to be the fibre product

$$\begin{array}{ccc} \mathrm{Flag}(\mathcal{X},x) & \longrightarrow & T \\ \downarrow & \lrcorner & \downarrow x \\ \mathrm{Filt}(\mathcal{X}) & \xrightarrow{\mathrm{ev}_1} & \mathcal{X} \end{array}$$

and the $\mathbb{Q}$-flag space $\mathrm{Flag}_{\mathbb{Q}}(\mathcal{X}, x)$ as the fibre product

$$\begin{array}{ccc} \mathrm{Flag}_{\mathbb{Q}}(\mathcal{X},x) & \longrightarrow & T \\ \downarrow & \lrcorner & \downarrow x \\ \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) & \xrightarrow{\mathrm{ev}_1} & \mathcal{X}. \end{array}$$

By representability and separatedness of $\mathrm{ev}_1\colon \mathrm{Filt}(\mathcal{X}) \to \mathcal{X}$ and $\mathrm{ev}_1\colon \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) \to \mathcal{X}$, the flag spaces $\mathrm{Flag}(\mathcal{X}, x)$ and $\mathrm{Flag}_{\mathbb{Q}}(\mathcal{X}, x)$ are separated algebraic spaces over T. Since $\mathrm{Filt}(\mathcal{X}) \to \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ is an open and closed immersion, we have a natural open and closed immersion $\mathrm{Flag}(\mathcal{X}, x) \to \mathrm{Flag}_{\mathbb{Q}}(\mathcal{X}, x)$ of flag spaces.

In the case of a field-valued point, we can talk about a *set of filtrations*.

Definition 2.2.18 (Set of filtrations of a point)**.** Let $\mathcal{X}$ be an algebraic stack over an algebraic space B satisfying Assumption 2.2.3. Let k be a field and let $x\colon \mathrm{Spec}(k) \to \mathcal{X}$ be a k-point. The *set of* $\mathbb{Z}$-*filtrations (or integral filtrations) of* x is defined to be

$$\mathbb{Z}\text{-}\mathrm{Filt}(\mathcal{X}, x) := \mathrm{Flag}(\mathcal{X}, x)(k),$$

the set of k-points of the $\mathbb{Z}$-flag space of x. Similarly, the *set of* $\mathbb{Q}$-*filtrations (or rational filtrations) of* x is

$$\mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, x) := \mathrm{Flag}_{\mathbb{Q}}(\mathcal{X}, x)(k).$$

The filtration of x given by the composition of $\mathbb{A}^1_k/\mathbb{G}_{m,k} \to \mathrm{Spec}\, k$ and $x\colon \mathrm{Spec}\, k \to \mathcal{X}$ is denoted 0 and referred to as the *trivial filtration*.

Remark **2.2.19** (Filtrations of a quotient stack). Let k be a field and let X be a separated scheme over k, endowed with an action $a\colon G \times X \to X$ by a linear algebraic group G. Form the quotient stack $\mathcal{X} = X/G$ and let $x \in X(k)$ be a k-point. We abusively also denote by x the composition $\operatorname{Spec} k \xrightarrow{x} X \to \mathcal{X}$. If $\lambda\colon \mathbb{G}_{m,k} \to G$ is a cocharacter, we say that $\lim_{t\to 0} \lambda(t)x$ *exists* in X if the map

$$\mathbb{G}_{m,k} \xrightarrow{\lambda} G \cong G \times \operatorname{Spec} k \xrightarrow{\mathrm{id}_G \times x} G \times X \xrightarrow{a} X$$

extends to a map $\overline{\lambda}_x\colon \mathbb{A}^1_k \to X$ (in which case it does so uniquely by separatedness of X), where we regard $\mathbb{G}_{m,k}$ as the open subscheme $\mathbb{A}^1_k \setminus \{0\}$ of $\mathbb{A}^1_k$. If $\lim_{t\to 0} \lambda(t)x$ exists, we let $\lim_{t\to 0} \lambda(t)x$ denote the k-point $\overline{\lambda}_x(0)$ of X. For $n \in \mathbb{Z}_{>0}$, we have that $\lim_{t\to 0} \lambda(t)x$ exists if and only if $\lim_{t\to 0} \lambda^n(t)x$ exists, so it makes sense to define this notion for a rational one-parameter subgroup $\lambda \in \Gamma^{\mathbb{Q}}(G)$. The k-points of X such that $\lim_{t\to 0} \lambda(t)x$ exists are in bijection with the k-points of the attractor $X^{\lambda,+}$. Thus it follows readily from [36, Theorem 1.4.8 and Remark 1.4.9] that we have an identification:

$$\mathbb{Q}\text{-}\operatorname{Filt}(\mathcal{X}, x) = \{\lambda \in \Gamma^{\mathbb{Q}}(G) \mid \lim_{t\to 0} \lambda(t)x \text{ exists}\}/\sim, \tag{2.1}$$

where $\lambda \sim \lambda'$ if there is $g \in P(\lambda)(k)$ such that $\lambda^g = \lambda'$.

We conclude this section with a couple of facts about maps induced on sets of filtrations.

Proposition 2.2.20. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a schematic proper morphism of algebraic stacks over an algebraic space B satisfying Assumption 2.2.3. Let k be a field, let $x \in \mathcal{X}(k)$ and $y \in \mathcal{Y}(k)$ be k-points, and let $f(x) \to y$ be an isomorphism. Then the induced maps $\mathbb{Z}\text{-}\operatorname{Filt}(\mathcal{X}, x) \to \mathbb{Z}\text{-}\operatorname{Filt}(\mathcal{Y}, y)$ and $\mathbb{Q}\text{-}\operatorname{Filt}(\mathcal{X}, x) \to \mathbb{Q}\text{-}\operatorname{Filt}(\mathcal{Y}, y)$ of sets of filtrations are bijective.*

Proof. It is enough to deal with the case of integral filtrations. An element of $\mathbb{Z}\text{-}\operatorname{Filt}(\mathcal{Y}, y)$ is a pair (λ, α) fitting in a commutative diagram

$$\begin{array}{ccc} \mathbb{A}^1_k/\mathbb{G}_{m,k} & \xrightarrow{\lambda} & \mathcal{Y} \\ \uparrow{\scriptstyle 1} & \overset{\alpha}{\Leftarrow} & \nearrow{\scriptstyle y} \\ \operatorname{Spec}(k) & & \end{array}$$

and similarly for $\mathbb{Z}\text{-}\operatorname{Filt}(\mathcal{X}, x)$. Fix such a $(\lambda, \alpha) \in \mathbb{Z}\text{-}\operatorname{Filt}(\mathcal{Y}, y)$. Now form the fibre product

$$\begin{array}{ccc} X/\mathbb{G}_{m,k} & \longrightarrow & \mathbb{A}^1_k/\mathbb{G}_{m,k} \\ {\scriptstyle r}\downarrow & \ulcorner & \downarrow{\scriptstyle \lambda} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y}. \end{array}$$

The base change is indeed of the form $X/\mathbb{G}_{m,k}$ for a scheme X because f is schematic, and $X/\mathbb{G}_{m,k} \to \mathbb{A}^1_k/\mathbb{G}_{m,k}$ is given by a $\mathbb{G}_{m,k}$-equivariant map $X \to \mathbb{A}^1_k$. There is a commutative solid square

$$\begin{array}{ccc} (\mathbb{A}^1_k \setminus 0)/\mathbb{G}_{m,k} & \longrightarrow & \mathbb{A}^1_k/\mathbb{G}_{m,k} \\ {\scriptstyle u}\downarrow & \overset{h}{\swarrow} & \downarrow{\scriptstyle 1_{\mathbb{A}^1/\mathbb{G}_m}} \\ X/\mathbb{G}_{m,k} & \longrightarrow & \mathbb{A}^1_k/\mathbb{G}_{m,k} \end{array} \tag{2.2}$$

and an isomorphism $r \circ u \to x$. An element of $\mathbb{Z}\text{-}\mathrm{Filt}(\mathcal{X}, x)$ mapping to (λ, u) is specified by a lift h of the square 2.2 in the 2-categorical sense. Thus we want to prove that there is a unique such lift. There is a unique lift g of

$$\begin{array}{ccc} \mathbb{A}^1_k \setminus 0 & \longrightarrow & \mathbb{A}^1_k \\ \downarrow & \overset{g}{\swarrow} & \downarrow{\scriptstyle 1_{\mathbb{A}^1}} \\ X & \longrightarrow & \mathbb{A}^1_k \end{array}$$

by properness, see [72, Tag 0BX7]. We just need to prove that g is $\mathbb{G}_{m,k}$-equivariant. Equivariance amounts to the commutativity of

$$\begin{array}{ccc} \mathbb{G}_{m,k} \times \mathbb{A}^1_k & \longrightarrow & \mathbb{A}^1_k \\ {\scriptstyle 1_{\mathbb{G}_m} \times g}\downarrow & & \downarrow{\scriptstyle g} \\ \mathbb{G}_{m,k} \times X & \longrightarrow & X. \end{array}$$

Both compositions agree when restricted to $\mathbb{G}_{m,k} \times (\mathbb{A}^1_k \setminus 0)$, which is schematically dense in $\mathbb{G}_{m,k} \times \mathbb{A}^1_k$. Thus, by separatedness of $X \to \mathbb{A}^1_k$, the square commutes. □

Proposition 2.2.21. *Suppose $f : \mathcal{X} \to \mathcal{Y}$ is a representable and separated morphism of algebraic stacks over an algebraic space B satisfying Assumption 2.2.3. Let k be a field, let $x \in \mathcal{X}(k)$ and $y \in \mathcal{Y}(k)$ be k-points, and let $f(x) \to y$ be an isomorphism. Then the induced maps $\mathbb{Z}\text{-}\mathrm{Filt}(\mathcal{X}, x) \to \mathbb{Z}\text{-}\mathrm{Filt}(\mathcal{Y}, y)$ and $\mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, x) \to \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{Y}, y)$ of sets of filtrations are injective.*

Proof. It is enough to prove the claim for integral filtrations. If λ_1, λ_2 are two filtrations of x that give the same filtration of y, we can form a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} k & \xrightarrow{x} & \mathcal{X} \\ {\scriptstyle 1}\downarrow & \nearrow & \downarrow{\scriptstyle \Delta_f} \\ \mathbb{A}^1_k/\mathbb{G}_{m,k} & \xrightarrow{(\lambda_1,\lambda_2)} & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}. \end{array}$$

Since Δ_f is a closed immersion and $1 : \operatorname{Spec} k \to \mathbb{A}^1_k/\mathbb{G}_{m,k}$ is a schematically dense open immersion, there is a unique dashed arrow filling the diagram, which gives the isomorphism between λ_1 and λ_2. □

2.3 NORMED STACKS

We now recall from [36] the notion of a norm on graded points of a stack. If G is an algebraic group, norms on graded points of BG are in bijection with norms on cocharacters of G (Proposition 2.3.10).

Definition 2.3.1 (Nondegenerate graded point)**.** Let $\mathcal{X}$ be a quasi-separated algebraic stack, let k be a field and let $x\colon B\mathbb{G}_{m,k}^n \to \mathcal{X}$ be a $\mathbb{Z}^n$-graded point. We say that the graded point x is *nondegenerate* if $\ker(\mathbb{G}_{m,k}^n \to \mathrm{Aut}(x|_{\mathrm{Spec}\, k}))$ is finite.

Suppose that $\mathcal{X}$ is defined over some algebraic space B and it satisfies Assumption 2.2.3. We say that a connected component $\mathcal{Z}$ of $\mathrm{Grad}^n(\mathcal{X})$ is *nondegenerate* if there is a field k and a point $x \in \mathcal{Z}(k)$ with x nondegenerate.

Remark **2.3.2.** If $\mathcal{Z}$ is a nondegenerate component, then by [36, Proposition 1.3.9] we have that for *all* fields k and points $x \in \mathcal{Z}(k)$, the point x is nondegenerate.

We recall the notion of norm on graded points of a stack from [36, Definition 4.1.12].

Definition 2.3.3 (Norm on graded points)**.** Let $\mathcal{X}$ be an algebraic stack over an algebraic space B, satisfying Assumption 2.2.3. A *(rational quadratic) norm* q on graded points of $\mathcal{X}$ (or simply a *norm* on $\mathcal{X}$) is a locally constant function

$$q\colon |\mathrm{Grad}(\mathcal{X})| \to \mathbb{Q}_{\geq 0}$$

such that for every field k and every nondegenerate $\mathbb{Z}^n$-graded point $x\colon B\mathbb{G}_{m,k}^n \to \mathcal{X}$, the induced map $q_x\colon \Gamma^{\mathbb{Z}}(\mathbb{G}_{m,k}^n) \to \mathbb{Q}$ is the quadratic form of a rational inner product on the finite free $\mathbb{Z}$-module $\Gamma^{\mathbb{Z}}(\mathbb{G}_{m,k}^n)$ of cocharacters of $\mathbb{G}_{m,k}^n$.

A *normed algebraic stack* is an algebraic stack endowed with a norm on graded points.

Let us clarify what the map q_x is. If $\lambda\colon \mathbb{G}_{m,k} \to \mathbb{G}_{m,k}^n$ is a cocharacter, the composition $B\mathbb{G}_{m,k} \xrightarrow{B\lambda} B\mathbb{G}_{m,k}^n \xrightarrow{x} \mathcal{X}$ defines a point $p \in |\mathrm{Grad}(\mathcal{X})|$, and we let $q_x(\lambda) = q(p)$.

Example **2.3.4.** We give two examples of norms on graded points that will also be discussed later.

1. Let G be an algebraic group over an algebraically closed field k, with a maximal torus T and Weyl group W. Then norms on graded points of BG are in bijection with W-invariant rational quadratic inner products on the set of rational cocharacters $\Gamma^{\mathbb{Q}}(T)$ of T (Proposition 2.3.10).

2. Let C be a smooth projective curve over the complex numbers, and let $\mathrm{Bun}(C)$ denote the stack of vector bundles on C. There is a natural norm q on cocharacters of $\mathrm{Bun}(C)$, defined as follows. A point $x \in \mathrm{Grad}(\mathrm{Bun}(C))(\mathbb{C})$ corresponds to a $\mathbb{Z}$-graded vector bundle $\bigoplus_{n\in\mathbb{Z}} E_n$. The norm $q(x)$ is defined by
$$q(x) = \sum_{n\in\mathbb{Z}} n^2 \,\mathrm{rk}(E_n).$$
Thus in this case the norm on graded points q encodes the information of the rank of vector bundles in a way that is intrinsic to the language of algebraic stacks. See Section 3.6.4 for more on this example.

Remark **2.3.5.** A norm on graded points $q\colon |\mathrm{Grad}(\mathcal{X})| \to \mathbb{Q}$ extends canonically to a map
$$q\colon |\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})| \to \mathbb{Q}$$
by setting $q(\frac{1}{n}\lambda) = \frac{1}{n^2}q(\lambda)$, for a rational graded point $\frac{1}{n}\lambda$.

Remark **2.3.6** (Notation for norms)**.** If $\mathcal{X}$ is endowed with a norm on graded points q and $\lambda \in \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})(k)$ is a graded point, with k a field, then we will often denote $\|\lambda\| := \sqrt{q(\lambda)}$.

In some circumstances we can pull back a norm under a morphism.

Proposition and Definition 2.3.7 (Pulling back norms)**.** Let $\mathcal{X}$ and $\mathcal{Y}$ be algebraic stacks over an algebraic space B, satisfying Assumption 2.2.3. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism such that the relative inertia $\mathcal{I}_f \to \mathcal{X}$ has proper fibres (for example if f is representable or separated). Let q be a norm on $\mathcal{Y}$ and denote f^*q the composition $|\mathrm{Grad}(\mathcal{X})| \longrightarrow |\mathrm{Grad}(\mathcal{Y})| \xrightarrow{q} \mathbb{Q}$. Then f^*q is a norm on $\mathcal{X}$, called the *pulled back norm*.

If $\mathcal{X}$ is endowed with a norm q', we say that the morphism f is *norm-preserving* if $f^*q = q'$.

Proof. We need to see that if $u\colon B\mathbb{G}^n_{m,k} \to \mathcal{X}$ is nondegenerate, then so is $f \circ u$. Let $x = u|_{\mathrm{Spec}\, k} \in \mathcal{X}(k)$ be the point that u grades. We have induced algebraic group homomorphisms
$$\begin{array}{ccc} \mathbb{G}^n_{m,k} & & \\ {\scriptstyle s}\downarrow & \overset{l}{\searrow} & \\ \mathrm{Aut}(x) & \xrightarrow[r]{} & \mathrm{Aut}(f(x)). \end{array}$$
The kernel of r is proper over $\mathrm{Spec}(k)$, since it is the fibre over x of the relative inertia morphism, and $\ker s$ is finite by hypothesis. We have a sequence
$$\ker l \xrightarrow{a} \ker l/\ker s \xrightarrow{b} \ker r$$

where a is finite and b is a closed immersion. Thus $\ker l$ is proper over k, and therefore finite, because $\mathbb{G}_{m,k}^n$ is affine. This proves that $f \circ u$ is nondegenerate. □

The notion of norm on graded points of an algebraic stack is a generalisation of the classical notion of norm on cocharacters of a group.

DEFINITION 2.3.8 (Norm on cocharacters of a group)**.** Let G be a smooth affine algebraic group, over a field k, that has a k-split maximal torus. A *(rational quadratic) norm on cocharacters of* G is a map $q\colon \Gamma^{\mathbb{Z}}(G) \to \mathbb{Q}$ that is invariant under the action of $G(k)$ on $\Gamma^{\mathbb{Z}}(G)$ by conjugation and such that for every k-split torus T of G, the restriction of q to $\Gamma^{\mathbb{Z}}(T)$ is the quadratic form of a rational inner product on $\Gamma^{\mathbb{Z}}(T)$.

Now fix a smooth connected linear algebraic group G over a field k. Any two maximal k-split tori are conjugate by an element of $G(k)$ [21, Theorem C.2.3], and there exists one such torus for dimension reasons. The Weyl group $W(G,T) = N_G(T)/Z_G(T)$, for a k-split torus T, is a finite constant group scheme [21, Proposition C.2.10], so we identify it with a finite abstract group. By conjugacy of maximal k-split tori and [65, Lemma 2.8], it follows that $\Gamma^{\mathbb{Z}}(T)/W(G,T) = \Gamma^{\mathbb{Z}}(G)/G(k)$ if T is a maximal k-split torus of G. Thus we deduce:

PROPOSITION 2.3.9. *If T is a k-split maximal torus of G, then the data of a norm on cocharacters of G is equivalent to a rational quadratic inner product on $\Gamma^{\mathbb{Z}}(T)$, invariant under the action of $W(G,T)$.* □

The link with the concept of a norm on graded points is given in the following well-known proposition.

PROPOSITION 2.3.10. *Suppose G has a k-split maximal torus T. Then norms on BG are in natural bijection with norms on cocharacters of G.*

Proof. Let $W = W(G,T)$ be the Weyl group. Then, by [36, Theorem 1.4.8], we can explicitly describe the stack of graded points as

$$\mathrm{Grad}(BG) = \bigsqcup_{\lambda \in \Gamma^{\mathbb{Z}}(T)/W} BL(\lambda),$$

where $L(\lambda)$ is the centraliser of a choice of representative of λ. Thus $|\mathrm{Grad}(BG)| = \Gamma^{\mathbb{Z}}(T)/W$, and the identification is compatible in the sense that if $\lambda\colon \mathbb{G}_m \to G$ is a cocharacter, the point in $|\mathrm{Grad}(BG)|$ defined by $B\lambda\colon B\mathbb{G}_m \to BG$ is the class of λ in $\Gamma^{\mathbb{Z}}(T)/W = \Gamma^{\mathbb{Z}}(G)/G(k)$. To conclude, just note that if $T \to T'$ is a map of k-split tori with finite kernel, then $\Gamma^{\mathbb{Z}}(T) \to \Gamma^{\mathbb{Z}}(T')$ is injective, so $\Gamma^{\mathbb{Z}}(T)$ inherits an inner product if $\Gamma^{\mathbb{Z}}(T')$ has one. □

Norms on cocharacters are a source of norms on quotient stacks.

Proposition 2.3.11 (Norms on quotient stacks)**.** *Suppose G has a k-split maximal torus and acts on a quasi-separated algebraic space X of finite type over k. If G is endowed with a norm on cocharacters q, then X/G is naturally endowed with a norm on graded points.*

Proof. Since $p\colon X/G \to BG$ is representable, the pullback p^*q is a norm on X/G by Proposition and Definition 2.3.7. □

Definition 2.3.12 (Norm on $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$)**.** Let $\mathcal{X}$ be an algebraic stack over an algebraic space B satisfying Assumption 2.2.3 and endowed with a norm on graded points q. We denote $\mathrm{Grad}(q)$ (resp. $\mathrm{Grad}_{\mathbb{Q}}(q)$) the norm on $\mathrm{Grad}(\mathcal{X})$ (resp. $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$) given as the pullback of q along the forgetful morphism $u\colon \mathrm{Grad}(\mathcal{X}) \to \mathcal{X}$ (resp. $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}) \to \mathcal{X}$), which is representable.

It follows from Proposition 2.3.7 that $\mathrm{Grad}(q)$ is a norm on $\mathrm{Grad}(\mathcal{X})$. If $\mathcal{X}$ is a normed stack, we will always regard $\mathrm{Grad}(\mathcal{X})$ (resp. $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$) as a normed stack, endowed with the norm $\mathrm{Grad}(q)$ (resp. $\mathrm{Grad}_{\mathbb{Q}}(q)$).

2.4 LINEAR FORMS ON STACKS

We now recall the notion of linear form on graded points of a stack from [36] and show how to get linear forms from line bundles.

Definition 2.4.1 (Linear form)**.** Let $\mathcal{X}$ be an algebraic stack over an algebraic space B, satisfying Assumption 2.2.3. A *rational linear form ℓ on graded points of $\mathcal{X}$* (or simply a *linear form on $\mathcal{X}$*) is a locally constant function

$$\ell\colon |\mathrm{Grad}(\mathcal{X})| \to \mathbb{Q}$$

such that, for every field k and every $\mathbb{Z}^n$-graded point $B\mathbb{G}^n_{m,k} \to \mathcal{X}$, the induced map $\Gamma^{\mathbb{Z}}(\mathbb{G}^n_{m,k}) \to \mathbb{Q}$ on cocharacters of the torus is a $\mathbb{Z}$-module homomorphism.

If $\lambda\colon B\mathbb{G}_{m,k} \to \mathcal{X}$ is a graded point, we denote by either $\langle \lambda, \ell \rangle$ or $\ell(\lambda)$ the value of ℓ at the point of $\mathrm{Grad}(\mathcal{X})$ defined by λ.

Remark **2.4.2.** A linear form $\ell\colon |\mathrm{Grad}(\mathcal{X})| \to \mathbb{Q}$ extends canonically to a map

$$\ell\colon |\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})| \to \mathbb{Q}$$

by setting $\langle \frac{1}{n}\lambda, \ell \rangle = \frac{1}{n}\langle \lambda, \mathcal{L} \rangle$, for a rational graded point $\frac{1}{n}\lambda$.

Line bundles are an important source of linear forms. The following definition essentially comes from [41] and [36].

DEFINITION 2.4.3 (Linear form associated to a line bundle)**.** Let $\mathcal{X}$ be an algebraic stack over an algebraic pace B, satisfying Assumption 2.2.3, and let $\mathcal{L}$ be a line bundle on $\mathcal{X}$. Define a map

$$\langle -, \mathcal{L}\rangle \colon |\mathrm{Grad}(\mathcal{X})| \to \mathbb{Q}$$

as follows. If $g \in |\mathrm{Grad}(\mathcal{X})|$ is represented by a field k and a map $\lambda\colon B\mathbb{G}_{m,k} \to \mathcal{X}$, let $\langle g, \mathcal{L}\rangle := \langle \lambda, \mathcal{L}\rangle := -\mathrm{wt}(\lambda^*\mathcal{L})$, the opposite of the weight of the one-dimensional representation $\lambda^*\mathcal{L}$ of $\mathbb{G}_{m,k}$.

Remark **2.4.4** (Sign conventions). Our sign convention for weights is as follows. Let k be a field. We denote by $\mathcal{O}_{B\mathbb{G}_{m,k}}(n)$, for $n \in \mathbb{Z}$, the representation of $\mathbb{G}_{m,k}$ with underlying vector space k and such that $t * 1 = t^n$ for $t \in k^\times$. By definition, $\mathrm{wt}\, \mathcal{O}_{B\mathbb{G}_{m,k}}(n) = n$.

The total space of $\mathcal{O}_{B\mathbb{G}_{m,k}}(n)$ is $\mathbb{A}(\mathcal{O}_{B\mathbb{G}_{m,k}}(n)) = \mathrm{Spec}_{B\mathbb{G}_{m,k}}(\mathrm{Sym}(\mathcal{O}_{B\mathbb{G}_{m,k}})^\vee)$. If we let $\mathbb{G}_{m,k}$ act on $\mathbb{A}^1_k$ by the formula $t * s = t^n s$, for any k-algebra R, $t \in \mathbb{G}_{m,k}(R) = R^\times$ and $s \in \mathbb{A}^1_k(R) = R$, then $\mathbb{A}(\mathcal{O}_{B\mathbb{G}_{m,k}}(n)) = \mathbb{A}^1_k/\mathbb{G}_{m,k}$ for this action. If we write $\mathbb{A}^1_k = \mathrm{Spec}\, k[x]$, the standard coordinate x has weight $-n$.

Let G be a linear algebraic group over k acting linearly on a finite dimensional vector space V, and let $p\colon \mathbb{P}(V)/G \to B\mathbb{G}_m$ be the map $\mathbb{P}(V)/G = (V \setminus \{0\})/G \times \mathbb{G}_m \to B(G \times \mathbb{G}_m) \to B\mathbb{G}_m$, where $\mathbb{G}_m$ acts by scaling on V. Then $p^*(\mathcal{O}_{B\mathbb{G}_{m,k}}(1)) = \mathcal{O}_{\mathbb{P}(V)/G}(1)$ is the standard ample line bundle on $\mathbb{P}(V)/G$. Let $\lambda\colon \mathbb{G}_{m,k} \to G$ be a one-parameter subgroup, and let $x \in \mathbb{P}(V)(k)$ be a point fixed by λ, defining a graded point $\lambda_x\colon B\mathbb{G}_{m,k} \to \mathbb{P}(V)/G$. The point x corresponds to a one-dimensional subspace $L \subset V$ invariant by λ. Thus there is $n \in \mathbb{Z}$ such that $\lambda(t)v = t^n v$ for all k-algebras R, $v \in R \otimes_k L$ and $t \in R^\times$. In other words, L is regarded as a $\mathbb{G}_{m,k}$-representation and we let $n = \mathrm{wt}\, L$. We can identify $L = \lambda_x^*\left(\mathcal{O}_{\mathbb{P}(V)/G}(-1)\right)$. Thus our sign conventions are such that $\langle \lambda_x, \mathcal{O}_{\mathbb{P}(V)/G}(1)\rangle = -\mathrm{wt}(L^\vee) = n$.

PROPOSITION 2.4.5. *Let $\mathcal{X}$ and $\mathcal{L}$ be as in Definition 2.4.3. Then $\langle -, \mathcal{L}\rangle$ is well-defined and a linear form on $\mathcal{X}$.*

Proof. If $k'|k$ is a field extension, inducing a map $g\colon B\mathbb{G}_{m,k'} \to B\mathbb{G}_{m,k}$, and U is a line bundle on $B\mathbb{G}_{m,k'}$, then $\mathrm{wt}\, U = \mathrm{wt}(g^*U)$, so $\langle -, \mathcal{L}\rangle$ is well-defined.

To see that $\langle -, \mathcal{L}\rangle$ is locally constant on $|\mathrm{Grad}(\mathcal{X})|$, it is enough to prove that for any map $f\colon \mathrm{Spec}\, A \to \mathrm{Grad}(\mathcal{X})$ with A any commutative ring, the composition $|\mathrm{Spec}\, A| \to |\mathrm{Grad}(\mathcal{X})| \to \mathbb{Q}$ of $|f|$ and $\langle -, \mathcal{L}\rangle$ is locally constant. Let $h\colon B\mathbb{G}_{m,A} \to \mathcal{X}$ be the map corresponding to f. We may assume that $h^*\mathcal{L}$ is trivial when restricted to $\mathrm{Spec}\, A$, so $h^*\mathcal{L}$ is an A-module direct sum decomposition $A = \bigoplus_{n\in\mathbb{Z}} A_n$. Let 1_n be the degree n part of $1 \in A$. Each A_n is generated by 1_n as an A-module. On the nonvanishing locus $D(1_n)$ of 1_n, we have that the restriction $A_n|_{D(1_n)} = A|_{D(1_n)}$ and $A_m|_{D(1_n)} = 0$ if $m \neq n$. Thus $\operatorname{Spec} A = \bigsqcup_{n\in\mathbb{Z}} D(1_n)$, and the composition $|D(1_n)| \to |\mathrm{Grad}(\mathcal{X})| \xrightarrow{\langle -,\mathcal{L}\rangle} \mathbb{Q}$ is constant with value $-n$.

Now let $\alpha\colon B\mathbb{G}_{m,k}^n \to \mathcal{X}$ be a $\mathbb{Z}^n$-graded point. The pullback $\alpha^*\mathcal{L}$ corresponds to a character $\chi \in \Gamma_{\mathbb{Z}}\left(\mathbb{G}_{m,k}^n\right)$, and the map $\Gamma^{\mathbb{Z}}\left(\mathbb{G}_{m,k}^n\right) \to \mathbb{Q}$ that $\langle -,\mathcal{L}\rangle$ induces is just the pairing $\lambda \mapsto -\langle \lambda, \chi\rangle$. It is thus linear. □

Remark **2.4.6.** Definition 2.4.3 naturally extends to *rational* line bundles $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})\otimes_{\mathbb{Z}} \mathbb{Q}$.

Now suppose that $\mathcal{X}$ is an algebraic stack over an algebraic space B satisfying Assumption 2.2.3, and endowed with a norm on graded points q.

Definition 2.4.7 (Canonical linear form of a norm)**.** The *canonical linear form* ℓ_q *on* $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ induced by q is the linear form on graded points of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ defined as follows. A graded point $\mu \in \mathrm{Grad}(\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}))(k)$ lying over a point $\lambda/n \in \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})(k)$, with $\lambda \in \mathrm{Grad}(\mathcal{X})(k)$ and $n \in \mathbb{Z}_{>0}$, gives a map $\alpha = (\mu,\lambda)\colon B\mathbb{G}_{m,k}^2 \to \mathcal{X}$ and q gives an inner product on the set $\Gamma^{\mathbb{Z}}\left(\mathbb{G}_{m,k}^2\right)$ of cocharacters of $\mathbb{G}_{m,k}^2$. Let e_1, e_2 be the standard basis of $\Gamma^{\mathbb{Z}}\left(\mathbb{G}_{m,k}^2\right)$. We define

$$\langle \lambda, \ell_q\rangle = \frac{1}{n}(e_1,e_2)_q.$$

Remark **2.4.8.** Note that ℓ_q determines q, since for a graded point $\lambda\colon B\mathbb{G}_{m,k} \to \mathcal{X}$, if $o\colon B\mathbb{G}_{m,k}^2 \to B\mathbb{G}_{m,k}$ is induced by $\mathbb{G}_{m,k}^2 \to \mathbb{G}_{m,k}\colon (t,t') \to tt'$, then $q(\lambda) = \ell_q(\lambda \circ o)$.

Definition 2.4.9 (Algebraic norm)**.** We say that the norm q on $\mathcal{X}$ is *algebraic* if the canonical linear form ℓ_q on $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ is induced, on each connected component $\mathcal{Z}$ of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ by a rational line bundle on $\mathcal{Z}$.

Remark **2.4.10.** If q is an algebraic norm on $\mathcal{X}$ and $f\colon \mathcal{Y} \to \mathcal{X}$ is a morphism with proper relative automorphism groups, then the pulled back norm f^*q (Proposition and Definition 2.3.7) is algebraic. Indeed, if $\ell_q = \langle -,\mathcal{M}\rangle$ for a rational line bundle $\mathcal{M}$ on $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$, then $\ell_{f^*q} = \langle -, \mathrm{Grad}_{\mathbb{Q}}(f)^*\mathcal{M}\rangle$. We will tacitly use this fact in the sequel.

2.5 Θ-STRATIFICATIONS

We now discuss the notion of Θ-*stratification* for algebraic stacks, which is a generalisation due to Halpern-Leistner of the Hesselink–Kempf–Kirwan–Ness stratification in GIT [55, Chapter 12] and of the stratification by Harder–Narasimhan type for vector bundles on a curve [10]. We follow [36] with some modifications. We use the stack $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ of rational filtrations instead of $\mathrm{Filt}(\mathcal{X})$ and we work with a reformulation of the original definition of Θ-stratification that is closer to our definition of *sequential stratification* (Definition 3.3.1). We will focus on Θ-stratifications induced by a pair (ℓ, q), where q is a norm and ℓ is a linear form on graded points of a stack $\mathcal{X}$.

We fix an algebraic stack $\mathcal{X}$ over an algebraic space B satisfying Assumption 2.2.3. The following is a variant of [36, Definition 2.1.2].

Definition 2.5.1 (Θ-stratification)**.** Let Γ be a partially ordered set. A *(weak) Θ-stratification* of $\mathcal{X}$ indexed by Γ is a family $(\mathcal{S}_c)_{c\in\Gamma}$ of open substacks of $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ satisfying:

1. For every $c \in \Gamma$, the composition $\mathcal{S}_c \to \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) \xrightarrow{\mathrm{ev}_1} \mathcal{X}$ is a locally closed immersion (resp. locally finite radicial[2]).
2. The $\mathrm{ev}_1(|\mathcal{S}_c|)$ are pairwise disjoint and cover $|\mathcal{X}|$.
3. For every $c \in \Gamma$, the stratum $\mathcal{S}_c$ is the preimage along $\mathrm{gr}\colon \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) \to \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ of an open substack $\mathcal{Z}_c$ of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$, called the *centre* of $\mathcal{S}_c$.
4. For every $c \in \Gamma$, the set $|\mathcal{X}_{\leq c}| := \bigcup_{c'\leq c} \mathrm{ev}_1(|\mathcal{S}_{c'}|)$ is open in $|\mathcal{X}|$, and it thus defines an open substack $\mathcal{X}_{\leq c}$ of $\mathcal{X}$.

Remark **2.5.2.** If $\sigma\colon \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}) \to \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ denotes the "split filtration" map, then for all $c \in \Gamma$ the centre of $\mathcal{S}_c$ is $\mathcal{Z}_c = \sigma^{-1}(\mathcal{S}_c)$ and thus it is uniquely determined.

The strata $\mathcal{S}_c$ in a Θ-stratification $(\mathcal{S}_c)_{c\in\Gamma}$ are locally closed Θ-strata in the following sense.

Definition 2.5.3. A *locally closed Θ-stratum* of $\mathcal{X}$ is an open substack $\mathcal{S}$ of $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ that is the preimage along gr of an open substack $\mathcal{Z}$ of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ (its *centre*) and such that the composition $\mathcal{S} \to \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) \xrightarrow{\mathrm{ev}_1} \mathcal{X}$ is a locally closed immersion.

The following explains the relation between our definition of Θ-stratification and Halpern-Leistner's original one [36, Definition 2.1.2].

Proposition 2.5.4. *Let Γ be a totally ordered set. Let $(\mathcal{S}_c)_{c\in\Gamma}$ be a (weak) Θ-stratification of $\mathcal{X}$. If each stratum $\mathcal{S}_c$ is contained in the closed and open substack* $\mathrm{Filt}(\mathcal{X}) \subset \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ *of integral filtrations, then $(\mathcal{S}_c)_{c\in\Gamma}$ and $(\mathcal{X}_{\leq c})_{c\in\Gamma}$ define a (weak) Θ-stratification in the sense of [36, Definition 2.1.2]. Conversely, any (weak) Θ-stratification in the sense of [36, Definition 2.1.2] defines a (weak) Θ-stratification.*

[2]We say that a morphism $\mathcal{Y} \to \mathcal{Z}$ is *locally finite radicial* if it factors as $\mathcal{Y} \to \mathcal{U} \to \mathcal{Z}$ with $\mathcal{Y} \to \mathcal{U}$ finite and radicial and $\mathcal{U} \to \mathcal{Z}$ an open immersion.

Proof. Let $(\mathcal{S}_c)_{c\in\Gamma}$ be a (weak) Θ-stratification of $\mathcal{X}$ in the sense of Definition 2.5.1. It is enough to show that each $\mathcal{S}_c$ is a (weak) Θ-stratum of $\mathcal{X}_{\leq c}$ [36, Definition 2.1.1], that is

1. Each $\mathcal{S}_c \to \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ factors through $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}_{\leq c})$ and $\mathcal{S}_c \to \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}_{\leq c})$ is an open and closed immersion.
2. The composition $\mathcal{S}_c \to \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}_{\leq c}) \to \mathcal{X}_{\leq c}$ is a closed immersion (resp. finite and radicial).

Then the conditions in [36, Definition 2.1.2] are trivially satisfied by construction. Note that Halpern-Leistner also demands the $\mathcal{S}_c$ to be integral and Γ to be a total order.

We have a diagram

$$\begin{array}{ccccc} \mathcal{Z}_c & \longrightarrow & \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}) & \longrightarrow & \mathcal{X} \\ \downarrow & \ulcorner & \downarrow{\scriptstyle \sigma} & & \| \\ \mathcal{S}_c & \longrightarrow & \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) & \xrightarrow{\mathrm{ev}_1} & \mathcal{X} \end{array}$$

and $\mathcal{S}_c \to \mathcal{X}$ factors through $\mathcal{X}_{\leq c}$. Therefore $\mathcal{Z}_c \to \mathcal{X}$ also factors through $\mathcal{X}_{\leq c}$ and thus $\mathcal{Z}_c \subset \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}_{\leq c})$ since the formation of $\mathrm{Grad}_{\mathbb{Q}}$ is compatible with immersions. Since the natural square

$$\begin{array}{ccc} \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}_{\leq c}) & \longrightarrow & \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) \\ \downarrow{\scriptstyle \mathrm{gr}} & \ulcorner & \downarrow{\scriptstyle \mathrm{gr}} \\ \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}_{\leq c}) & \longrightarrow & \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}) \end{array}$$

is cartesian [36, Proposition 1.3.1 (3)] and $\mathcal{S}_c = \mathrm{gr}^{-1}(\mathcal{Z}_c)$, we have $\mathcal{S}_c \subset \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}_{\leq c})$.

Now the composition $\mathcal{S}_c \to \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}_{\leq c}) \to \mathcal{X}_{\leq c}$ is a locally closed immersion (resp. locally finite radicial) and has closed image. It is thus a closed immersion (resp. finite and radicial). Since $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}_{\leq c}) \to \mathcal{X}_{\leq c}$ is representable and separated, the map $\mathcal{S}_c \to \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}_{\leq c})$ is also a closed immersion (resp. finite and radicial). Since $\mathcal{S}_c \to \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}_{\leq c})$ is also an open immersion, it has to be an open and closed immersion.

For the converse, if $(\mathcal{S}_c)_{c\in\Gamma}$ and $(\mathcal{X}_{\leq c})_{c\in\Gamma}$ define a Θ-stratification in the sense of [36, Definition 2.1.1], just note that $\mathcal{S}_c \to \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}_{\leq c})$ being an open and closed immersion implies that $\mathcal{S}_c = \mathrm{gr}^{-1}(\mathcal{Z}_c)$ for some $\mathcal{Z}_c \subset \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}_{\leq c})$ open and closed, because $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}_{\leq c})$ and $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}_{\leq c})$ have the same connected components [36, Lemma 1.3.8]. The other conditions of Definition 2.5.1 are readily seen to be satisfied. □

Remark **2.5.5.** If each $\mathcal{S}_c$ intersects only a finite number of connected components of $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$, then it becomes integral after scaling up, that is, after replacing $\mathcal{S}_c$ by its image under the "rising to the nth power" map $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) \to \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ for big enough n. After suitably subdividing each $\mathcal{S}_c$, the condition is always satisfied.

A Θ-stratification of $\mathcal{X}$ assigns, for each geometric point x of $\mathcal{X}$, a canonical rational filtration of x:

Definition 2.5.6 (HN filtrations, [36, Lemma 2.1.4])**.** Consider a Θ-stratification $(\mathcal{S}_c)_{c\in\Gamma}$ of $\mathcal{X}$, and let $x\colon \mathrm{Spec}(k) \to \mathcal{X}$ be a field-valued point. There is a unique $c \in \Gamma$ such that the image of $\mathcal{S}_c \to \mathcal{X}$ contains the point of $|\mathcal{X}|$ defined by x; and there is a unique, up to unique isomorphism, lift of x to $\mathcal{S}_c$, which gives a filtration $\lambda \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, x)$ called the *Harder–Narasimhan filtration* (or the *HN filtration*) of x.

Remark **2.5.7.** In the case of a weak Θ-stratification, the HN filtration of a k-point is defined over a finite purely inseparable extension of k. We will only use the case of Θ-stratifications.

The following is a reformulation of [36, Definition 2.3.1], convenient for our purposes.

Proposition and Definition 2.5.8 (Induced Θ-stratifications)**.** Let $\mathcal{X}'$ be an algebraic stack over an algebraic space B, satisfying Assumption 2.2.3, and let $h\colon \mathcal{X}' \to \mathcal{X}$ be either a closed immersion or a base change of a map between algebraic spaces like in Proposition 2.2.14. Let $(\mathcal{S}_c)_{c\in\Gamma}$ be a (weak) Θ-stratification of $\mathcal{X}$. For each $c \in \Gamma$ let $h^*\mathcal{S}_c$ be the pullback

$$\begin{array}{ccc} h^*\mathcal{S}_c & \longrightarrow & \mathcal{S}_c \\ \big\downarrow & \lrcorner & \big\downarrow \\ \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}') & \longrightarrow & \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}). \end{array}$$

Then the family $(h^*\mathcal{S}_c)_{c\in\Gamma}$ is a (weak) Θ-stratification of $\mathcal{X}'$ called the *Θ-stratification induced by* h and $(\mathcal{S}_c)_{c\in\Gamma}$.

Proof. This is the content of [36, Lemmas 2.3.2 and 2.3.3] in the case of integral filtration, from which the case of rational filtrations follows easily. □

Now we fix a rational quadratic norm q and a rational linear form ℓ on graded points of $\mathcal{X}$. We regard q and ℓ as maps $|\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})| \to \mathbb{Q}$ by Remarks 2.3.5 and 2.4.2. Moreover, q and ℓ induce functions on $|\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})|$ by precomposing along $|\mathrm{gr}| : |\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})| \to |\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})|$ that we will still denote q and ℓ.

The norm q and the linear form ℓ give rise to two other interesting functions.

Definition 2.5.9 (Associated numerical invariant [36, 4.1.1 and 4.1.14])**.** We define the *numerical invariant* μ associated to q and ℓ to be the function $\mu\colon |\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})| \to \mathbb{R}_{\geq 0}$ such that

1. on the open and closed substack $\sigma\colon \mathcal{X} \to \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ defined by the "trivial grading" map, μ takes the value 0; and
2. on $|\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})| \setminus |\mathcal{X}|$, we set $\mu = \dfrac{\ell}{\sqrt{q}}$.

We extend μ to a function on $|\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})|$ by taking the composition

$$|\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})| \xrightarrow{|\mathrm{gr}|} |\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})| \xrightarrow{\mu} \mathbb{R}_{\geq 0},$$

which we will still denote μ.

Definition 2.5.10 (Stability function [36, 4.1.1])**.** The *stability function* $M^{\mu}\colon |\mathcal{X}| \to [0, \infty]$ associated to μ is defined by

$$M^{\mu}(x) := \sup\{\mu(\lambda) \mid \lambda \in |\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})|\,,\ \mathrm{ev}_1(\lambda) = x\}.$$

Remark **2.5.11.** In [36], the stack $\mathrm{Grad}(\mathcal{X})$ is used instead of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$. This is not an important difference because the numerical invariant μ is scale-invariant.

Definition 2.5.12 (Semistable locus)**.** The *semistable locus* $|\mathcal{X}^{\mathrm{ss}}|$ with respect to the linear form ℓ on $\mathcal{X}$ is the subset

$$\{x \in |\mathcal{X}| \mid \ell(\lambda) \leq 0, \text{ for all } \lambda \in |\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})| \text{ with } \mathrm{ev}_1(\lambda) = x\} \subset |\mathcal{X}|\,.$$

If the semistable locus $|\mathcal{X}^{\mathrm{ss}}|$ is open, then it defines an open substack of $\mathcal{X}$ denoted $\mathcal{X}^{\mathrm{ss}}$.

Remark **2.5.13.** If k is a field and $x\colon \mathrm{Spec}\, k \to \mathcal{X}$ is a point, to see whether x is semistable it suffices to check that $\ell(\lambda) \leq 0$ for $\lambda \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, x|_{\bar{k}})$. This follows from $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) \to \mathcal{X}$ being representable and locally of finite presentation and ℓ being locally constant on $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$.

Definition 2.5.14 (Θ-stratification defined by a linear form and a norm)**.** We say that the pair (ℓ, q) *defines a (weak) Θ-stratification* if the following holds:

1. The semistable locus $|\mathcal{X}^{\mathrm{ss}}| \subset |\mathcal{X}|$ is open. If $s\colon \mathcal{X} \to \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ is the "trivial filtration", which is an open and closed immersion, we denote $\mathcal{S}_0 = s(\mathcal{X}^{\mathrm{ss}})$, an open substack of $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ isomorphic to $\mathcal{X}^{\mathrm{ss}}$.
2. For all $c \in \mathbb{Q}_{>0}$, the subset $|\mathcal{S}_c| := \{\lambda \in |\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})| \mid \langle \lambda, \ell \rangle = 1 \text{ and } \mu(\lambda) = M^{\mu}(\mathrm{ev}_1(\lambda)) = \sqrt{c}\}$ of $|\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})|$ is open, and thus it defines an open substack $\mathcal{S}_c$ of $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$.

3. The family $(\mathcal{S}_c)_{c\in\mathbb{Q}_{\geq 0}}$ is a (weak) Θ-stratification of $\mathcal{X}$ indexed by $\mathbb{Q}_{\geq 0}$, referred to as the Θ-stratification *induced by* (ℓ, q).

Remark **2.5.15.** If (ℓ, q) defines a Θ-stratification, then for all $c \in \mathbb{Q}_{\geq 0}$ we have the equality $|\mathcal{X}_{\leq c}| = \{x \in |\mathcal{X}| \mid M^\mu(x) \leq \sqrt{c}\}$ of $|\mathcal{X}|$, using the notation of Definition 2.5.1. In particular $\mathcal{X}^{\mathrm{ss}} = \mathcal{X}_{\leq 0}$, and note as well that the centre $\mathcal{Z}_0$ of the minimal stratum $\mathcal{S}_0 = \mathcal{X}^{\mathrm{ss}}$ is canonically isomorphic to $\mathcal{X}^{\mathrm{ss}}$.

Remark **2.5.16.** In Halpern-Leistner's definition, the stack $\mathrm{Filt}(\mathcal{X})$ is used instead of $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$. This is not an important difference, as we now explain. In [36, Definition 4.1.3], the Θ-stratification depends, in principle, on the choice of a complete set of representatives for $\pi_0(\mathrm{Filt}(\mathcal{X}))/\mathbb{N}^*$, although two different choices give rise to isomorphic locally closed Θ-strata. Since $\pi_0(\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}))/\mathbb{N}^* = \pi_0(\mathrm{Filt}(\mathcal{X}))/\mathbb{N}^*$, and since two connected components a and b of $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ and $\mathrm{Filt}(\mathcal{X})$ respectively that represent the same class are isomorphic via the action of $\mathbb{N}^*$, we can instead take a complete set of representatives in $\pi_0(\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}))$. We are using canonical set of representatives for the unstable strata, namely the components of $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ on which ℓ takes the value 1. This observation, together with Proposition 2.5.4, implies that the pair (ℓ, q) defines a Θ-stratification in the sense of [36, Definition 4.1.3] if and only if it does so in the sense above, and that in that case the locally closed Θ-strata that we get are isomorphic to Halpern-Leistner's. However, Definition 2.5.14 has the advantage that it does not depend on noncanonical choices. Moreover, for Conjecture 1.7.1 it is important to take the canonical choice of HN filtration and not just consider it up to scaling.

Remark **2.5.17** (Conventions on HN filtrations). Suppose that (ℓ, q) define a Θ-stratification of $\mathcal{X}$. For a filtration $\lambda \in \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})(k)$, where k is some field, we denote

$$\widehat{\lambda} = \begin{cases} \dfrac{\lambda}{\|\lambda\|^2}, & \|\lambda\| \neq 0, \\ \lambda, & \|\lambda\| = 0. \end{cases}$$

We have $\widehat{\widehat{\lambda}} = \lambda$. Since $\mathbb{Q}_{>0}$ acts on $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$, by open and closed immersion, the assignment $\lambda \mapsto \widehat{\lambda}$ gives an involution $\iota\colon \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) \to \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ commuting with $\mathrm{ev}_1\colon \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) \to \mathcal{X}$. Denoting $\widehat{\mathcal{S}}_c = \iota(\mathcal{S}_c)$ for $c \in \mathbb{Q}_{\geq 0}$, we have that $\left(\widehat{\mathcal{S}}_c\right)_{c\in\mathbb{Q}_{\geq 0}}$ defines a Θ-stratification, that we call the Θ-stratification associated to (ℓ, q) with *direct convention*, while we refer to the Θ-stratification $(\mathcal{S}_c)_{c\in\mathbb{Q}_{\geq 0}}$ in Definition 2.5.14 as the one with *inverse convention*. Unless otherwise stated, we will use the inverse convention in this thesis for Θ-stratifications induced by a linear form and a norm on graded points.

Remark **2.5.18** (Characterisation of the HN filtration)**.** It follows from the definitions that, if (ℓ, q) defines a Θ-stratification on $\mathcal{X}$, then the HN filtration with inverse convention (or simply *inverse HN filtration*) of a field-valued unstable point $x \in \mathcal{X}(k)$ is the unique $\lambda \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, x)$ such that $\langle \lambda, \ell \rangle \geq 1$ (equivalently, $\langle \lambda, \ell \rangle = 1$) and for all $\gamma \in \mathbb{Q}\text{-}\mathrm{Filt}\left(\mathcal{X}, x|_{\bar{k}}\right)$ such that $\langle \gamma, \ell \rangle \geq 1$ we have $\|\lambda\|^2 \leq \|\gamma\|^2$.

The HN filtration of x with direct convention (or simply *direct HN filtration*) is the unique $\eta \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, x)$ such that $\langle \eta, \ell \rangle \geq \|\eta\|^2$ (equivalently, $\langle \eta, \ell \rangle = \|\eta\|^2$) and such that for all $\gamma \in \mathbb{Q}\text{-}\mathrm{Filt}\left(\mathcal{X}, x|_{\bar{k}}\right)$ with $\langle \gamma, \ell \rangle \geq \|\gamma\|^2$ we have $\|\gamma\|^2 \leq \|\eta\|^2$. It can also be characterised as the unique element of $\mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, x)$ (equivalently, of $\mathbb{Q}\text{-}\mathrm{Filt}\left(\mathcal{X}, x_{\bar{k}}\right)$) maximising the function

$$\gamma \mapsto \langle \gamma, \ell \rangle - \frac{1}{2}\|\gamma\|^2$$

(see [38]). Note that, contrary to the case of the inverse HN filtration, this characterisation of the direct HN filtration treats semistable points and unstable points uniformly.

Proposition 2.5.19 (Compatibility with pullback)**.** *Let $\mathcal{X}'$ be an algebraic stack over an algebraic space B, satisfying Assumption 2.2.3, and let $h \colon \mathcal{X}' \to \mathcal{X}$ be either a closed immersion or a base change of a map between algebraic spaces like in Proposition 2.2.14. Suppose that the pair (ℓ, q) defines a Θ-stratification $(\mathcal{S}_c)_{c \in \mathbb{Q}_{\geq 0}}$. Then $(h^*\ell, h^*q)$ defines a Θ-stratification of $\mathcal{X}'$, equal to the induced stratification $(h^*\mathcal{S}_c)_{c \in \mathbb{Q}_{\geq 0}}$ of Proposition and Definition 2.5.8.*

Proof. The claim follows at once from the observation that, for every field k and point $x \in \mathcal{X}'(k)$, the induced map $\mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}', x) \to \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, h(x))$ is a bijection, compatible with the values of (ℓ, q) and $(h^*\ell, h^*q)$. □

2.6 Θ-STRATIFICATIONS FOR STACKS PROPER OVER A NORMED GOOD MODULI STACK

We now get to the main result (Theorem 2.6.4) about existence and properties of Θ-stratifications that we will use. It is an extension of [36, Theorem 5.6.1] where we also establish existence of good moduli spaces for the centres of the strata, along with other improvements. We first introduce the notion of *positive linear form on graded points*, a slight variant of [36, Definition 5.3.1].

Definition 2.6.1 (Positive linear form)**.** Let $\mathcal{X}$ and $\mathcal{Y}$ be algebraic stacks over an algebraic space B, satisfying Assumption 2.2.3. Let $f : \mathcal{Y} \to \mathcal{X}$ be a proper representable morphism and let ℓ be a linear form on graded points of $\mathcal{Y}$. We say that ℓ is *f-positive* provided that for all fields k and all diagrams

$$\begin{array}{ccc} \mathbb{P}^1_k/\mathbb{G}_{m,k} & \xrightarrow{\varphi} & \mathcal{Y} \\ \downarrow & & \downarrow f \\ B\mathbb{G}_{m,k} & \longrightarrow & \mathcal{X} \end{array} \tag{2.3}$$

such that the induced map $\mathbb{P}^1_k/\mathbb{G}_{m,k} \to B\mathbb{G}_{m,k} \times_{\mathcal{X}} \mathcal{Y}$ is finite, where the action of $\mathbb{G}_{m,k}$ on $\mathbb{P}^1_k$ is given by $t[a,b] = [ta,b]$ in projective coordinates and we denote $0 = [0,1]$ and $\infty = [1,0]$, we have

$$\ell(\varphi|_{\{0\}/\mathbb{G}_{m,k}}) < \ell(\varphi|_{\{\infty\}/\mathbb{G}_{m,k}}).$$

Remark **2.6.2.** Note that if a square (2.3) satisfying that $\mathbb{P}^1_k/\mathbb{G}_{m,k} \to B\mathbb{G}_{m,k} \times_{\mathcal{X}} \mathcal{Y}$ is finite exists, then the graded point $B\mathbb{G}_{m,k} \to \mathcal{X}$ is nondegenerate.

Example **2.6.3.** If $\mathcal{L}$ is a rational line bundle on $\mathcal{Y}$ that is f-ample and $\ell = \langle -, \mathcal{L} \rangle$, then ℓ is f-positive. Indeed, for all commutative diagrams as in Definition 2.6.1, the pullback $\varphi^*\mathcal{L}$ is ample relative to $\mathbb{P}^1_k/\mathbb{G}_{m,k} \to B\mathbb{G}_{m,k}$, and the claim follows after embedding $\mathbb{P}^1_k$ in a bigger projective space and looking at the weights of the corresponding $\mathbb{G}_{m,k}$-representation.

Theorem 2.6.4. *Let $\mathcal{X}$ be a noetherian algebraic stack with affine diagonal and a good moduli space $\pi\colon \mathcal{X} \to X$, endowed with a norm on graded points q. Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a proper representable morphism and let ℓ be an f-positive linear form on graded points of $\mathcal{Y}$. Then*

1. *The pair (ℓ, f^*q) defines a Θ-stratification $(\mathcal{S}_c)_{c \in \mathbb{Q}_{\geq 0}}$ of $\mathcal{Y}$.*
2. *For every $c \in \mathbb{Q}_{\geq 0}$, the centre $\mathcal{Z}_c$ has a good moduli space $\mathcal{Z}_c \to Z_c$. We denote $\mathcal{Y}^{ss} = \mathcal{Z}_0$ and $Y^{ss} = Z_0$.*
3. *For $c \in \mathbb{Q}_{\geq 0}$, let $\mathcal{X}_c$ be the union of connected components of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ intersecting the image of $\mathcal{Z}_c \to \mathrm{Grad}_{\mathbb{Q}}(\mathcal{Y}) \to \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$. Then $\mathcal{X}_c$ is quasi-compact and has a good moduli space $\mathcal{X}_c \to X_c$.*
4. *For every $c \in \mathbb{Q}_{\geq 0}$, the induced map $Z_c \to X_c$ is proper.*
5. *If $\ell = \langle -, \mathcal{L} \rangle$ for an f-ample rational line bundle $\mathcal{L}$ on $\mathcal{Y}$, then $Y^{ss} \to X$ is projective. If in addition the norm on graded points q is algebraic (Definition 2.4.9), then for all $c \in \mathbb{Q}_{\geq c}$ the map $Z_c \to X_c$ is projective.*

Remark **2.6.5.** The fact that we get a Θ-stratification instead of a weak Θ-stratification implies in particular rationality of HN filtrations. This comes at the expense of demanding $\mathcal{X}$ to have a good moduli space instead of an adequate moduli space [3], which is the less restrictive notion in positive characteristic.

Remark **2.6.6.** The proof of Theorem 2.6.4 expresses $Z_c \to X_c$ explicitly as Proj of a graded algebra on X_c.

Remark **2.6.7.** Let k be an algebraically closed field and let G be a smooth linearly reductive algebraic group over k. Then every norm on cocharacters of BG is algebraic by [28, Proposition 2]. In particular, if $\mathcal{X}$ is of the form $\mathcal{X} = U/G$ for a separated noetherian algebraic space U endowed with a G-action and if the norm q on $\mathcal{X}$ is induced by a norm on BG, then q is algebraic.

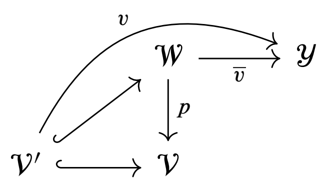
Example **2.6.8.** Let $\mathcal{X}$ be as in the statement of the theorem, and let $f : \mathcal{Y} \to \mathcal{X}$ be the blow-up of $\mathcal{X}$ along a closed substack $\mathcal{Z}$. Let $\mathcal{E} = f^{-1}(\mathcal{Z})$ be the exceptional divisor. The ideal sheaf $\mathcal{O}_{\mathcal{Y}}(-\mathcal{E})$ of the exceptional divisor is f-ample. Therefore we may apply the theorem with $\ell = \langle -, \mathcal{O}_{\mathcal{Y}}(-\mathcal{E})\rangle$ to get a Θ-stratification of $\mathcal{Y}$. In this case, the semistable locus $\mathcal{Y}^{\mathrm{ss}}$ is the *saturated blow-up* of $\mathcal{X}$ along $\mathcal{Z}$ [27, Definition 3.2], by [27, Proposition 3.17] and [36, Theorem 5.6.1, (2)].

More generally, if f is projective and $\ell = \langle -, \mathcal{L}\rangle$, with $\mathcal{L}$ an f-ample line bundle. Then $\mathcal{Y}^{\mathrm{ss}}$ is the *saturated* Proj [27, Definition 3.1], $\mathcal{Y}^{\mathrm{ss}} = \mathrm{Proj}^{\pi}_{\mathcal{X}}\left(\bigoplus_{n\in\mathbb{N}} f_*(\mathcal{L}^{\otimes n})\right)$.

Proof of Theorem 2.6.4. Note that $\mathcal{X}$ satisfies Assumption 2.2.3 with $B = X$ (Example 2.2.4). Let μ denote the numerical invariant defined by (ℓ, f^*q).

Step 1. The numerical invariant μ is strictly Θ-monotone over X [36, Definition 5.2.1] and strictly S-monotone [36, Definition 5.5.7] over X.

We first recall the definitions of Θ-monotonicity and S-monotonicity. Let R be a discrete valuation ring over X with uniformiser π, residue field k and field of fractions K. Let $\overline{\mathrm{ST}}_R = \mathrm{Spec}(R[s,t]/(st-\pi))/\mathbb{G}_m$, where s has weight 1 and t has weight -1 [8, Section 3.5.1]. Let $\mathcal{V}$ be the stack Θ_R (resp. the stack $\overline{\mathrm{ST}}_R$), and let $\mathcal{V}' = \mathcal{V} \setminus \{(0,0)\}$. The numerical invariant μ is said to be strictly Θ-monotone (resp. strictly S-monotone) if, for every such DVR R and for every morphism $v : \mathcal{V}' \to \mathcal{Y}$ such that $v|_{B\mathbb{G}_{m,K}}$ is nondegenerate and $\mu(v|_{B\mathbb{G}_{m,K}}) \geq 0$, there exists a diagram



satisfying the following properties:

1. The algebraic stack $\mathcal{W}$ is reduced and irreducible, the morphism $\overline{v}$ has quasi-finite relative inertia, and the morphism p is proper, relatively representable by Deligne–Mumford stacks and it is an isomorphism over $\mathcal{V}'$.

2. For any commutative square

$$\begin{array}{ccc} \mathbb{P}^1_k/\mathbb{G}_{m,k} & \xrightarrow{\varphi} & \mathcal{W} \\ \downarrow & & \downarrow p \\ B\mathbb{G}_{m,k} & \xrightarrow{g} & \mathcal{V} \end{array} \tag{2.4}$$

such that the induced map $\mathbb{P}^1_k/\mathbb{G}_{m,k} \to B\mathbb{G}_{m,k} \times_{\mathcal{V}} \mathcal{W}$ is finite and g is a positive multiple of the canonical graded point $\{(0,0)\}/\mathbb{G}_{m,k} \to \mathcal{V}$, we have the inequality

$$\mu(\overline{v} \circ \varphi|_{\{0\}/\mathbb{G}_{m,k}}) < \mu(\overline{v} \circ \varphi|_{\{\infty\}/\mathbb{G}_{m,k}}). \tag{2.5}$$

We follow the argument in the proof of [36, Proposition 5.3.3]. Suppose given a morphism $v\colon \mathcal{V}' \to \mathcal{Y}$. Since it has a good moduli space, the stack $\mathcal{X}$ is Θ-reductive [8, Definition 3.10] and S-complete [8, Definition 3.38] by [8, Theorem 5.4]. This means that any morphism $\mathcal{V}' \to \mathcal{X}$ extends uniquely to a morphism $\mathcal{V} \to \mathcal{X}$. Therefore the morphism $f \circ v\colon \mathcal{V}' \to \mathcal{X}$ extends to a map $u\colon \mathcal{V} \to \mathcal{X}$. Let $\mathcal{W}$ be the schematic image of $\mathcal{V}'$ inside the pullback $\mathcal{V} \times_{u,\mathcal{X},f} \mathcal{Y}$. We have a diagram

$$\begin{array}{ccccc} & & \mathcal{W} & \xrightarrow{\overline{v}} & \mathcal{Y} \\ & \nearrow & \downarrow p & & \downarrow f \\ \mathcal{V}' & \hookrightarrow & \mathcal{V} & \xrightarrow{u} & \mathcal{X} \end{array} \qquad (v\colon \mathcal{V}' \to \mathcal{Y})$$

and we want to show that the two conditions above are satisfied. The first condition follows trivially by construction. To check the second, suppose given a commutative square (2.4) as in the statement of the second condition. Since $\|\overline{v} \circ \varphi|_{\{0\}/\mathbb{G}_{m,k}}\| = \|\overline{v} \circ \varphi|_{\{\infty\}/\mathbb{G}_{m,k}}\|$, the norm being induced from $\mathcal{X}$, the inequality (2.5) is equivalent to

$$\ell(\overline{v} \circ \varphi|_{\{0\}/\mathbb{G}_{m,k}}) < \ell(\overline{v} \circ \varphi|_{\{\infty\}/\mathbb{G}_{m,k}}).$$

The map $\mathbb{P}^1_k/\mathbb{G}_{m,k} \to B\mathbb{G}_{m,k} \times_{\mathcal{X}} \mathcal{Y}$ is also finite, so the inequality follows from ℓ being f-positive.

Step 2. The pair (ℓ, f^*q) defines a weak Θ-stratification.

Since $\mathcal{Y}$ is quasi-compact, the numerical invariant μ trivially satisfies the *HN-boundedness* property, defined in [36, Proposition 4.4.2]. Therefore (ℓ, f^*q) defines a weak Θ-stratification if μ is strictly Θ-monotone [36, Theorem 5.2.3]. Thus the claim follows from Step 1.

Step 3. The weak Θ-stratification defined by (ℓ, f^*q) is a Θ-stratification.

Checking that a finite morphism $\mathcal{U} \to \mathcal{V}$ is a closed immersion can be done after base change along all geometric points $\operatorname{Spec} k \to \mathcal{V}$. Since the stratification is preserved by base change along any map $X' \to X$ Proposition 2.5.19, we may assume that $X = \operatorname{Spec} k$ is the spectrum of an algebraically closed field. If $\operatorname{Spec} k$ is of characteristic 0, then the weak Θ-stratification is a Θ-stratification by [36, Theorem 5.6.1]. Suppose k is of positive characteristic p. The stack $\mathcal{X}$ is of the form $\mathcal{X} = \operatorname{Spec} A/G$, where G is a linearly reductive group over k, by 2.1.5. Therefore we have $\mathcal{Y} = Y/G$, where $Y \to \operatorname{Spec} A$ is G-equivariant and projective. By Nagata's Theorem [24, Chapter IV, Section 3, Theorem 3.6], the identity component G° is a group of multiplicative type and p does not divide the order of G/G°. By [36, Lemma 2.1.7, (2)] it is enough to prove that for any $c \in \mathbb{Q}_{\geq 0}$ and for any k-point λ of $\mathcal{S}_c$, the induced map $\varphi\colon \operatorname{Lie}(\operatorname{Aut}_{\mathcal{S}_c}(\lambda)) \to \operatorname{Lie}(\operatorname{Aut}_{\mathcal{Y}}(\mathrm{ev}_1(\lambda))$ on Lie algebras is surjective. The filtration λ is contained in some open substack of $\operatorname{Filt}_{\mathbb{Q}}(\mathcal{Y})$ of the form $Y^{\gamma,+}/P(\gamma)$ for some one-parameter subgroup γ of G, by [36, Theorem 1.4.8], and it corresponds to a point $z \in Y^{\gamma,+}(k)$. Thus $\operatorname{Aut}_{\mathcal{S}_c}(\lambda) = \operatorname{Stab}_{P(\gamma)}(z)$, while $\operatorname{Aut}_{\mathcal{X}}(\mathrm{ev}_1(\lambda)) = \operatorname{Stab}_G(z)$. Both groups have the same identity component, equal to $\operatorname{Stab}_{G^\circ}(z)^\circ$, since G° is contained in $P(\lambda)$. Thus the map φ on Lie algebras is actually an isomorphism.

Step 4. The semistable locus $\mathcal{Y}^{ss}$ has a good moduli space $\mathcal{Y}^{ss} \to Y^{ss}$ and $Y^{ss} \to X$ is separated.

By [8, Theorem 5.4], $\mathcal{X}$ is Θ-reductive and S-complete. Therefore μ is strictly Θ-monotone and strictly S-monotone over X by Step 1. By [36, Theorem 5.5.8], the semistable locus $\mathcal{Y}^{\mathrm{ss}}$ is Θ-reductive and S-complete over X. Therefore it is enough to show, by [8, Theorem 5.4], that the stabiliser of every closed point of $\mathcal{Y}^{\mathrm{ss}}$ is linearly reductive. Let k be an algebraically closed field and let $x \in \mathcal{Y}^{\mathrm{ss}}(k)$ be a closed point. By [8, Proposition 3.47], the automorphism group $\operatorname{Aut}(x)$ is geometrically reductive. If the characteristic of k is 0, then $\operatorname{Aut}(x)$ is also linearly reductive. If k is of positive characteristic p, then $f(x)$ specialises to a k-point y closed in the fibre of $\pi\colon \mathcal{X} \to X$, whose stabiliser $\operatorname{Aut}(y)$ is thus linearly reductive. By Nagata's Theorem [24, Chapter IV, Section 3, Theorem 3.6], the identity component $\operatorname{Aut}(y)^\circ$ is a group of multiplicative type and p does not divide the order of $\operatorname{Aut}(y)/\operatorname{Aut}(y)^\circ$. Since $\operatorname{Aut}(x)$ is a subgroup of $\operatorname{Aut}(y)$, the same holds for $\operatorname{Aut}(x)$ and it is thus linearly reductive.

Step 5. The map $Y^{ss} \to X$ is proper.

Since $\mathcal{Y} \to \mathcal{X}$ is proper and $\mathcal{X} \to X$ is universally closed [2, Theorem 4.16], we have that $\mathcal{Y} \to X$ is universally closed. Now the Semistable Reduction Theorem [8, Corollary 6.12] implies that $\mathcal{Y}^{\mathrm{ss}} \to X$ is also universally closed. Therefore $Y^{\mathrm{ss}} \to X$

is universally closed. Since it is also separated, by Step 4, and of finite type, because f and π are, it is proper.

Step 6. The map $Y^{ss} \to X$ is projective if $\ell = \langle -, \mathcal{L} \rangle$ for a rational line bundle $\mathcal{L}$.

We may scale $\mathcal{L}$ up to assume that it is a line bundle, since this does not change the semistable locus. Then we have, by [36, Theorem 5.6.1, (3), (b)], that

$$Y^{\text{ss}} = \operatorname{Proj}_X \left(\pi_* f_* (\oplus_{n \in \mathbb{N}} \mathcal{L}^{\otimes n}) \right).$$

Step 7. For all $c \in \mathbb{Q}_{\geq 0}$, the stack $\mathcal{X}_c$ is quasi-compact and has a good moduli space $\mathcal{X}_c \to X_c$.

Since $\mathcal{Z}_c$ is quasi-compact, and because of the existence of a norm on graded points on $\mathcal{X}$, we have by [36, Proposition 3.8.2] that $\mathcal{X}_c$ is quasi-compact. The claim follows from Lemma 2.6.10.

Step 8. The centres $\mathcal{Z}_c$ have good moduli spaces $\mathcal{Z}_c \to Z_c$ for all $c \in \mathbb{Q}_{c \geq 0}$ and $Z_c \to X_c$ is proper.

For $c = 0$, the claim is the content of Step 4 and Step 5. If $c \in \mathbb{Q}_{>0}$, let $\overline{\mathcal{Z}}_c$ be the union of those connected components of $\operatorname{Grad}_{\mathbb{Q}}(\mathcal{Y})$ intersecting $\mathcal{Z}_c$. Again, because of the existence of a norm on graded points on $\mathcal{X}$ and $\mathcal{Y}$, we have by [36, Proposition 3.8.2] that $\overline{\mathcal{Z}}_c$ is quasi-compact. Let $f_c \colon \overline{\mathcal{Z}}_c \to \mathcal{X}_c$ be the restriction of $\operatorname{Grad}_{\mathbb{Q}}(f)$ to $\overline{\mathcal{Z}}_c$ and $\mathcal{X}_c$. By Proposition 2.2.16, the map f_c is representable and proper.

Let us denote $\ell|_{\overline{\mathcal{Z}}_c}$ the pullback of the linear form ℓ on $\mathcal{Y}$ along $\overline{\mathcal{Z}}_c \to \operatorname{Grad}_{\mathbb{Q}}(\mathcal{Y}) \to \mathcal{Y}$. We denote ℓ_{f^*q} the linear form on $\operatorname{Grad}_{\mathbb{Q}}(\mathcal{Y})$ induced by the norm f^*q on $\mathcal{Y}$ (Definition 2.4.7), and $\ell_{f^*q}|_{\overline{\mathcal{Z}}_c}$ its restriction to $\overline{\mathcal{Z}}_c$. Note that $\ell_{f^*q}|_{\overline{\mathcal{Z}}_c} = (f_c)^*(\ell_q|_{\mathcal{X}_c})$ is the pullback of the linear form $\ell_q|_{\mathcal{X}_c}$ on $\mathcal{X}_c$. We will consider the *shifted linear form*

$$\ell_c := \ell|_{\overline{\mathcal{Z}}_c} - c\,\ell_{f^*q}|_{\overline{\mathcal{Z}}_c} \tag{2.6}$$

on $\overline{\mathcal{Z}}_c$. We will use the following result, whose proof can be found in [38].

Theorem 2.6.9 (Linear Recognition Theorem)**.** *The centre $\mathcal{Z}_c$ is the semistable locus inside $\overline{\mathcal{Z}}_c$ with respect to the shifted linear form ℓ_c on $\overline{\mathcal{Z}}_c$.*

Therefore, we have a representable proper morphism $f_c \colon \overline{\mathcal{Z}}_c \to \mathcal{X}_c$ and a linear form ℓ_c and a norm on graded points $q|_{\mathcal{X}_c}$ on $\mathcal{X}_c$. By steps 4 and 5 applied to f_c in place of f, it is enough to show that ℓ_c is f_c-positive. For this, suppose given a commutative square

$$\begin{array}{ccc} \mathbb{P}^1_k/\mathbb{G}_{m,k} & \xrightarrow{\varphi} & \overline{\mathcal{Z}}_c \\ \downarrow & & \downarrow \\ B\mathbb{G}_{m,k} & \xrightarrow{g} & \mathcal{X}_c \end{array}$$

where the induced $\mathbb{P}^1_k/\mathbb{G}_{m,k} \to B\mathbb{G}_{m,k} \times_{\mathcal{X}_c} \overline{\mathcal{Z}}_c$ is finite. We want to show that

$$\ell_c(\varphi|_{\{0\}/\mathbb{G}_{m,k}}) < \ell_c(\varphi|_{\{\infty\}/\mathbb{G}_{m,k}}).$$

First, note that

$$\left(\ell_{f^*q}|_{\overline{\mathcal{Z}}_c}\right)(\varphi|_{\{0\}/\mathbb{G}_{m,k}}) = \ell_q(g) = \left(\ell_{f^*q}|_{\overline{\mathcal{Z}}_c}\right)(\varphi|_{\{\infty\}/\mathbb{G}_{m,k}}).$$

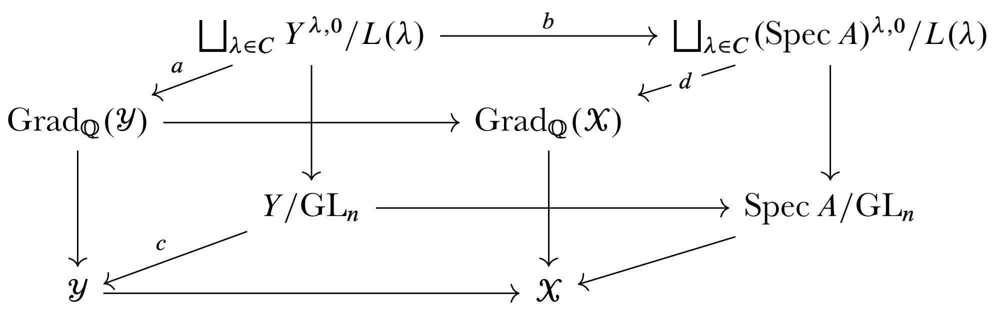
On the other hand, the map $\overline{\mathcal{Z}}_c \to \mathcal{X}_c \times_{\mathcal{X}} \mathcal{Y}$ is proper, since $\overline{\mathcal{Z}}_c \to \mathcal{X}_c$ and $\mathcal{Y} \to \mathcal{X}$ are proper; and also affine, since $\overline{\mathcal{Z}}_c \to \mathcal{Y}$ and $\mathcal{X}_c \to \mathcal{X}$ are affine. Thus the induced map $\mathbb{P}^1_k/\mathbb{G}_{m,k} \to B\mathbb{G}_{m,k} \times_{\mathcal{X}} \mathcal{Y}$ is finite. Therefore we have

$$\begin{aligned} &\ell_c(\varphi|_{\{\infty\}/\mathbb{G}_{m,k}}) - \ell_c(\varphi|_{\{0\}/\mathbb{G}_{m,k}}) = \\ &\ell|_{\overline{\mathcal{Z}}_c}(\varphi|_{\{\infty\}/\mathbb{G}_{m,k}}) - \ell|_{\overline{\mathcal{Z}}_c}(\varphi|_{\{0\}/\mathbb{G}_{m,k}}) > 0, \end{aligned}$$

because ℓ is f-positive.

Step 9. If $\ell = \langle -, \mathcal{L}\rangle$ for an f-ample line bundle $\mathcal{L}$ and q is algebraic, then for all $c \in \mathbb{Q}_{\geq 0}$ the map $Z_c \to X_c$ is projective.

We begin by showing that the pullback $\mathcal{M} := (\mathrm{Grad}_{\mathbb{Q}}(\mathcal{Y}) \to \mathcal{Y})^*\mathcal{L}$ is relatively ample with respect to $g = \mathrm{Grad}_{\mathbb{Q}}(f)\colon \mathrm{Grad}_{\mathbb{Q}}(\mathcal{Y}) \to \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$. For this, we may assume that $\mathcal{L}$ is a line bundle and we want to show that the canonical map $\mathcal{Y} \to \mathrm{Proj}\left(\bigoplus_{n\in\mathbb{N}} g_*(\mathcal{M}^{\otimes n})\right)$ is everywhere defined and an open immersion (although it is actually an isomorphism by properness of g). This can be checked étale locally on $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$. Chose a surjective, affine and strongly étale morphism $\rho\colon \mathrm{Spec}\, A/\mathrm{GL}_n \to \mathcal{X}$ (Theorem 2.1.3). Pulling back along ρ we get a cube



where we are using [36, Theorem 1.4.7] for the description of the stack of graded points of a quotient stack. Here, Y is a scheme acted on by GL_n with an equivariant map into $\mathrm{Spec}\, A$. We are denoting $C = \Gamma^{\mathbb{Z}}(T)/W$, where T is the standard maximal torus of GL_n and W is the Weyl group. For $\lambda \in \Gamma^{\mathbb{Z}}(T)$, $Y^{\lambda,0}$ and $(\mathrm{Spec}\, A)^{\lambda,0}$ denote the fixed point loci by the cocharacter λ, and $L(\lambda)$ is the centraliser of λ in GL_n.

The arrow d is an étale cover of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$. Thus we want to show that $a^*\mathcal{M}$ is ample relative to b. In fact, since the $(\mathrm{Spec}\, A)^{\lambda,0}$ are affine, it is enough to show that

$a^*\mathcal{M}|_{Y^{\lambda,0}}$ is ample on the scheme $Y^{\lambda,0}$. Now, $a^*\mathcal{M}|_{Y^{\lambda,0}} = (\mathcal{L}|_Y)|_{Y^{\lambda,0}}$, the line bundle $\mathcal{L}|_Y$ is ample and $Y^{\lambda,0} \to Y$ is affine, so $a^*\mathcal{M}|_{Y^{\lambda,0}}$ is ample too. This shows that $\mathcal{M}$ is relatively g-ample. In particular, if we let $\mathcal{M}_c = \mathcal{M}|_{\overline{\mathcal{Z}}_c}$, then $\mathcal{M}_c$ is relatively ample with respect to f_c.

Since q is algebraic, the linear form $\ell_{f^*q}|_{\overline{\mathcal{Z}}_c}$ is induced by a rational line bundle of the form $f_c^*\mathcal{N}$, where $\mathcal{N}$ is a rational line bundle on $\mathcal{X}_c$. Thus ℓ_c is induced by the rational line bundle $\mathcal{M}_c - cf_c^*\mathcal{N}$, which is relatively f_c-ample because $\mathcal{M}_c$ is. Therefore the claim follows from Step 6 applied to f_c in place of f. □

In the proof we used the following lemmas.

Lemma 2.6.10. *Let $\mathcal{X}$ be a noetherian algebraic stack with affine diagonal and a good moduli space $\pi\colon \mathcal{X} \to X$. For every quasi-compact open and closed substack $\mathcal{Z}$ of $\operatorname{Grad}_{\mathbb{Q}}(\mathcal{X})$, the composition*

$$\mathcal{Z} \to \operatorname{Grad}_{\mathbb{Q}}(\mathcal{X}) \xrightarrow{u} \mathcal{X}$$

is affine. In particular, $\mathcal{Z}$ has a good moduli space Z which is affine over X.

Proof. Let $\operatorname{Spec} A/\mathrm{GL}_n \to \mathcal{X}$ be affine, surjective and strongly étale (Theorem 2.1.3). We have a cartesian diagram

$$\begin{array}{ccc} \operatorname{Grad}_{\mathbb{Q}}(\operatorname{Spec} A/\mathrm{GL}_n) & \longrightarrow & \operatorname{Grad}_{\mathbb{Q}}(\mathcal{X}) \\ \downarrow & \ulcorner & \downarrow \\ \operatorname{Spec} A/\mathrm{GL}_n & \longrightarrow & \mathcal{X} \end{array}$$

and $\operatorname{Grad}_{\mathbb{Q}}(\operatorname{Spec} A/\mathrm{GL}_n)$ is a disjoint union of schemes of the form $\operatorname{Spec}(A)^{\lambda,0}/L(\lambda)$ with λ a rational cocharacter of GL_n. We conclude by Lemma 2.6.11 and descent. □

Lemma 2.6.11. *Let A be a commutative ring, and consider an action of GL_N on $X = \operatorname{Spec} A$ (over $\mathbb{Z}$) such that X/GL_N has a good moduli space. Let $\lambda\colon \mathbb{G}_m \to \mathrm{GL}_N$ be a cocharacter. Then the natural map $X^{\lambda,0}/L(\lambda) \to X/\mathrm{GL}_N$ is affine, where $X^{\lambda,0}$ is the fixed point locus of the $\mathbb{G}_m$-action on X induced by λ and $L(\lambda)$ is the centraliser of λ.*

Proof. There is a cartesian square

$$\begin{array}{ccc} \mathrm{GL}_N \times^{L(\lambda)} X^{\lambda,0} & \longrightarrow & X \\ \downarrow & \ulcorner & \downarrow \\ X^{\lambda,0}/L(\lambda) & \longrightarrow & X/\mathrm{GL}_N \end{array}$$

where $\mathrm{GL}_N \times^{L(\lambda)} X^{\lambda,0}$ is the stack quotient of $\mathrm{GL}_N \times X^{\lambda,0}$ by the diagonal action of $L(\lambda)$. Since the action is free, $\mathrm{GL}_N \times^{L(\lambda)} X^{\lambda,0}$ is an algebraic space. Now, $L(\lambda)$ is isomorphic to a product of GL_{N_i}'s and it is thus geometrically reductive [3, Definition 9.1.1]. Since $\mathrm{GL}_N \times X^{\lambda,0} = \operatorname{Spec} B$ is affine, the $L(\lambda)$-invariants give an adequate moduli space $\mathrm{GL}_N \times^{L(\lambda)} X^{\lambda,0} \to \operatorname{Spec}\left(B^{L(\lambda)}\right)$ [3, Theorem 9.1.4]. By universality for adequate moduli spaces [7, Theorem 3.12], we get an isomorphism $\mathrm{GL}_N \times^{L(\lambda)} X^{\lambda,0} = \operatorname{Spec} B^{L(\lambda)}$. Therefore $\mathrm{GL}_N \times^{L(\lambda)} X^{\lambda,0}$ is affine and we are done by descent. □

Remark **2.6.12.** Since $L(\lambda)$ is geometrically reductive and $X^{\lambda,0}$ is affine, taking $L(\lambda)$-invariants gives an adequate moduli space for $X^{\lambda,0}/L(\lambda)$. However, unless A is of characteristic 0, an extra argument is needed to show that the adequate moduli space is indeed a good moduli space.

2.7 FORMAL FANS AND THE DEGENERATION FAN

The set $\mathbb{Z}\text{-}\mathrm{Filt}(\mathcal{X}, x)$ of filtrations of a point x in an algebraic stack $\mathcal{X}$ can be endowed with some extra structure closely related both to fans in toric geometry and to spherical buildings of reductive groups. This extra structure can be encapsulated using Halpern-Leistner's combinatorial notion of formal fan [36, Definition 3.1.1], yielding the *degeneration fan* $\mathbf{DF}(\mathcal{X}, x)_\bullet$ [36, Definition 3.2.2]. We recall these notions here, with slightly modified conventions. We then study some special properties of the degeneration fan in the case when $\mathcal{X}$ has a good moduli space. In Chapter 6, we will define the *degeneration fan* $\mathbf{DF}(L)_\bullet$ *of an artinian lattice* L, in analogy with $\mathbf{DF}(\mathcal{X}, x)_\bullet$. The degeneration fan $\mathbf{DF}(\mathcal{X}, x)_\bullet$ can detect whether x is polystable (Proposition 2.7.14), while $\mathbf{DF}(L)_\bullet$ can detect whether L is complemented (Proposition 6.5.34). This will be used in the proof of the comparison between the iterated balanced filtration and the iterated HKKP filtration (Theorem 7.5.9).

Definition 2.7.1 (Category of cones)**.** Let A be a subring of $\mathbb{R}$, endowed with the inherited order. We let $\mathbf{Cone}_A$ be the subcategory of the category of A-modules whose objects are the finite free A-modules A^n, with $n \in \mathbb{N}$, and where a map $\varphi\colon A^n \to A^m$ is a homomorphism such that $\varphi\left(A^n_{\geq 0}\right) \subset A^m_{\geq 0}$. We denote $\mathbf{Cone} := \mathbf{Cone}_{\mathbb{Z}}$. It is naturally a subcategory of $\mathbf{Cone}_{\mathbb{Q}}$ consisting of maps $\varphi\colon \mathbb{Q}^n \to \mathbb{Q}^m$ such that $\varphi(\mathbb{Z}^n) \subset \mathbb{Z}^m$.

Definition 2.7.2 (Formal fan)**.** An *A-linear formal fan* $F_\bullet$ is a functor $F_\bullet\colon \mathbf{Cone}_A^{\mathrm{op}} \to \mathbf{Set}$.

If $F_\bullet$ is an A-linear formal fan, we denote $F_n = F(A^n)$. A $\mathbb{Z}$-linear formal fan will simply be called a *formal fan*, while by *rational formal fan* we mean a $\mathbb{Q}$-linear formal fan. A-linear formal fans form a category, the corresponding functor category.

Remark **2.7.3.** Note that in [36, Definition 3.1.1] a different category of cones is used where $A = \mathbb{Z}$ always and only injective homomorphisms $\varphi\colon \mathbb{Z}^n \to \mathbb{Z}^m$ are considered. This yields a slightly different category of $\mathbb{Z}$-linear formal fans.

If $F_\bullet$ is a formal fan, then the multiplicative monoid $\mathbb{Z}_{>0}$ acts on each F_n — if $l \in \mathbb{Z}_{>0}$, multiplication by l gives a map $\mathbb{Z}^n \to \mathbb{Z}^n$ in **Cone**, which in turns gives a map $F_n \to F_n$. We can localise this action by setting $F_n^{\mathbb{Q}} := \operatorname{colim}_{l \in \mathbb{Z}_{>0}} F_n$, which is the set of symbols a/l with $a \in F_n$ and $l \in \mathbb{Z}_{>0}$, and were we identify $a/l = a'/l'$ if there is $m \in \mathbb{Z}_{>0}$ such that $ml'a = mla'$. For a map $\varphi\colon \mathbb{Q}^m \to \mathbb{Q}^n$ in $\mathbf{Cone}_{\mathbb{Q}}$, there exists some $k \in \mathbb{Z}_{>0}$ such that $k\varphi$ sends $\mathbb{Z}^m$ into $\mathbb{Z}^n$. We define $\varphi^*\colon F_n^{\mathbb{Q}} \to F_m^{\mathbb{Q}}$ by the formula $\varphi^*(a/l) = (k\varphi)^*(a)/(kl)$. This gives a rational formal fan $F_\bullet^{\mathbb{Q}}$, that we will refer to as the *rational formal fan associated to* F.

The following definition is [36, Definition 3.2.2] with slightly modified conventions (see Remark 2.7.3).

Definition 2.7.4 (Degeneration fan)**.** Let $\mathcal{X}$ be an algebraic stack satisfying Assumption 2.2.3 and let $x \in \operatorname{Spec} k \to \mathcal{X}$ be a geometric point. The *degeneration fan* $\mathbf{DF}(\mathcal{X}, x)_\bullet$ is the formal fan given by $\mathbf{DF}(\mathcal{X}, x)_n = \operatorname{Hom}\big((\Theta_k^n, 1), (\mathcal{X}, x)\big)$, the set of pointed morphisms from $(\Theta_k^n, 1)$ to $(\mathcal{X}, x)$.

In principle, $\operatorname{Hom}\big((\Theta_k^n, 1), (\mathcal{X}, x)\big)$ is a groupoid, but because $\mathcal{X}$ has separated inertia, it is equivalent to a set. Note that we consider all morphisms $\lambda\colon (\Theta_k^n, 1) \to (\mathcal{X}, x)$ and not just nondegenerate ones, i.e. those for which the induced homomorphism $\mathbb{G}_{m,k}^n \to \operatorname{Aut}(\lambda(0))$ has finite kernel. The associated rational formal fan $\mathbf{DF}^{\mathbb{Q}}(\mathcal{X}, x)_\bullet$ will be called the *rational degeneration fan* of x. The definitions are such that $\mathbf{DF}(\mathcal{X}, x)_1 = \mathbb{Z}\text{-}\mathrm{Filt}(\mathcal{X}, x)$ and $\mathbf{DF}^{\mathbb{Q}}(\mathcal{X}, x)_1 = \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, x)$, and we will use both notations indistinctively.

For $\gamma \in \mathbf{DF}^{\mathbb{Q}}(\mathcal{X}, x)_n$, we denote $v_1\gamma, \dots, v_n\gamma \in \mathbf{DF}^{\mathbb{Q}}(\mathcal{X}, x)_1$ the pullbacks of γ along the maps $\mathbb{Z} \to \mathbb{Z}^n$ given by the standard basis of $\mathbb{Z}^n$. We have the following explicit description of the degeneration fan of a quotient stack, extending Remark 2.2.19.

Proposition 2.7.5. *Let k be a field, let G be a smooth affine algebraic group over k admitting a split maximal torus, let X be a separated scheme over k endowed with an action of G and let $\mathcal{X} = X/G$. Let $x \in \mathcal{X}(k)$ and let $x' \in X(k)$ be a point representing x. For a homomorphism $\gamma\colon \mathbb{G}_{m,k}^n \to G$, which is given by commuting cocharacters $\gamma_1, \dots, \gamma_n$, we say that* $\lim \gamma x'$ exists *if the map $\mathbb{G}_{m,k}^n \to X\colon t \mapsto \gamma(t)x'$ extends to $\mathbb{A}_k^n$, in which case the extension is unique by separatedness* *of X. Then*

$$\mathbf{DF}(\mathcal{X}, x)_n = \{\gamma\colon \mathbb{G}^n_{m,k} \to G \mid \lim \gamma x' \text{ exists}\}/\sim,$$

where $\gamma \sim \gamma^g$ if $g \in P(\gamma)(k)$. Here $P(\gamma)$ is the attractor [26, Definition 1.3.2] for the conjugation action of $\mathbb{G}^n_{m,k}$ on G induced by γ.

Proof. This follows directly from [36, Theorem 1.4.8]. □

Proposition 2.7.6. *Let $\mathcal{X}$ be an algebraic stack satisfying Assumption 2.2.3, and assume that $\mathcal{X}$ has a good moduli space $\pi\colon \mathcal{X} \to X$. Let $x\colon \mathrm{Spec}(k) \to \mathcal{X}$ be a geometric point. Then the map $\mathbf{DF}(\mathcal{X}, x)_n \to \mathbf{DF}(\mathcal{X}, x)_1^n\colon \gamma \mapsto (v_1\gamma, \dots, v_n\gamma)$ is injective for all $n \in \mathbb{Z}_{>0}$ and a bijection for $n = 2$. The same holds for $\mathbf{DF}^{\mathbb{Q}}(\mathcal{X}, x)_n \to (\mathbf{DF}^{\mathbb{Q}}(\mathcal{X}, x)_1)^n$.*

Proof. It is enough to prove the result for $\mathbf{DF}(\mathcal{X}, x)_\bullet$. Since $\mathbf{DF}(\mathcal{X}, x)_\bullet$ only depends on the fibre of the good moduli space of $\mathcal{X}$ containing x, we may assume by Corollary 2.1.5 that $\mathcal{X} = X/\mathrm{GL}_{N,k}$, where $X = \mathrm{Spec}\, A$ is affine. Let $x' \in X(k)$ be a point representing x. By Proposition 2.7.5 we have

$$\mathbf{DF}(\mathcal{X}, x)_n = \{\gamma\colon \mathbb{G}^n_{m,k} \to \mathrm{GL}_{N,k} \mid \lim \gamma x' \text{ exists}\}/\sim .$$

Since X is affine, $\lim \gamma x'$ exists if and only if $\lim \gamma_i x'$ exists for all i. To see this, we look at the $\mathbb{Z}^n$-grading $\oplus_{l \in \mathbb{Z}^n} A_l = A$ that γ induces on A. Let $\varphi\colon A \to k$ be the homomorphism defining x. We have that $\lim \gamma x'$ exists if and only if $\varphi(A_l) = 0$ unless $l \in \mathbb{Z}^n_{\geq 0}$, and that $\lim \gamma_i x'$ exists if and only if $\varphi(A_l) = 0$ unless $l_i \geq 0$. From this the statement follows. In particular, we have $P(\gamma) = P(\gamma_1) \cap \cdots \cap P(\gamma_n)$.

Now suppose that $\gamma'\colon \mathbb{G}^n_{m,k} \to \mathrm{GL}_{N,k}$ is such that γ_i and γ'_i define the same element of $\mathbf{DF}(\mathcal{X}, x)_1$ for each i, that is, there are $g_i \in P(\gamma_i)$ such that $\gamma'_i = \gamma_i^{g_i}$. The γ'_i are contained in some maximal torus T' of $P(\gamma)$. If T is another maximal torus of $P(\gamma)$ containing the γ_i and $g \in P(\gamma)$ is such that $gTg^{-1} = T'$, then, for each i, γ_i^g and γ'_i are commuting representatives of the same element of $\mathbf{DF}(\mathcal{X}, x)_\bullet$, so $\gamma_i^g = \gamma'_i$. Therefore $\gamma^g = \gamma'$. This proves injectivity of $\mathbf{DF}(\mathcal{X}, x)_n \to (\mathbf{DF}(\mathcal{X}, x)_1)^n$.

Now, if $\alpha_1, \alpha_2 \in \mathbf{DF}(\mathcal{X}, x)_1$, then there is a maximal torus of $\mathrm{GL}_{N,k}$ contained in $P(\alpha_1) \cap P(\alpha_2)$ [20, Theorem 10.3.6]. Thus there are two commuting representatives of α_1 and α_2, which gives $\gamma \in \mathbf{DF}(\mathcal{X}, x)_2$ such that $v_1\gamma = \alpha_1$ and $v_2\gamma = \alpha_2$. □

For $a, b \in \mathbb{N}$, we denote $\binom{a}{b}$ the homomorphism $\mathbb{Z} \to \mathbb{Z}^2\colon l \mapsto (la, lb)$, which is a Map in **Cone**.

Definition 2.7.7 (Sum of two filtrations)**.** Let $\mathcal{X}$ be an algebraic stack satisfying Assumption 2.2.3 and assume that $\mathcal{X}$ has a good moduli space. Let $x\colon \mathrm{Spec}(k) \to \mathcal{X}$ be a geometric point. For $\lambda_1, \lambda_2 \in \mathbf{DF}(\mathcal{X}, x)_1$ (resp. $\mathbf{DF}^{\mathbb{Q}}(\mathcal{X}, x)_1$), let γ be the unique element of $\mathbf{DF}(\mathcal{X},x)_2$ (resp. $\mathbf{DF}^{\mathbb{Q}}(\mathcal{X},x)_2$) such that $v_1\gamma = \lambda_1$ and $v_2\gamma = \lambda_2$. We define the *sum* $\lambda_1 + \lambda_2 := \binom{1}{1}^* \gamma$, which is again an element of $\mathbf{DF}(\mathcal{X},x)_1$ (resp. $\mathbf{DF}^{\mathbb{Q}}(\mathcal{X},x)_1$).

Remark **2.7.8.** The sum in $\mathbf{DF}(\mathcal{X},x)_1$ satisfies the following properties:

1. $\forall \lambda, \mu \in \mathbf{DF}(\mathcal{X},x)_1, \quad \lambda + \mu = \mu + \lambda,$
2. $\forall \lambda \in \mathbf{DF}(\mathcal{X},x)_1, \quad \lambda + 0 = \lambda,$
3. $\forall a, b \in \mathbb{Z}_{\geq 0}, \forall \lambda \in \mathbf{DF}(\mathcal{X},x)_1, \quad (a+b)\lambda = a\lambda + b\lambda.$

The same properties hold for the sum in $\mathbf{DF}^{\mathbb{Q}}(\mathcal{X},x)_1$. Thus we can see $\mathbf{DF}(\mathcal{X},x)_1$, $+, 0)$ (resp. $(\mathbf{DF}^{\mathbb{Q}}(\mathcal{X},x)_1, +, 0)$) as a commutative magma with zero on which $\mathbb{Z}_{\geq 0}$ (resp. $\mathbb{Q}_{\geq 0}$) acts in a compatible way. Note that the addition $+$ on $\mathbf{DF}(\mathcal{X},x)_1$ need not be associative.

Definition 2.7.9. Let $\mathcal{X}$ be a good moduli stack satisfying Assumption 2.2.3 and let $x\colon \mathrm{Spec}(k) \to \mathcal{X}$ be a geometric point. We say that $\gamma_1, \dots, \gamma_n \in \mathbf{DF}(\mathcal{X},x)_1$ *commute* if there is $\gamma \in \mathbf{DF}(\mathcal{X},x)_n$ such that $v_i\gamma = \gamma_i$ for $i \in \{1, \dots, n\}$. In that case, γ is unique by Proposition 2.7.6 and we denote $\gamma = \gamma_1 \boxplus \dots \boxplus \gamma_n$.

An *apartment* of $\mathbf{DF}(\mathcal{X},x)_1$ is a subset $S \subset \mathbf{DF}(\mathcal{X},x)_1$ such that any finite subset of S commutes and S is maximal with this property.

Similar definitions apply to $\mathbf{DF}^{\mathbb{Q}}(\mathcal{X},x)_1$.

Remark **2.7.10.** Using the language of the proof of Proposition 2.7.6, we have that if $\gamma_1, \dots, \gamma_n \in \mathbf{DF}(\mathcal{X},x)_1$ are represented by one-parameter subgroups $\gamma_i'\colon \mathbb{G}_{m,k} \to \mathrm{GL}_{N,k}$ then $\gamma_1, \dots, \gamma_n$ commute if and only if $\gamma_1', \dots, \gamma_n'$ factor through a common torus (that is, if they commute as one-parameter subgroups of $\mathrm{GL}_{N,k}$). If T is a maximal torus of $\mathrm{GL}_{N,k}$, then the set S_T of filtrations $\gamma \in \mathbf{DF}(\mathcal{X},x)_1$ that can be represented by a one-parameter subgroup of T is an apartment, and all apartments are of the form S_T for some T. Sum and multiplication of scalars in S_T coincide with the corresponding operations in the set $\Gamma^{\mathbb{Z}}(T)$ of cocharacters of T. Therefore, the sum of filtrations inside a given apartment is associative. Thus if $\gamma_1, \dots, \gamma_n$ are filtrations in the same apartment and $a_1, \dots, a_n \in \mathbb{Z}_{\geq 0}$, then the expression $a_1\gamma_1 + \dots + a_n\gamma_n$ is unambiguously defined.

From the description of filtrations used in the proof of Proposition 2.7.6, we easily deduce the following.

Proposition 2.7.11. *Let $\mathcal{X}$ be a good moduli stack satisfying Assumption 2.2.3 and let $x \in \mathcal{X}(k)$ be a geometric point. Let $h\colon \mathbb{Z}^n \to \mathbb{Z}^m$ be a map in* **Cone** *given by a matrix $(a_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$ and* *let* $\gamma = \gamma_1 \boxplus \cdots \boxplus \gamma_m \in \mathbf{DF}(\mathcal{X}, x)_m$. *Then the pullback of* γ *along* h *is*

$$h^*\gamma = \left(\sum_i a_{i1}\gamma_i\right) \boxplus \cdots \boxplus \left(\sum_i a_{in}\gamma_i\right). \qquad \square$$

Remark **2.7.12.** From Propositions 2.7.6 and 2.7.11 we see that the degeneration fan $\mathbf{DF}(\mathcal{X}, x)_\bullet$ of a geometric point in a good moduli stack determines and is determined by the following data:

1. The set $\mathbf{DF}(\mathcal{X}, x)_1$ of integral filtrations.
2. The apartments of $\mathbf{DF}(\mathcal{X}, x)_1$.
3. The sum of filtrations in $\mathbf{DF}(\mathcal{X}, x)_1$.
4. The multiplication of scalars $a \in \mathbb{Z}_{\geq 0}$ in $\mathbf{DF}(\mathcal{X}, x)_1$.

Remark **2.7.13.** The degeneration fan $\mathbf{DF}(\mathcal{X}, x)_\bullet$ of a geometric point in a good moduli stack is analogous to the spherical building of a reductive group G [74]. The apartments of $\mathbf{DF}(BG, \mathrm{pt})_1$ coincide with the apartments of the spherical building of G in the sense of Tits. Note however that $\mathbf{DF}(BG, \mathrm{pt})_\bullet$ does not contain the information of which are the chambers of the building.

On the other hand, if X is a toric variety for a torus T and x is a point in the open orbit, then $\mathbf{DF}^{\mathbb{Q}}(X/T, x)_\bullet$ is the fan of X without a choice of embedding into $\Gamma^{\mathbb{Q}}(T)$. See [36, Section 3.2.1].

We say that $\mu \in \mathbf{DF}(\mathcal{X}, x)_1$ is an *opposite* of $\lambda \in \mathbf{DF}(\mathcal{X}, x)_1$ if $\lambda + \mu = 0$. An element $\lambda \in \mathbf{DF}(\mathcal{X}, x)_1$ may have several opposites (see Remark 6.5.33 and Proposition 7.1.3).

The degeneration fan encodes geometric information about the stack around a point. The following proposition is an example of this.

Proposition 2.7.14. *Let* $\mathcal{X}$ *be an algebraic stack satisfying Assumption 2.2.3 and assume that* $\mathcal{X}$ *has a good moduli space* $\pi\colon \mathcal{X} \to X$. *Let* $x\colon \mathrm{Spec}(k) \to \mathcal{X}$ *be a geometric point. Then* x *is polystable (that is, it is closed in the fibre* $\pi^{-1}\pi(x)$*) if and only if every element of* $\mathbf{DF}(\mathcal{X}, x)_1$ *has an opposite.*

Proof. We can write $\pi^{-1}\pi(x) = \mathrm{Spec}\, A/G$ by Corollary 2.1.5, where G is linearly reductive and x is given by a closed point $x' \in (\mathrm{Spec}\, A)(k)$ whose stabiliser is G.

If x is closed, then $\mathbf{DF}(\mathcal{X}, x)_\bullet$ is isomorphic to the degeneration fan of a point of BG. Every element λ of $\mathbf{DF}(\mathcal{X}, x)_1$ has thus an opposite, which is given by the inverse of a cocharacter of G defining λ.

If x is not closed, then there is $\lambda \in \mathbf{DF}(\mathcal{X}, x)_1$ such that $\lambda(0)$ is closed [8, Lemma 3.24]. Choose a representative $\lambda' \in \Gamma^{\mathbb{Z}}(G)$ of λ such that $(\lambda')^{-1}$ represents an opposite of λ. This gives a map $u\colon \mathbb{A}^1_k \to \operatorname{Spec} A$ corresponding to λ' and a map $v\colon \mathbb{A}^1_k \to \operatorname{Spec} A$ corresponding to $(\lambda')^{-1}$. The two maps glue to give $w\colon \mathbb{P}^1_k \to \operatorname{Spec} A$. The image of w is a single point, since the target is affine. Thus $\lambda(0) = \lambda(1)$, a contradiction. $\square$

CHAPTER 3

SEQUENTIAL STRATIFICATIONS AND THE ITERATED BALANCED FILTRATION

The goal of this chapter is the construction of the *balancing stratification* for every noetherian normed good moduli stack $\mathcal{X}$ with affine diagonal (Theorem 3.5.2 and Definition 3.5.3) and derive from it the definition of the *iterated balanced filtration* of a point of $\mathcal{X}$ (Definition 3.5.8). From this, we define the *refined Harder-Narasimhan stratification* induced by a norm and a linear form on a stack by reducing to the centres of the Θ-strata (Definition 3.5.10).

We start by defining a first approximation of the iterated balanced filtration, simply called the *balanced filtration* (Definition 3.1.6). We then construct a stack of sequential filtrations $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ for a stack $\mathcal{X}$, and define a notion of *sequential stratification* (Definition 3.3.1), roughly a weak analogue of Θ-stratification with the stack $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ used instead of $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$. For the purpose of using induction in the construction of the balancing stratification, we introduce the concept of *central rank* of a stack (Definition 3.4.1) and show that it increases after taking the centre of an unstable stratum in a blow-up (Lemma 3.4.7). After our main construction (Theorem 3.5.2), we prove some functorial properties of the balancing stratification (Proposition 3.5.5). We finish the section with a collection of natural examples of normed good moduli stacks in moduli theory, including GIT quotients, moduli of Bridgeland semistable objects and moduli of K-semistable Fano varieties.

3.1 THE BALANCED FILTRATION

Let $\mathcal{X}$ be an algebraic stack over an algebraic space B, satisfying Assumption 2.2.3. Let $\mathcal{Z} \to \mathcal{X}$ be a closed substack, let k be a field, let x be a k-point of $\mathcal{X}$ and let $\lambda \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, x)$ be a rational filtration of x. We would like to formalise the idea of velocity at which $\lambda(t)$ tends to $\mathcal{Z}$ when t tends to 0. For that, we first write $\lambda = \gamma/m$ with $\gamma \in \mathbb{Z}\text{-}\mathrm{Filt}(\mathcal{X}, x)$ an integral filtration and $m \in \mathbb{Z}_{>0}$. We then form the pullback

$$\begin{array}{ccc} R/\mathbb{G}_{m,k} & \longrightarrow & \mathcal{Z} \\ \downarrow & & \downarrow \\ \mathbb{A}^1_k/\mathbb{G}_{m,k} & \xrightarrow{\gamma} & \mathcal{X} \end{array}$$

of $\mathcal{Z} \to \mathcal{X}$ along γ, which is given by a $\mathbb{G}_{m,k}$-equivariant closed subscheme R of $\mathbb{A}^1_k$, thus necessarily of the form $R = \mathrm{Spec}(k[x]/(x^n))$ for some $n \in \mathbb{N} \cup \{\infty\}$. The following concept is used by Kempf in [53].

Definition 3.1.1 (Kempf's intersection number)**.** We define *Kempf's intersection number* (or simply *Kempf's number*) to be $\langle \lambda, \mathcal{Z} \rangle := n/m \in \mathbb{Q}_{\geq 0} \cup \{\infty\}$.

The definition does not depend on the presentation $\lambda = \gamma/m$ because if $l \in \mathbb{Z}_{>0}$, then $\langle l\gamma, \mathcal{Z} \rangle = ln$. More generally, the linearity property $\langle c\lambda, \mathcal{Z} \rangle = c\langle \lambda, \mathcal{Z} \rangle$ holds for all $c \in \mathbb{Q}_{>0}$. We have $\langle \lambda, \mathcal{Z} \rangle = \infty$ precisely when x is in $\mathcal{Z}$, and $\langle \lambda, \mathcal{Z} \rangle = 0$ if and only if $\mathrm{ev}_0(\lambda)$ is not in $\mathcal{Z}$. If x is not in $\mathcal{Z}$ but $\mathrm{ev}_0(\lambda)$ is in $\mathcal{Z}$, then $\langle \lambda, \mathcal{Z} \rangle$ is a positive rational number that can be thought of as the velocity at which λ approaches $\mathcal{Z}$.

Proposition 3.1.2. *Suppose x is not in $\mathcal{Z}$. Let $p \colon \mathrm{Bl}_{\mathcal{Z}} \mathcal{X} \to \mathcal{X}$ be the blow-up of $\mathcal{X}$ along $\mathcal{Z}$, let $\mathcal{L}$ be the standard p-ample line bundle on $\mathrm{Bl}_{\mathcal{Z}} \mathcal{X}$, and let $x' \in \mathrm{Bl}_{\mathcal{Z}} \mathcal{X}(k)$ be the lift of x to $\mathrm{Bl}_{\mathcal{Z}} \mathcal{X}$. Let $\lambda' \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathrm{Bl}_{\mathcal{Z}} \mathcal{X}, x')$ be the unique lift of λ to a rational filtration of x', which exists by Proposition 2.2.20. Then $\langle \lambda', \mathcal{L} \rangle = \langle \lambda, \mathcal{Z} \rangle$.*

Proof. By linearity of $\langle -, \mathcal{Z} \rangle$, we may assume that λ is integral. By definition, $\mathcal{L}$ is the ideal sheaf of the exceptional divisor $\mathcal{E} = p^{-1}(\mathcal{Z})$. We have a diagram

$$\begin{array}{ccccc} R/\mathbb{G}_{m,k} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{Z} \\ \downarrow & & \downarrow & & \downarrow \\ \Theta_k & \xrightarrow{\lambda'} & \mathrm{Bl}_{\mathcal{Z}} \mathcal{X} & \xrightarrow{p} & \mathcal{X}, \end{array}$$

where $R = \mathrm{Spec}\, k[x]/(x^n)$ for some $n \in \mathbb{N}$, the variable x having weight -1. There is a natural injection $\mathcal{L} \to \mathcal{O}_{\mathrm{Bl}_{\mathcal{Z}} \mathcal{X}}$. Pulling back along λ' we get a map $(\lambda')^*\mathcal{L} \to \mathcal{O}_{\Theta_k}$ whose image $\mathcal{I} = (x^n)$ is the ideal sheaf of $R/\mathbb{G}_{m,k}$. Since there is a surjective map $(\lambda')^*\mathcal{L} \to \mathcal{I}$ and both source and target are line bundles, the map should be an isomorphism. Thus $(\lambda')^*\mathcal{L} \cong \mathcal{I} \cong \mathcal{O}_{\Theta_k}(-n)$, where $\mathcal{O}_{\Theta_k}(-n)$ is the pullback of $\mathcal{O}_{B\mathbb{G}_{m,k}}(-n)$ along the structure map $\Theta_k \to B\mathbb{G}_{m,k}$, because x^n has weight $-n$. Therefore $\langle \lambda', \mathcal{L} \rangle = -\mathrm{wt}\big(((\lambda')^*\mathcal{L})|_{B\mathbb{G}_{m,k}}\big) = n = \langle \lambda, \mathcal{Z} \rangle$, as desired. □

The following result is a generalisation of Kempf's Theorem [53, Theorem 3.4] to stacks with good moduli space. See also [36, Example 5.3.7].

THEOREM 3.1.3 (Kempf)**.** *Let $\mathcal{X}$ be a noetherian algebraic stack with affine diagonal and a good moduli space $\pi\colon \mathcal{X} \to X$. Let k be a field, let $\mathcal{Z} \to \mathcal{X}$ be a closed substack and let x be a k-point of $\mathcal{X} \setminus \mathcal{Z}$. Then:*

1. *The intersection $|\mathcal{Z}| \cap \left|\pi^{-1}\pi(x)\right|$ is nonempty if and only if there is a filtration $\lambda \in \mathbb{Z}\text{-}\mathrm{Filt}(\mathcal{X}, x)$ with $\mathrm{ev}_0(\lambda)$ in $\mathcal{Z}$.*
2. *Suppose that $\mathcal{X}$ is endowed with a norm on graded points. If $|\mathcal{Z}| \cap \left|\pi^{-1}\pi(x)\right| \neq \varnothing$, then there is a unique $\lambda \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, x)$ with $\langle \lambda, \mathcal{Z}\rangle \geq 1$ and such that for all $\gamma \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, x|_{\bar{k}})$ with $\langle \gamma, \mathcal{Z}\rangle \geq 1$ we have $\|\lambda\| \leq \|\gamma\|$.*

Remark **3.1.4.** Note that we do not require k to be perfect as in the original Kempf's Theorem. This is possible because we are working with good moduli spaces instead of adequate moduli spaces, that are more general in positive characteristic.

Proof. By replacing $\mathcal{X}$ by $\pi^{-1}\pi(x)$ we may assume that $X = \operatorname{Spec} k$. By Corollary 2.1.4, $\mathcal{X} = \operatorname{Spec} A/\mathrm{GL}_{n,k}$, where A is a k-algebra of finite type.

We first show Item 1 assuming k is algebraically closed. Suppose $\mathcal{Z} \neq \varnothing$. Since $\mathcal{X}$ has a unique closed point [2, Proposition 9.1], we also have that $\overline{\{x\}} \cap |\mathcal{Z}|$ is nonempty. By [53, Theorem 1.4], there exists $\lambda \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, x)$ such that $\mathrm{ev}_1(\lambda) = x$ and $\mathrm{ev}_0(\lambda)$ is in $\mathcal{Z}$.

Now we show Item 2 for any k. We are given a norm on graded points on $\mathcal{X}$. Let x' be a lift of x to the blow-up $\mathrm{Bl}_{\mathcal{Z}}\,\mathcal{X}$. By Theorem 2.6.4 and Example 2.6.8 there is a Θ-stratification on $\mathrm{Bl}_{\mathcal{Z}}\,\mathcal{X}$. By Proposition 2.2.20 we have $\mathbb{Q}\text{-}\mathrm{Filt}\left(\mathcal{X}, x|_{\bar{k}}\right) = \mathbb{Q}\text{-}\mathrm{Filt}\left(\mathrm{Bl}_{\mathcal{Z}}\,\mathcal{X}, x'|_{\bar{k}}\right)$, and by Item 1 in the algebraically closed case and Proposition 3.1.2 we have that x' is semistable if and only if $|\mathcal{Z}| = \varnothing$. If x' is unstable, then its HN filtration is the unique $\lambda \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, x)$ in Item 2, also by Proposition 3.1.2.

By choosing any norm on cocharacters of $\mathrm{GL}_{n,k}$ and pulling it back to $\mathcal{X}$, we see that Item 2 readily implies Item 1 for any k. □

We now shift attention to the case where $\mathcal{Z}$ is the locus $\mathcal{X}^{\max}$ of maximal dimension of stabiliser groups of $\mathcal{X}$, whose definition we recall below. Suppose $\mathcal{X}$ is noetherian with affine diagonal. For each $d \in \mathbb{N}$, the set $\{x \in |\mathcal{X}| \mid \dim \mathrm{Aut}(x) \geq d\}$ is closed [25, Exposé VIb, Proposition 4.1]. Therefore, by quasi-compactness of $\mathcal{X}$, it makes sense to define:

DEFINITION 3.1.5 (Maximal stabiliser dimension)**.** Let $\mathcal{X}$ be a noetherian algebraic stack with affine diagonal. The *maximal stabiliser dimension* $d(\mathcal{X}) \in \mathbb{N}$ of $\mathcal{X}$ is the maximal dimension of the stabiliser group of a point of $\mathcal{X}$. For the empty stack we set $d(\varnothing) = -\infty$.

As a topological space, $|\mathcal{X}^{\max}| = \{x \in |\mathcal{X}| \mid \dim \operatorname{Aut}(x) = d(\mathcal{X})\}$, and it is a closed subset of $|\mathcal{X}|$. It is a nontrivial result [27, Proposition C.5] that if $\mathcal{X}$ is a noetherian good moduli stack with affine diagonal, then $|\mathcal{X}^{\max}|$ can be given a natural structure of closed substack of $\mathcal{X}$, denoted $\mathcal{X}^{\max}$ and called the *maximal dimension stabiliser locus* of $\mathcal{X}$. It can be characterised étale locally by the property that, if $\mathcal{X} = X/G$, where X is an algebraic space and $G \to \operatorname{Spec} \mathbb{Z}$ is an affine flat group scheme of finite type that is either diagonalisable or a Chevalley group, with fibres of dimension $d(\mathcal{X})$, then $\mathcal{X}^{\max} = X^{G_\circ}/G$, where $G_\circ$ is the reduced identity component of G [27, Section C.2]. In general, the functorial definition of $\mathcal{X}^{\max}$ is as follows. A map $f : T \to \mathcal{X}$ factors through $\mathcal{X}^{\max}$ if and only if the pullback $f^*\mathcal{I}_{\mathcal{X}}$ of the inertia of $\mathcal{X}$ has a smooth closed subgroup all whose fibres are connected and of dimension $d(\mathcal{X})$.

One reason why Edidin–Rydh's stack structure on $\mathcal{X}^{\max}$ is better behaved than the reduced structure is that it behaves well with respect to base change. If $\mathcal{Y} \to \mathcal{X}$ is a closed immersion and $d(\mathcal{Y}) = d(\mathcal{X})$, then $\mathcal{Y}^{\max} = \mathcal{Y} \times_{\mathcal{X}} \mathcal{X}^{\max}$. Similarly, if $\mathcal{Y}$ has a good moduli space, $\mathcal{Y} \to \mathcal{X}$ is representable, étale and separated, and $d(\mathcal{Y}) = d(\mathcal{X})$, then $\mathcal{Y}^{\max} = \mathcal{Y} \times_{\mathcal{X}} \mathcal{X}^{\max}$. See [27, Proposition C.5]. We get to the main definition of this section.

Definition 3.1.6 (The balanced filtration)**.** Let $\mathcal{X}$ be a normed noetherian algebraic stack with affine diagonal and a good moduli space $\pi : \mathcal{X} \to X$. Let $x : \operatorname{Spec} k \to \mathcal{X}$ be a field-valued point and denote $\mathcal{F} = \pi^{-1}\pi(x)$. We define the *balanced filtration* $\lambda_{\mathrm{b}}(x)$ of x to be the unique element λ of $\mathbb{Q}\text{-}\operatorname{Filt}(\mathcal{X}, x)$ satisfying $\langle \lambda, \mathcal{F}^{\max} \rangle \geq 1$ and such that for all filtrations $\gamma \in \mathbb{Q}\text{-}\operatorname{Filt}(\mathcal{X}, x|_{\bar{k}})$ with $\langle \gamma, \mathcal{F}^{\max} \rangle \geq 1$ we have $\|\lambda\| \leq \|\gamma\|$.

Note that we have an identification $\mathbb{Q}\text{-}\operatorname{Filt}(\mathcal{X}, x) = \mathbb{Q}\text{-}\operatorname{Filt}(\mathcal{F}, x)$ and that existence and uniqueness of the balanced filtration is guaranteed by Theorem 3.1.3. The balanced filtration of x is 0 precisely when x is closed in $\mathcal{F}$ or, equivalently, when x lies in $\mathcal{F}^{\max}$.

In the case where $d(\mathcal{F}) = d(\mathcal{X})$, from the fact that $\mathcal{X}^{\max} \cap \mathcal{F} = \mathcal{F}^{\max}$ it follows that $\langle \lambda, \mathcal{F}^{\max} \rangle = \langle \lambda, \mathcal{X}^{\max} \rangle$. By Proposition 3.1.2, if x is not in $\mathcal{X}^{\max}$ then the balanced filtration of x coincides with the inverse HN filtration of x in the blow-up $\operatorname{Bl}_{\mathcal{X}^{\max}} \mathcal{X}$. We have a Θ-stratification $(\mathcal{S}_c)_{c \in \mathbb{Q}_{\geq 0}}$ of $\operatorname{Bl}_{\mathcal{X}^{\max}} \mathcal{X}$ by type of inverse HN filtration (Theorem 2.6.4), and thus a stratification of $\mathcal{X}$ where the strata are $\mathcal{X}^{\max}$ and the $\mathcal{S}_c \setminus \mathcal{E}$ with $c \geq 0$, where $\mathcal{E}$ is the exceptional divisor. This can be seen as a stratification of $\mathcal{X}$ by type of balanced filtration. Our goal is to refine this stratification by iterating the blowing-up process from the centres $\mathcal{Z}_c$ of the strata $\mathcal{S}_c$ (Theorem 3.5.2). The iterated strata will naturally live inside a stack of sequential filtrations, defined in the following section.

We finish this section with a useful technical fact that we will need later. We thank Daniel Halpern-Leistner for pointing out that Kempf's theorem can be used to prove it.

Proposition 3.1.7. *Let k be an algebraically closed field, let G be a linearly reductive algebraic group over k and let $X = \operatorname{Spec} A$ be an affine scheme of finite type over k, endowed with an action of G. Suppose that there is a point $x_0 \in X(k)$ fixed by G and that the map $X/G \to \operatorname{Spec} k$ is a good moduli space. Then the maps*

$$\pi_0(\operatorname{Grad}(X/G)) \to \pi_0(\operatorname{Grad}(BG))$$

and

$$\pi_0(\operatorname{Grad}_{\mathbb{Q}}(X/G)) \to \pi_0(\operatorname{Grad}_{\mathbb{Q}}(BG))$$

induced by $X/G \to BG$ are bijective. In particular, every norm on graded points of X/G is the pullback along $X/G \to BG$ of a unique norm on graded points of BG.

Proof. The morphism $BG_{\text{red}} \to BG$ is finite and a universal homeomorphism so, by [36, Proposition 1.3.2], the induced map $\operatorname{Filt}(X/G_{\text{red}}) \to \operatorname{Filt}(X/G)$ is a universal homeomorphism too. Since $\operatorname{gr}\colon \operatorname{Filt}(X/G) \to \operatorname{Grad}(X/G)$ induces a bijection on connected components [36, Lemma 1.3.8], and similarly for X/G_{red}, we have that $\pi_0(\operatorname{Grad}(X/G_{\text{red}})) \to \pi_0(\operatorname{Grad}(X/G))$ is a bijection. As a particular case, $\pi_0(\operatorname{Grad}(BG_{\text{red}})) \to \pi_0(\operatorname{Grad}(BG))$ is a bijection. Therefore, we may replace G by G_{red} and assume that G is smooth.

It is enough to prove the claim for Grad. By Example 2.2.13, the statement is equivalent to the fixed point locus $X^{\alpha,0}$ being connected for every $\alpha\colon \mathbb{G}_{m,k} \to G$. Let $y \in X^{\alpha,0}(k)$ be a point different from x_0 and let $\lambda \in \Gamma^{\mathbb{Q}}(G)$ be a representative of the balanced filtration of y in X/G for some choice of norm on graded points of G. For every $t \in \mathbb{G}_{m,k}(k)$, the conjugate cocharacter $\lambda^{\alpha(t)}$ is also a representative of the balanced filtration of y. Therefore λ and $\lambda^{\alpha(t)}$ are conjugate inside $P(\lambda)$ and, since this holds for all $t \in \mathbb{G}_{m,k}(k)$, we have that α factors through $P(\lambda)$. If T_1 and T_2 are maximal tori inside $P(\lambda)$ containing α and λ respectively, there is $g \in P(\lambda)(k)$ such that $gT_2g^{-1} = T_1$. Then α and λ^g commute, so the morphism $\mathbb{A}^1 \to X\colon t \mapsto \lambda^g(t)y$ (extended by taking the limit when t tends to 0) factors through $X^{\alpha,0}$ and connects y and x_0. □

3.2 STACKS OF SEQUENTIAL FILTRATIONS

The main goal of this section is to define a stack $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ of $\mathbb{Q}^\infty$-*filtrations* (or *sequential filtrations*) for a suitable algebraic stack $\mathcal{X}$. Here, the symbol $\mathbb{Q}^\infty$ denotes the direct sum $\mathbb{Q}^{\oplus\mathbb{N}}$ of countably many copies of $\mathbb{Q}$ with lexicographic order. We start with a simpler version of the stack $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$.

Definition 3.2.1 (Stack of $\mathbb{Q}^n_{\mathrm{lex}}$-filtrations)**.** Let $\mathcal{X}$ be an algebraic stack over an algebraic space B, satisfying Assumption 2.2.3. We define the stack $\mathrm{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X})$ of $\mathbb{Q}^n$-*filtrations of $\mathcal{X}$ with lexicographic order* to be the limit of the diagram

$$\begin{array}{ccccccc}
\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) & & \mathrm{Filt}_{\mathbb{Q}}\,\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}) & & \cdots & & \mathrm{Filt}_{\mathbb{Q}}\,\mathrm{Grad}^{n-1}_{\mathbb{Q}}(\mathcal{X}) \\
& \searrow^{\mathrm{gr}} & \downarrow{\scriptstyle \mathrm{ev}_1} & \searrow^{\mathrm{gr}} & \downarrow & \searrow & \downarrow{\scriptstyle \mathrm{ev}_1} \\
& & \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}) & & \mathrm{Grad}^2_{\mathbb{Q}}(\mathcal{X}) & & \mathrm{Grad}^{n-1}_{\mathbb{Q}}(\mathcal{X}).
\end{array}$$

Thus $\mathrm{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X})$ is just the fibre product

$$\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) \underset{\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})}{\times} \mathrm{Filt}_{\mathbb{Q}}\,\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}) \underset{\mathrm{Grad}^2_{\mathbb{Q}}(\mathcal{X})}{\times} \cdots \underset{\mathrm{Grad}^{n-1}_{\mathbb{Q}}(\mathcal{X})}{\times} \mathrm{Filt}_{\mathbb{Q}}\,\mathrm{Grad}^{n-1}_{\mathbb{Q}}(\mathcal{X}).$$

We define, for each $n \in \mathbb{Z}_{>0}$, a map $\mathrm{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X}) \to \mathrm{Filt}_{\mathbb{Q}^{n+1}_{\mathrm{lex}}}(\mathcal{X})$ by

$$\begin{array}{ccc}
\mathrm{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X}) & & \mathrm{Filt}_{\mathbb{Q}^{n+1}_{\mathrm{lex}}}(\mathcal{X}) \\
\| & & \| \\
\mathrm{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X}) \times_{\mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X})} \mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X}) & \xrightarrow{1\times o} & \mathrm{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X}) \times_{\mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X})} \mathrm{Filt}_{\mathbb{Q}}\,\mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X}),
\end{array}$$

where $o\colon \mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X}) \to \mathrm{Filt}_{\mathbb{Q}}\,\mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X})$ is the "trivial filtration" map. If $\mathcal{X}$ satisfies Assumption 2.2.3, then so does $\mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X})$, so by [36, Proposition 1.3.9] and the argument in [36, Proposition 1.3.11], the morphism o is an open and closed immersion. Thus each of the maps $\mathrm{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X}) \to \mathrm{Filt}_{\mathbb{Q}^{n+1}_{\mathrm{lex}}}(\mathcal{X})$ is an open immersion. We also have morphisms $\mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X}) \to \mathrm{Grad}_{\mathbb{Q}}\,\mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X}) = \mathrm{Grad}^{n+1}_{\mathbb{Q}}(\mathcal{X})$ given by the "trivial grading" maps, that are also open and closed immersions.

Definition 3.2.2 (Stack of sequential filtrations)**.** Let $\mathcal{X}$ be an algebraic stack over an algebraic space B, satisfying Assumption 2.2.3. We define the stack $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ of $\mathbb{Q}^\infty$-*filtrations* (or *sequential filtrations*) of $\mathcal{X}$ as the colimit

$$\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) = \mathrm{colim}_{n\in\mathbb{Z}_{>0}}\, \mathrm{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X})$$

in the 2-category $\mathbf{St}_{\mathrm{fppf}}$ of stacks for the fppf site of schemes. Similarly, we define the stack $\mathrm{Grad}_{\mathbb{Q}^\infty}(\mathcal{X})$ of $\mathbb{Q}^\infty$-graded points of $\mathcal{X}$ as

$$\mathrm{Grad}_{\mathbb{Q}^\infty}(\mathcal{X}) = \mathrm{colim}_{n\in\mathbb{Z}_{>0}}\, \mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X}).$$

As a direct application of Lemma 2.2.10, we get:

PROPOSITION 3.2.3. *Let $\mathcal{X}$ be an algebraic stack over an algebraic space B, satisfying Assumption 2.2.3. Then the stacks* $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ *and* $\mathrm{Grad}_{\mathbb{Q}^\infty}(\mathcal{X})$ *are algebraic, naturally defined over B, and also satisfy Assumption 2.2.3.* □

Remark **3.2.4.** Here we are regarding $\mathbb{Q}^\infty$ as having lexicographic order. It would be more precise to use the notation $\mathrm{Filt}_{\mathbb{Q}^\infty_{\mathrm{lex}}}(\mathcal{X})$ for what we called $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$, to distinguish it from a stack of $\mathbb{Q}^\infty$-filtrations with product order, which can also be defined. Since we will not use such a stack, our notation will not be problematic.

For each n, we have an associated graded map $\mathrm{gr}\colon \mathrm{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X}) \to \mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X})$, defined as the composition of the projection $\mathrm{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X}) \to \mathrm{Filt}_{\mathbb{Q}}\,\mathrm{Grad}^{n-1}_{\mathbb{Q}}(\mathcal{X})$ and the associated graded map $\mathrm{Filt}_{\mathbb{Q}}\,\mathrm{Grad}^{n-1}_{\mathbb{Q}}(\mathcal{X}) \to \mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X})$. These maps glue to a morphism $\mathrm{gr}\colon \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \to \mathrm{Grad}_{\mathbb{Q}^\infty}(\mathcal{X})$. Similarly, we get an "evaluation at 1" map $\mathrm{ev}_1\colon \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \to \mathcal{X}$, a "forgetful" map $u\colon \mathrm{Grad}_{\mathbb{Q}^\infty}(\mathcal{X}) \to \mathcal{X}$, and a "split filtration" map $\sigma\colon \mathrm{Grad}_{\mathbb{Q}^\infty}(\mathcal{X}) \to \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ as in Section 2.2. We will also consider the "trivial filtration" map $\mathcal{X} \to \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$, defined as the composition of the usual trivial filtration map $\mathcal{X} \to \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ and the map $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) = \mathrm{Filt}_{\mathbb{Q}^1_{\mathrm{lex}}}(\mathcal{X}) \to \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ given by the colimit. It is an open and closed immersion. Similarly, we have a "trivial grading" map $\mathcal{X} \to \mathrm{Grad}_{\mathbb{Q}^\infty}(\mathcal{X})$, and it is also an open and closed immersion.

PROPOSITION 3.2.5. *Let $\mathcal{X}$ be an algebraic stack over an algebraic space B, satisfying Assumption 2.2.3. Then the "evaluation at 1" map* $\mathrm{ev}_1\colon \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \to \mathcal{X}$ *is representable and separated.*

Proof. It is enough to prove that $\mathrm{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X}) \to \mathcal{X}$ is representable and separated for each n. There is a cartesian square

$$\begin{array}{ccc} \mathrm{Filt}_{\mathbb{Q}^{n+1}_{\mathrm{lex}}}(\mathcal{X}) & \xrightarrow{a_n} & \mathrm{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X}) \\ \downarrow & \ulcorner & \downarrow \\ \mathrm{Filt}_{\mathbb{Q}}\,\mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X}) & \xrightarrow{b_n} & \mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X}) \end{array}$$

for each n, where b_n is the "evaluation at 1" map. Thus by [36, Proposition 1.1.13], the map a_n is representable and separated, being a base change of b_n. Expressing $\mathrm{ev}_1\colon \mathrm{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X}) \to \mathcal{X}$ as a composition of the a_n, we get the result. □

Remark **3.2.6.** One can define stacks $\mathrm{Filt}_{\mathbb{Z}^\infty}(\mathcal{X})$ and $\mathrm{Grad}_{\mathbb{Z}^\infty}(\mathcal{X})$ in a similar vein. The monoid $(\mathbb{N}, \cdot, 1)$ acts on these stacks, and $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ and $\mathrm{Grad}_{\mathbb{Q}^\infty}(\mathcal{X})$ are obtained from these by localising the action as in Definition 2.2.7.

Remark **3.2.7.** The formation of $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ is functorial in $\mathcal{X}$. If $f\colon \mathcal{X} \to \mathcal{Y}$ is a morphism, then there is an obvious induced map $\mathrm{Filt}_{\mathbb{Q}^\infty}(f)\colon \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \to \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{Y})$.

Remark **3.2.8** (Stack of polynomial filtrations). Our definition of the stack $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ as a colimit of the stack $\mathrm{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X})$ is justified by the fact that the poset $\mathbb{Q}^\infty$ can be written as the colimit $\mathbb{Q}^\infty = \mathrm{colim}_n\, \mathbb{Q}^n_{\mathrm{lex}}$ where the maps are $\mathbb{Q}^n_{\mathrm{lex}} \to \mathbb{Q}^{n+1}_{\mathrm{lex}}\colon (a_0, \ldots, a_{n-1}) \mapsto (a_0, \ldots, a_{n-1}, 0)$. Alternatively, we can consider the diagram where the maps are $\mathbb{Q}^n_{\mathrm{lex}} \to \mathbb{Q}^{n+1}_{\mathrm{lex}}\colon (a_0, \ldots, a_{n-1}) \mapsto (0, a_0, \ldots, a_{n-1})$, and the colimit is now $\mathbb{Q}[t]$, the set of polynomials in one variable with rational coefficients, where $p \le q$ if $p(n) \le q(n)$ for $n \gg 0$. We define the stack $\mathrm{Filt}_{\mathbb{Q}[t]}(\mathcal{X})$ of *polynomial filtrations* of $\mathcal{X}$ to be the colimit of the corresponding diagram of open and closed immersions $\mathrm{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X}) \to \mathrm{Filt}_{\mathbb{Q}^{n+1}_{\mathrm{lex}}}(\mathcal{X})$, which are given as the composition

$$\begin{aligned}\mathrm{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X}) \to \mathrm{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})) &= \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}) \times_{\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})} \mathrm{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})) \\ &\xrightarrow{\sigma\times\mathrm{id}} \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) \times_{\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})} \mathrm{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})) = \mathrm{Filt}_{\mathbb{Q}^{n+1}_{\mathrm{lex}}}(\mathcal{X}).\end{aligned}$$

Everything we have said about $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ applies also to $\mathrm{Filt}_{\mathbb{Q}[t]}(\mathcal{X})$ with similar arguments.

The stack of sequential filtrations behaves well with respect to pullback from an algebraic space.

Proposition 3.2.9. *Let $\mathcal{X} \to B$ and $\mathcal{X}' \to B'$ satisfy Assumption 2.2.3, and let*

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ \big\downarrow & \lrcorner & \big\downarrow \\ X' & \longrightarrow & X \end{array}$$

be a cartesian square, with X, X' algebraic spaces. Then

$$\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}') \cong \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \times_{\mathrm{ev}_1, \mathcal{X}} \mathcal{X}' \cong \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \times_X X'.$$

Proof. Since $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}')$ is an increasing union of the stacks $\mathrm{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X}')$, it is enough to show the analogue claim for these stacks. For $n = 1$, this is Proposition 2.2.14. For $n > 1$, we have

$$\begin{aligned}\mathcal{X}' \times_{\mathcal{X},\mathrm{ev}_1} \mathrm{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X}) &= \mathcal{X}' \times_{\mathcal{X},\mathrm{ev}_1} \mathrm{Filt}_{\mathbb{Q}^{n-1}_{\mathrm{lex}}}(\mathcal{X}) \times_{\mathrm{gr},\mathrm{Grad}^{n-1}_{\mathbb{Q}}(\mathcal{X}),\mathrm{ev}_1} \mathrm{Filt}_{\mathbb{Q}}(\mathrm{Grad}^{n-1}_{\mathbb{Q}}(\mathcal{X})) \\ &= \mathrm{Filt}_{\mathbb{Q}^{n-1}_{\mathrm{lex}}}(\mathcal{X}') \times_{\mathrm{gr},\mathrm{Grad}^{n-1}_{\mathbb{Q}}(\mathcal{X}')} \mathrm{Grad}^{n-1}_{\mathbb{Q}}(\mathcal{X}') \times_{\mathrm{Grad}^{n-1}_{\mathbb{Q}}(\mathcal{X}),\mathrm{ev}_1} \mathrm{Filt}_{\mathbb{Q}}\left(\mathrm{Grad}^{n-1}_{\mathbb{Q}}(\mathcal{X})\right) \\ &= \mathrm{Filt}_{\mathbb{Q}^{n-1}_{\mathrm{lex}}}(\mathcal{X}') \times_{\mathrm{gr},\mathrm{Grad}^{n-1}_{\mathbb{Q}}(\mathcal{X}'),\mathrm{ev}_1} \mathrm{Filt}_{\mathbb{Q}}\left(\mathrm{Grad}^{n-1}_{\mathbb{Q}}(\mathcal{X}')\right) = \mathrm{Filt}_{\mathbb{Q}^{n-1}_{\mathrm{lex}}}(\mathcal{X}')\end{aligned}$$

again by Proposition 2.2.14. ∎

PROPOSITION 3.2.10. *Let $\mathcal{X}$ be an algebraic stack defined over an algebraic space B, satisfying Assumption 2.2.3, and let $\mathcal{X}' \to \mathcal{X}$ be a closed immersion. Then*

$$\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}') \cong \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \times_{\mathrm{ev}_1,\mathcal{X}} \mathcal{X}'.$$

Proof. The statement follows in the same way as Proposition 3.2.9, but using Proposition 2.2.15 instead of Proposition 2.2.14. □

PROPOSITION 3.2.11. *Let $\mathcal{X}$ be an algebraic stack over an algebraic space B, satisfying Assumption 2.2.3. Then there is a cartesian diagram*

$$\begin{array}{ccc} \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) & \longrightarrow & \mathrm{Filt}_{\mathbb{Q}^\infty}\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}) \\ \downarrow & \lrcorner & \downarrow \mathrm{ev}_1 \\ \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) & \xrightarrow{\mathrm{gr}} & \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}) \end{array}$$

Proof. The claim follows from cartesianity of the diagram

$$\begin{array}{ccc} \mathrm{Filt}_{\mathbb{Q}^{n+1}_{\mathrm{lex}}}(\mathcal{X}) & \longrightarrow & \mathrm{Filt}_{\mathbb{Q}^{n}_{\mathrm{lex}}}\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}) \\ \downarrow & \lrcorner & \downarrow \mathrm{ev}_1 \\ \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) & \xrightarrow{\mathrm{gr}} & \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}) \end{array}$$

by taking the colimit when n tends to ∞. □

We define $\mathbb{Q}^\infty$-flag spaces in analogy with Definition 2.2.17.

DEFINITION 3.2.12 ($\mathbb{Q}^\infty$-Flag spaces)**.** Let $\mathcal{X}$ be an algebraic stack over an algebraic space B, satisfying Assumption 2.2.3, and let $x\colon T \to \mathcal{X}$ be a scheme-valued point. We define the *space of* $\mathbb{Q}^\infty$*-flags* $\mathrm{Flag}_{\mathbb{Q}^\infty}(\mathcal{X}, x)$ *of* x as the fibre product

$$\begin{array}{ccc} \mathrm{Flag}_{\mathbb{Q}^\infty}(\mathcal{X}, x) & \longrightarrow & T \\ \downarrow & \ulcorner & \downarrow x \\ \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) & \xrightarrow{\mathrm{ev}_1} & \mathcal{X}, \end{array}$$

which is, by Proposition 3.2.5, a separated algebraic space over T locally of finite presentation.

In the case of a field-valued point x, it is reasonable to talk about $\mathbb{Q}^\infty$-filtrations.

DEFINITION 3.2.13 ($\mathbb{Q}^\infty$-filtrations of a point)**.** Let $\mathcal{X}$ be an algebraic stack over an algebraic space B, satisfying Assumption 2.2.3. Let k be a field and let $x\colon \mathrm{Spec}\, k \to \mathcal{X}$ point. We define the set $\mathbb{Q}^\infty\text{-}\mathrm{Filt}(\mathcal{X}, x)$ of $\mathbb{Q}^\infty$*-filtrations* (or *sequential filtrations*) of x to be $\mathbb{Q}^\infty\text{-}\mathrm{Filt}(\mathcal{X}, x) := \mathrm{Flag}_{\mathbb{Q}^\infty}(\mathcal{X}, x)(k)$, the set of k-points of the space of $\mathbb{Q}^\infty$-flags of x.

Remark **3.2.14.** The set $\mathbb{Q}^\infty$- $\mathrm{Filt}(\mathcal{X}, x)$ can be described as follows. An element λ of $\mathbb{Q}^\infty$- $\mathrm{Filt}(\mathcal{X}, x)$ is uniquely determined by a sequence $(\lambda_n)_{n\in\mathbb{N}}$ where

1. $\lambda_0 \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, x)$,
2. $\lambda_n \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X}), \mathrm{gr}\,\lambda_{n-1})$ for $n \geq 1$, and
3. $\lambda_n = 0$ for $n \gg 0$.[1]

The trivial $\mathbb{Q}^\infty$-filtration corresponds to the sequence where $\lambda_n = 0$ for all n. In general,

$$N := \min\{n \mid \lambda_i = 0, \ \forall i \geq n\}$$

is the minimal natural number such that λ is in $\mathrm{Filt}_{\mathbb{Q}^N_{\mathrm{lex}}}(\mathcal{X}) \subset \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$. The description follows at once from the definition of $\mathrm{Filt}_{\mathbb{Q}^N_{\mathrm{lex}}}(\mathcal{X})$ as a fibre product.

Remark **3.2.15.** If $f\colon \mathcal{X}' \to \mathcal{X}$ is either a closed immersion or a base change from a map of algebraic spaces, and if $x \in \mathcal{X}'(k)$ is a field-valued point, then, by Propositions 3.2.9 and 3.2.10, we have a canonical bijection $\mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}', x) \cong \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, f(x))$. We will use this fact throughout.

Remark **3.2.16** (Sequential filtrations on quotient stacks). Let k be a field, and consider a quotient stack $\mathcal{X} = X/G$ where X is a separated scheme of finite type over k and G is a linear algebraic group over k. Let $x \in X(k)$ be a k-point and call also $x \in \mathcal{X}(k)$ its image in $\mathcal{X}$. From Remarks 2.2.19 and 3.2.14, it follows that we have an identification of the set $\mathbb{Q}^\infty$- $\mathrm{Filt}(\mathcal{X}, x)$ of sequential filtrations of x with the set of sequences $(\lambda_n)_{n\in\mathbb{N}}$ where

1. each $\lambda_n \in \Gamma^{\mathbb{Q}}(G)$;
2. for all $n, m \in \mathbb{N}$, λ_n and λ_m commute;
3. $\lambda_n = 0$ for $n \gg 0$;
4. for every $n \in \mathbb{Z}_{>0}$, the iterated limit

$$\lim_{t_n\to 0} \lambda_n(t_n) \left(\lim_{t_{n-1}\to 0} \lambda_{n-1}(t_{n-1}) \left(\cdots \lim_{t_0 \to 0} \lambda_0(t_0)x \right) \cdots \right)$$

   exists in X;

subject to the equivalence relation that identifies $(\lambda_n)_{n\in\mathbb{N}} \sim (\lambda'_n)_{n\in\mathbb{N}}$ if there are $g_n \in P_{L(\lambda_0,\cdots,\lambda_{n-1})}(\lambda_n)$ such that $(\lambda_n)^{g_n g_{n-1}\cdots g_0} = \lambda'_n$ for all $n \in \mathbb{N}$. To see this, note that we can explicitly describe components of $\mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X})$ as $X^{\lambda_0,\cdots,\lambda_n,0}/L(\lambda_0, \cdots, \lambda_n)$, where $\lambda_0, \cdots, \lambda_n$ are commuting rational cocharacters of G, $X^{\lambda_0,\cdots,\lambda_n,0}$ denotes the fixed-point locus by $\lambda_0, \cdots, \lambda_n$ and $L(\lambda_0, \cdots, \lambda_n)$ is the centraliser of $\lambda_0, \cdots, \lambda_n$ inside G, and we can apply Remark 2.2.19 to these quotient stacks.

[1] Here, 0 denotes the trivial filtration in $\mathbb{Q}\text{-}\mathrm{Filt}(\mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X}), \mathrm{gr}\,\lambda_{n-1})$, see Definition 2.2.18.

3.3 SEQUENTIAL STRATIFICATIONS

In this section, we give the definition of *sequential stratification*, and we study some pullback and pushforward operations for sequential strata. All algebraic stacks are defined over an algebraic space B and are assumed to satisfy Assumption 2.2.3.

DEFINITION 3.3.1 (Sequential stratification)**.** Let $\mathcal{X}$ be an algebraic stack and let Γ be a partially ordered set. A *sequential stratification* (or $\mathbb{Q}^\infty$*-stratification*) of $\mathcal{X}$ indexed by Γ is a family $(\mathcal{S}_i)_{i\in\Gamma}$ of locally closed substacks of $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ such that:

1. Each composition $r_i \colon \mathcal{S}_i \to \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \to \mathcal{X}$ is a locally closed immersion.
2. The $r_i(|\mathcal{S}_i|)$ are pairwise disjoint and cover $|\mathcal{X}|$.
3. For each $i \in \Gamma$, the union $\bigcup_{j\le i} r_j\left(|\mathcal{S}_j|\right)$ is open in $|\mathcal{X}|$.

A sequential stratification defines a canonical sequential filtration for every point.

DEFINITION 3.3.2 (Induced sequential filtration)**.** Let $\mathcal{X}$ be an algebraic stack and let $(\mathcal{S}_i)_{i\in\Gamma}$ be a sequential stratification of $\mathcal{X}$ indexed by a poset Γ. For a field-valued point $x \in \mathcal{X}(k)$, we define the *sequential filtration* λ *of* x *induced by the stratification* $(\mathcal{S}_i)_{i\in\Gamma}$ as follows. Let $i \in \Gamma$ be the unique element such that $x \colon \mathrm{Spec}\, k \to \mathcal{X}$ factors through $\mathcal{S}_i \hookrightarrow \mathcal{X}$. We have cartesian squares

$$\begin{array}{ccc} \mathrm{Spec}\, k & \longrightarrow & \mathcal{S}_i \\ {\scriptstyle\lambda}\downarrow & \ulcorner & \downarrow \\ \mathrm{Flag}_{\mathbb{Q}^\infty}(\mathcal{X}, x) & \longrightarrow & \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \\ \downarrow & \ulcorner & \downarrow{\scriptstyle \mathrm{ev}_1} \\ \mathrm{Spec}\, k & \xrightarrow[x]{} & \mathcal{X}, \end{array}$$

and we define λ to be the top left downward arrow $\lambda \colon \mathrm{Spec}\, k \to \mathrm{Flag}_{\mathbb{Q}^\infty}(\mathcal{X}, x)$, which is by definition an element of $\mathbb{Q}^\infty\text{-}\,\mathrm{Filt}(\mathcal{X}, x)$.

Remark **3.3.3** (Sequential and polynomial Θ-stratifications). This definition is not quite the analogue of the notion of Θ-stratification for $\mathbb{Q}^\infty$-filtrations. That would require in addition that each $\mathcal{S}_i$ is open in $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ and is the preimage of an open substack of $\mathrm{Grad}_{\mathbb{Q}^\infty}(\mathcal{X})$, but these conditions do not hold for the balancing stratification, that we will construct later (Definition 3.5.3). We may call this stronger notion *sequential* Θ*-stratifications*. Similarly, if we use the stack $\mathrm{Filt}_{\mathbb{Q}[t]}(\mathcal{X})$ of polynomial filtrations (Remark 3.2.8) instead of $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$, we get a notion of *polynomial* Θ*-stratification*. The stratification of the stack of pure coherent sheaves on a polarised projective scheme over a noetherian base by polynomial Harder–Narasimhan filtration [66] should be a polynomial Θ-stratification. Another example should be given by the stratifications for moduli spaces of principal ρ-sheaves considered in [32].

Definition 3.3.4 (Pulling back sequential stratifications)**.** Let $f\colon \mathcal{X}' \to \mathcal{X}$ be a morphism of algebraic stacks such that

$$\begin{array}{ccc} \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}') & \longrightarrow & \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \\ {\scriptstyle \mathrm{ev}_1}\downarrow & \ulcorner & \downarrow{\scriptstyle \mathrm{ev}_1} \\ \mathcal{X}' & \xrightarrow{\quad f \quad} & \mathcal{X} \end{array}$$

is cartesian (for example a closed immersion or a base change from a map of algebraic spaces, see Propositions 3.2.9 and 3.2.10). Let $(\mathcal{S}_i)_{i\in\Gamma}$ be a sequential stratification of $\mathcal{X}$. For each $i \in \Gamma$, set $f^*\mathcal{S}_i := \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}') \times_{\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})} \mathcal{S}_i$, which is a locally closed substack of $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}')$. Then $(f^*\mathcal{S}_i)_{i\in\Gamma}$ is a sequential stratification of $\mathcal{X}'$, called the *pulled back* sequential stratification.

To see that $(f^*\mathcal{S}_i)_{i\in\Gamma}$ is indeed a sequential stratification, just note that we have $f^*\mathcal{S}_i = \mathcal{X}' \times_{\mathcal{X}} \mathcal{S}_i$ for all $i \in \Gamma$. We refer to the $\mathcal{S}_i$ as *sequential strata*. More generally, we define:

Definition 3.3.5. A *sequential stratum* $\mathcal{S}$ of $\mathcal{X}$ is a locally closed substack of $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ such that the composition

$$\mathcal{S} \to \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \to \mathcal{X}$$

is a locally closed immersion. We refer to the morphism $\mathcal{S} \to \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ as the *structure map*.

Remark **3.3.6.** If $a\colon \mathcal{S} \to \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ is a morphism such that the composition

$$\mathcal{S} \to \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \to \mathcal{X}$$

is a locally closed immersion, then a is a locally closed immersion as well, since $\mathrm{ev}_1\colon \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \to \mathcal{X}$ is representable and separated, so its diagonal is a closed immersion.

It will be useful to have a few ways of constructing sequential strata from given ones.

Definition 3.3.7 (Pushforward along a locally closed immersion)**.** If $\iota\colon \mathcal{X} \to \mathcal{Y}$ is a locally closed immersion and $\mathcal{S}$ is a sequential stratum of $\mathcal{X}$, then we define a sequential stratum $\iota_*\mathcal{S}$ as follows. As a stack, $\iota_*\mathcal{S} = \mathcal{S}$, and the structure map is the composition

$$\mathcal{S} \to \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \to \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{Y}),$$

which is a locally closed immersion because $\mathrm{Filt}_{\mathbb{Q}^\infty}(\iota)$ is[2]. The map $\mathcal{S} \to \mathcal{Y}$ factors as $\mathcal{S} \to \mathcal{X} \to \mathcal{Y}$, so it is a locally closed immersion.

Definition 3.3.8 (Induction of a sequential stratum from centre of a Θ-stratum)**.** Suppose $\mathcal{S}$ is a locally closed Θ-stratum of an algebraic stack $\mathcal{X}$, with centre $\mathcal{Z}$ (Definition 2.5.3), and let $\mathcal{S}'$ be a sequential stratum of $\mathcal{Z}$. We define a sequential stratum $\mathrm{ind}_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{S}')$ of $\mathcal{X}$, as follows. As a stack, $\mathrm{ind}_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{S}')$ is the pullback

$$\begin{array}{ccc} \mathrm{ind}_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{S}') & \longrightarrow & \mathcal{S}' \\ \big\downarrow & \ulcorner & \big\downarrow \\ \mathcal{S} & \xrightarrow{\mathrm{gr}} & \mathcal{Z}. \end{array}$$

The structure morphism a is obtained by pulling back the square

$$\begin{array}{ccc} \mathcal{S}' & \longrightarrow & \mathrm{Filt}_{\mathbb{Q}^\infty}\,\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}) \\ \big\downarrow & & \big\downarrow \\ \mathcal{Z} & \longrightarrow & \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}) \end{array}$$

along $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) \xrightarrow{\mathrm{gr}} \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$, obtaining a square of the form

$$\begin{array}{ccc} \mathrm{ind}_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{S}') & \xrightarrow{a} & \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \\ \big\downarrow & & \big\downarrow \\ \mathcal{S} & \longrightarrow & \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) \end{array}$$

by Proposition 3.2.11. The induced morphism $\mathrm{ind}_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{S}') \to \mathcal{X}$ is the composition of the locally closed immersions $\mathrm{ind}_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{S}') \to \mathcal{S}$ and $\mathcal{S} \to \mathcal{X}$, and it is thus also a locally closed immersion.

Definition 3.3.9 (Pushforward along a blow-up)**.** Let $p\colon \mathcal{Y} \to \mathcal{X}$ be a blow-up with exceptional divisor $\mathcal{E} \subset \mathcal{Y}$. Let $\mathcal{S}$ be a sequential stratum of $\mathcal{Y}$. We define a sequential stratum $p_*(\mathcal{S} \setminus \mathcal{E})$ of $\mathcal{X}$ as follows. If $r\colon \mathcal{S} \to \mathcal{Y}$ is the locally closed immersion, then we set $p_*(\mathcal{S} \setminus \mathcal{E}) = \mathcal{S} \setminus r^{-1}(\mathcal{E})$ as a stack. The structure map is the composition

$$\mathcal{S} \setminus r^{-1}(\mathcal{E}) \to \mathcal{S} \to \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{Y}) \to \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}).$$

We have a diagram

$$\begin{array}{ccccc} \mathcal{S} \setminus r^{-1}(\mathcal{E}) & \longrightarrow & \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{Y}) & \longrightarrow & \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \\ {\scriptstyle a}\big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathcal{Y} \setminus \mathcal{E} & \xrightarrow[b]{} & \mathcal{Y} & \xrightarrow[p]{} & \mathcal{X}. \end{array}$$

[2]The fact that $\mathrm{Filt}_{\mathbb{Q}}$ and $\mathrm{Grad}_{\mathbb{Q}}$ preserve closed immersions [36, Proposition 1.3.1] easily implies that $\mathrm{Filt}_{\mathbb{Q}^\infty}(\iota)$ is a closed immersion.

The map a is a locally closed immersion, and $p \circ b$ is an open immersion. Thus $\mathcal{S} \setminus r^{-1}(\mathcal{E}) \to \mathcal{X}$ is a locally closed immersion. By Remark 3.3.6, the structure map is also a locally closed immersion.

3.4 CENTRAL RANK AND $B\mathbb{G}_m^n$-ACTIONS

The concept of *central rank* of an algebraic stack introduced here will be fundamental in our construction of sequential stratifications for good moduli stacks (Theorem 3.5.2). Naively, the central rank of $\mathcal{X}$ is the maximal n such that all stabiliser groups of $\mathcal{X}$ contain a central torus of rank n. The correct implementation of this idea is Definition 3.4.1 below. Let $\mathcal{X}$ be an algebraic stack over an algebraic space B, satisfying Assumption 2.2.3, and assume further that $\mathcal{X}$ is noetherian and has affine diagonal.

Definition 3.4.1 (Central rank)**.** We define the *central rank* $z(\mathcal{X}) \in \mathbb{N}$ of $\mathcal{X}$ to be the biggest $n \in \mathbb{N}$ such that there is a union $\mathcal{Z}$ of nondegenerate components of $\mathrm{Grad}^n(\mathcal{X})$ (Definition 2.3.1) such that the composition $\mathcal{Z} \to \mathrm{Grad}^n(\mathcal{X}) \xrightarrow{u} \mathcal{X}$ is an isomorphism. For the empty stack we define $z(\varnothing) = \infty$.

Remark **3.4.2** (Relation with Donaldson–Thomas theory). The concept of central rank is important in extending Donaldson–Thomas theory to non-linear moduli problems, that is, to moduli stacks not parametrising objects in an abelian category. Donaldson–Thomas invariants were first defined by Thomas [73] for moduli spaces $\mathcal{X}^{\mathrm{ss}}$ of semistable sheaves on a Calabi–Yau threefold in the case where all semistable sheaves are stable. In [49, Definition 8.1], Joyce defines a stack function [50], denoted ϵ, on the stack $\mathcal{X}^{\mathrm{ss}}$ of semistable objects in a suitable abelian category with stability condition. In the case of sheaves on a Calabi–Yau threefold, when ϵ is integrated against the Behrend function [11] it gives rise to a generalised Donaldson–Thomas invariant, defined also when there are strictly semistable sheaves [51]. In upcoming work with Bu and Kinjo [17], we generalise Joyce's ϵ function to general algebraic stacks, not necessarily parametrising objects in an abelian category. The concept of central rank plays an important role in our theory, particularly in the proof of the no-pole theorem [49, Theorem 8.7] in this new setting.

The stack $B\mathbb{G}_m^n$ is a group stack. It turns out that the data of a $B\mathbb{G}_m^n$-action on a stack $\mathcal{X}$ satisfying Assumption 2.2.3 is equivalent to the data of a section $s\colon \mathcal{X} \to \mathrm{Grad}^n(\mathcal{X})$ of $u\colon \mathrm{Grad}^n(\mathcal{X}) \to \mathcal{X}$ such that s is a closed and open immersion. This is [35, Corollary 1.4.2.1] in the case of the monoid stack $\mathbb{A}^1/\mathbb{G}_m$, but the same proof works for $B\mathbb{G}_m^n$. We say that the $B\mathbb{G}_m^n$-action is *nondegenerate* if all components of $s(\mathcal{X})$

are nondegenerate. If $m\colon B\mathbb{G}_m^n \times \mathcal{X} \to \mathcal{X}$ is the action map, then the action is nondegenerate if, for all $x \in \mathcal{X}(k)$, the homomorphism $\mathbb{G}_{m,k}^n \to \mathrm{Aut}(x)$ induced by m has finite kernel.

Using the stack $\mathrm{Grad}_{\mathbb{Q}}^n(\mathcal{X})$ instead of $\mathrm{Grad}^n(\mathcal{X})$ we can talk about rational $B\mathbb{G}_m^n$-actions.

DEFINITION 3.4.3. A *rational $B\mathbb{G}_m^n$-action* on $\mathcal{X}$ is a section $s\colon \mathcal{X} \to \mathrm{Grad}_{\mathbb{Q}}^n(\mathcal{X})$ of

$$u\colon \mathrm{Grad}_{\mathbb{Q}}^n(\mathcal{X}) \to \mathcal{X}$$

that is a closed and open immersion.

Remark **3.4.4.** It is tacitly understood that an isomorphism $u \circ s \sim \mathrm{id}_{\mathcal{X}}$ is part of the data of a section s.

LEMMA 3.4.5. *Suppose that $\mathcal{X}$ has a good moduli space $\pi\colon \mathcal{X} \to X$, and let $p\colon \mathcal{Y} \to \mathcal{X}$ be a blow-up. Then any (rational) $B\mathbb{G}_m^n$-action on $\mathcal{X}$ lifts canonically to $\mathcal{Y}$.*

Proof. We prove the lemma for integral actions, the proof for rational actions being identical after replacing Grad with $\mathrm{Grad}_{\mathbb{Q}}$. The $B\mathbb{G}_m^n$-action corresponds to a section $s\colon \mathcal{X} \to \mathrm{Grad}^n(\mathcal{X})$ of $\mathrm{Grad}^n(\mathcal{X}) \to \mathcal{X}$ that is an open and closed immersion. With this language, what we want to show is that the preimage $\mathcal{Y}'$ of $s(\mathcal{X})$ along $\mathrm{Grad}^n(\mathcal{Y}) \to \mathrm{Grad}^n(\mathcal{X})$ is a closed and open substack of $\mathrm{Grad}^n(\mathcal{Y})$ such that the composition $\mathcal{Y}' \to \mathrm{Grad}^n(\mathcal{Y}) \to \mathcal{Y}$ is an isomorphism.

If $\mathcal{X}$ is of the form $\mathcal{X} = \mathrm{Spec}\, A/\mathrm{GL}_l$, then

$$\mathrm{Grad}^n(\mathcal{X}) = \bigsqcup_{\lambda \in \mathrm{Hom}(\mathbb{G}_m^n, T)/W} (\mathrm{Spec}\, A)^{\lambda,0}/L(\lambda),$$

where T is the standard maximal torus of GL_l and W is the symmetric group of degree l [36, Theorem 1.4.7]. Thus $\mathcal{X}$ is isomorphic to a union of connected components of one of the stacks $(\mathrm{Spec}\, A)^{\lambda,0}/L(\lambda)$, with $\lambda\colon \mathbb{G}_m^n \to \mathrm{GL}_l$ having finite kernel. Any blow-up $p\colon \mathcal{Y} \to \mathcal{X}$ is then of the form $Y/L(\lambda)$ with $\lambda(\mathbb{G}_m^l)$ acting trivially on Y. This proves the lemma in the case $\mathcal{X} = \mathrm{Spec}\, A/\mathrm{GL}_l$.

For general $\mathcal{X}$, the claim can be checked étale locally on $\mathcal{X}$, since for $\mathcal{X}' \to \mathcal{X}$ representable and étale, the square

$$\begin{array}{ccc} \mathrm{Grad}^n(\mathcal{X}') & \longrightarrow & \mathrm{Grad}^n(\mathcal{X}) \\ \downarrow & \ulcorner & \downarrow \\ \mathcal{X}' & \longrightarrow & \mathcal{X} \end{array}$$

is cartesian [36, Corollary 1.1.7], and since blow-ups commute with flat base change. We can cover $\mathcal{X}$ by representable étale neighbourhoods of the form $\mathrm{Spec}\, A/\mathrm{GL}_l$ (Theorem 2.1.3), which proves the lemma for general $\mathcal{X}$. □

Remark **3.4.6.** It should not be essential that $\mathcal{X}$ has a good moduli space for Lemma 3.4.5, but this is all we will need.

Lemma 3.4.7. *Let $\mathcal{X}$ be a normed noetherian good moduli stack with affine diagonal. Let $p\colon \mathcal{Y} \to \mathcal{X}$ be a blow-up of $\mathcal{X}$ at some closed substack, and let $\mathcal{E}$ be the exceptional divisor. The stack $\mathcal{Y}$ is endowed with the Θ-stratification induced by the norm on $\mathcal{X}$ and the p-ample line bundle $\mathcal{O}_{\mathcal{Y}}(-\mathcal{E})$ (Theorem 2.6.4 and Example 2.6.8). Let $\mathcal{Z}_c$ be the centre of some unstable stratum of $\mathcal{Y}$. Then $z(\mathcal{X}) < z(\mathcal{Z}_c)$.*

Proof. Let $n = z(\mathcal{X})$. There is a nondegenerate $B\mathbb{G}_m^n$ action on $\mathcal{X}$ that, since p is a blow-up, lifts canonically to $\mathcal{Y}$ by Lemma 3.4.5. The $B\mathbb{G}_m^n$ action gives a closed and open immersion $\mathcal{Y} \to \mathrm{Grad}^n(\mathcal{Y})$.

Since $\mathcal{Z}_c$ is the centre of a stratum, it comes equipped with a rational $B\mathbb{G}_m$-action inherited from a closed and open immersion $\mathcal{Z}_c \to \mathrm{Grad}_{\mathbb{Q}}(\mathcal{Y})$. Scaling up, we get a closed and open immersion $\mathcal{Z}_c \to \mathrm{Grad}(\mathcal{Y})$ and thus an integral $B\mathbb{G}_m$-action on $\mathcal{Z}_c$. Applying Grad to the closed and open immersion $\mathcal{Y} \to \mathrm{Grad}^n(\mathcal{Y})$ and composing with $\mathcal{Z}_c \to \mathrm{Grad}(\mathcal{Y})$, we get a closed and open immersion $\mathcal{Z}_c \to \mathrm{Grad}^{n+1}(\mathcal{Y})$, which gives a $B\mathbb{G}_m \times B\mathbb{G}_m^n$-action on $\mathcal{Z}_c$.

Let $x \in \mathcal{Z}_c(k)$ be a k-point for some field k. The $B\mathbb{G}_m \times B\mathbb{G}_m^n$-action provides cocharacters $\lambda, \beta_1, \ldots, \beta_n$ of the centre $Z(\mathrm{Aut}(x))$. Since the $\beta_1, \ldots, \beta_n$ come from $\mathcal{X}$, we have $\langle \beta_i, \mathcal{O}_{\mathcal{Y}}(-\mathcal{E})\rangle = 0$, where $\mathcal{E}$ is the exceptional divisor, while since $\mathcal{Z}_c$ is the centre of an *unstable* stratum, we have $\langle \lambda, \mathcal{O}_{\mathcal{Y}}(-\mathcal{E})\rangle > 0$. Therefore $\lambda(\mathbb{G}_m)$ is not contained in the image of $(\beta_1, \ldots, \beta_n)\colon \mathbb{G}_m^n \to \mathrm{Aut}(x)$, and thus $(\lambda, \beta_1, \ldots, \beta_n)\colon \mathbb{G}_m^{n+1} \to \mathrm{Aut}(x)$ has finite kernel. Therefore $z(\mathcal{Z}_c) \geq n + 1$, as desired. □

3.5 THE BALANCING STRATIFICATION AND THE ITERATED BALANCED FILTRATION

We now get to the main construction of the thesis, namely the *balancing stratification* for normed good moduli stacks (Theorem 3.5.2), and the canonical sequential filtration it defines for every point (the *iterated balanced filtration*, Definition 3.5.8). We also show that the balancing stratification is preserved under certain pullbacks (Proposition 3.5.5).

Here is the idea of the construction. If $\mathcal{X}$ is a normed noetherian good moduli stack, and all stabilisers of $\mathcal{X}$ have the same dimension, then we want the balancing stratification of $\mathcal{X}$ to consists only of the stratum $\mathcal{X}^{\max}$. Note that, while $\mathcal{X}^{\max} \hookrightarrow \mathcal{X}$ could fail to be an isomorphism, it is a surjective closed immersion in this case. Now, if $\mathcal{X}^{\max}$ does not cover $\mathcal{X}$, then the blow-up $\mathcal{Y} = \mathrm{Bl}_{\mathcal{X}^{\max}} \mathcal{X}$ is endowed canonically with a Θ-stratification $(\mathcal{S}_c)_{c\in\mathbb{Q}_{\geq 0}}$. By an inductive argument, the balancing stratification is defined for the centres $\mathcal{Z}_c$ with $c > 0$. Thus the $\mathcal{S}_c$ inherit the stratification along the associated graded map $\mathcal{S}_c \to \mathcal{Z}_c$. Denoting $\mathcal{E} \subset \mathcal{Y}$ the exceptional divisor, each of the locally closed substacks $\mathcal{S}_c \setminus \mathcal{E}$ of $\mathcal{X}$ with $c > 0$ is thus stratified. The stack $\mathcal{S}_0 \setminus \mathcal{E}$ equals $\mathcal{X} \setminus \pi^{-1}\pi(\mathcal{X}^{\max})$, and thus it has a good moduli space and we can assume by induction that its balancing stratification is defined. In this way, the union of the strata of each of the $\mathcal{S}_c \setminus \mathcal{E}$ for all $c \geq 0$, together with $\mathcal{X}^{\max}$, defines a stratification of $\mathcal{X}$ by locally closed substacks, and this is the balancing stratification of $\mathcal{X}$. In this section we make this idea precise by keeping track of the structure of sequential stratum of each of the pieces and proving that the inductive argument can be made to work.

We now introduce the indexing poset labelling the stratification.

Definition 3.5.1 (Indexing poset for the balancing stratification)**.** We define a totally ordered set $\mathbf{\Gamma}$ as follows. As a set, $\mathbf{\Gamma}$ consists of the sequences

$$((d_0, c_0), (d_1, c_1), \ldots, (d_n, c_n))$$

with

1. $n \in \mathbb{N}$,
2. $d_0 \geq d_1 \geq \cdots \geq d_n$ in $\mathbb{N}$,
3. $d_i \geq i$ for all $0 \leq i \leq n$,
4. $c_0, \ldots, c_{n-1} \in \mathbb{Q}_{>0}$,
5. $c_n = \infty$.

As a poset, we write

$$((d_0, c_0), \ldots, (d_n, c_n)) < ((d'_0, c'_0), \ldots, (d'_m, c'_m))$$

if there is $0 \leq i \leq \min(n, m)$ such that $(d_j, c_j) = (d'_j, c'_j)$ for $j < i$ and either $d_i < d'_i$ or $d_i = d'_i$ and $c_i < c'_i$. This makes $\mathbf{\Gamma}$ a totally ordered set.

If $\alpha \in \mathbf{\Gamma}$, we use the notation

$$\alpha = \left(\left(d_0^\alpha, c_0^\alpha\right), \ldots, \left(d_{n(\alpha)}^\alpha, c_{n(\alpha)}^\alpha\right)\right).$$

Theorem 3.5.2 (Existence of the balancing stratification)**.** *There is a unique way of assigning, to every normed noetherian good moduli stack $\mathcal{X}$ with affine diagonal, a sequential stratification $\left(\mathcal{S}_\alpha^{\mathcal{X}}\right)_{\alpha\in\mathbf{\Gamma}}$ of $\mathcal{X}$ indexed by $\mathbf{\Gamma}$ in such a way that the following properties are satisfied for every such $\mathcal{X}$:*

1. *The stratum indexed by $(d(\mathcal{X}), \infty)$ is $\mathcal{S}^{\mathcal{X}}_{(d(\mathcal{X}),\infty)} = \mathcal{X}^{\max}$, with structure map*

$$\mathcal{X}^{\max} \to \mathcal{X} \to \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}),$$

where the second arrow is the "trivial filtration" morphism.

2. *Let* $\pi\colon \mathcal{X} \to X$ *be the good moduli space and let* $\mathcal{U} = \mathcal{X} \setminus \pi^{-1}\pi(\mathcal{X}^{\max})$*. Denote* $j\colon \mathcal{U} \to \mathcal{X}$ *the open immersion. Then for all* $\alpha \in \mathbf{\Gamma}$ *with* $d_0^\alpha < d(\mathcal{X})$ *we have* $\mathcal{S}_\alpha^{\mathcal{X}} = j_*(\mathcal{S}_\alpha^{\mathcal{U}})$*.*
3. *Let* $p\colon \mathcal{Y} = \mathrm{Bl}_{\mathcal{X}^{\max}} \mathcal{X} \to \mathcal{X}$ *be the blow-up of* $\mathcal{X}$ *along* $\mathcal{X}^{\max}$ *and let* $\mathcal{E}$ *be the exceptional divisor. Let* $\mathcal{Z}_c$ *be the centres of the* Θ*-stratification (with inverse convention, see Remark 2.5.17) on* $\mathcal{Y}$ *induced by the norm on* $\mathcal{X}$ *and the line bundle* $\mathcal{O}_{\mathcal{Y}}(-\mathcal{E})$ *(Example 2.6.8). The* $\mathcal{Z}_c$ *are endowed with the induced norm and are thus also normed noetherian good moduli stacks with affine diagonal by Theorem 2.6.4. Then, for all* $c \in \mathbb{Q}_{>0}$ *and for all* $\alpha \in \mathbf{\Gamma}$ *such that the concatenation* $((d(\mathcal{X}), c), \alpha)$ *of* $(d(\mathcal{X}), c)$ *and* α *also belongs to* $\mathbf{\Gamma}$*, we have the equality*

$$\mathcal{S}^{\mathcal{X}}_{((d(\mathcal{X}),c),\alpha)} = p_*\left(\mathrm{ind}^{\mathcal{Y}}_{\mathcal{Z}_c}(\mathcal{S}_\alpha^{\mathcal{Z}_c}) \setminus \mathcal{E}\right).$$

Moreover, for every $\alpha \in \mathbf{\Gamma}$ *such that* $\mathcal{S}_\alpha^{\mathcal{X}}$ *is nonempty, we have:*

4. $d_0^\alpha \le d(\mathcal{X})$*; and*
5. $d_i^\alpha \ge i + z(\mathcal{X})$*, for all* $0 \le i \le n(\alpha)$*.*

Proof. First we note that if $\mathcal{X}$ is empty, then the only stratification is given by $\mathcal{S}_\alpha^{\mathcal{X}} = \varnothing$ for all α, and it satisfies the required properties.

For a stack $\mathcal{X}$ as in the statement of the theorem, define the number $N(\mathcal{X}) = d(\mathcal{X}) - z(\mathcal{X})$. We will use induction on $N(\mathcal{X})$.

Assume $\mathcal{X} \neq \varnothing$ and let $\mathcal{U}$ be as in 2. Then clearly $d(\mathcal{U}) < d(\mathcal{X})$ and $z(\mathcal{U}) \ge z(\mathcal{X})$, so $N(\mathcal{U}) < N(\mathcal{X})$. If $\mathcal{Z}_c$ is as in 3, with $c > 0$, then $d(\mathcal{Z}_c) \le d(\mathcal{X})$, because $\mathcal{Z}_c \to \mathcal{X}$ is representable, and $z(\mathcal{Z}_c) > z(\mathcal{X})$ by Lemma 3.4.7. Thus $N(\mathcal{Z}_c) < N(\mathcal{X})$. Therefore the statement of the theorem makes sense if we fix an $N \in \mathbb{N}$ and we restrict to the class of normed noetherian good moduli stacks $\mathcal{X}$ with affine diagonal and with $N(\mathcal{X}) \le N$. We prove the theorem for this class of stacks by induction on N.

If $N = 0$, and $N(\mathcal{X}) = 0$, then the identity component of every stabiliser group of a point of $\mathcal{X}$ is a split torus of dimension $d(\mathcal{X})$. Therefore $|\mathcal{X}^{\max}| = |\mathcal{X}|$ and $\mathcal{S}^{\mathcal{X}}_{(d(\mathcal{X}),\infty)} = \mathcal{X}^{\max}$ is the only nonempty stratum. This gives the desired sequential stratification.

Fix $N > 0$ and assume the theorem is true for $N - 1$. For $\mathcal{X}$ with $N(\mathcal{X}) = N$, define $\mathcal{S}_\alpha^{\mathcal{X}}$ as in the statement of the theorem when $d_0^\alpha \le d(\mathcal{X})$, which makes sense because $N(\mathcal{U}) < N$ and $N(\mathcal{Z}_c) < N$. Define $\mathcal{S}_\alpha^{\mathcal{X}} = \varnothing$ otherwise. We show that $(\mathcal{S}_\alpha^{\mathcal{X}})_{\alpha\in\mathbf{\Gamma}}$ is a sequential stratification of $\mathcal{X}$.

Denote $r_\alpha\colon \mathcal{S}_\alpha^{\mathcal{X}} \to \mathcal{X}$ the induced locally closed immersions. First we show that the $r_\alpha\left(|\mathcal{S}_\alpha^{\mathcal{X}}|\right)$ are pairwise disjoint and cover $\mathcal{X}$. For $p \in |\mathcal{X}|$, there is a unique $p' \in \overline{\{x\}}$ that is closed in $\pi^{-1}\pi(p)$ [2, Proposition 9.1]. One and only one of the following situations takes place:

1. The dimension $\dim \operatorname{Aut}(p') < d(\mathcal{X})$. In this case $p \in |\mathcal{U}|$ and it is contained in exactly one of the $j_* \mathcal{S}_\alpha^{\mathcal{U}} = \mathcal{S}_\alpha^{\mathcal{X}}$ with $d_0^{\mathcal{X}} < d(\mathcal{X})$, by induction hypothesis.
2. We have $\dim \operatorname{Aut}(p') = d(\mathcal{X})$ and $p = p'$. Then $p \in |\mathcal{X}^{\max}| = \left|\mathcal{S}^{\mathcal{X}}_{d(\mathcal{X}),\infty)}\right|$ and this is the only stratum containing p.
3. Again, $\dim \operatorname{Aut}(p') = d(\mathcal{X})$, but this time $p \neq p'$. In this case, $p \in |\mathcal{X}| \setminus (|\mathcal{U}| \cup |\mathcal{X}^{\max}|)$ and there is a unique point $q \in |\mathcal{Y}| \setminus |\mathcal{E}|$ mapping to p. The point q lies in a unique stratum $\mathcal{S}_c$ ($c \in \mathbb{Q}_{\geq 0}$) of the Θ-stratification of $\mathcal{Y}$. There exists a map $\lambda\colon \Theta_k \to \mathcal{X}$ with $\lambda(0) = p'$ and $\lambda(1) = p$ for some field k [8, Lemma 3.24], and Kempf's number $\langle \lambda, \mathcal{X}^{\max}\rangle > 0$ is positive because $\lambda(0) \in |\mathcal{X}^{\max}|$ and $\lambda(1) \notin |\mathcal{X}^{\max}|$ (Definition 3.1.1). The filtration λ lifts uniquely to $\lambda\colon \Theta_k \to \mathcal{Y}$ (Proposition 2.2.20), and $\langle \lambda, \mathcal{O}_{\mathcal{Y}}(-\mathcal{E})\rangle = \langle \lambda, \mathcal{X}^{\max}\rangle > 0$ (Proposition 3.1.2). Therefore q is unstable in $\mathcal{Y}$ and thus $c > 0$. Since the $\mathcal{S}_\alpha^{\mathcal{Z}_c}$ stratify $\mathcal{Z}_c$, the $\operatorname{ind}^{\mathcal{Y}}_{\mathcal{Z}_c}(\mathcal{S}_\alpha^{\mathcal{Z}_c})$ stratify $\mathcal{S}_c$, so q is contained in $\operatorname{ind}^{\mathcal{Y}}_{\mathcal{Z}_c}(\mathcal{S}_\alpha^{\mathcal{Z}_c})$ for a unique $\alpha \in \mathbf{\Gamma}$ and thus p is contained in a unique $p_*\left(\operatorname{ind}^{\mathcal{Y}}_{\mathcal{Z}_c}(\mathcal{S}_\alpha^{\mathcal{Z}_c}) \setminus \mathcal{E}\right)$. It is left to check that $((d(\mathcal{X}), c), \alpha) \in \mathbf{\Gamma}$. This follows because $d_i^\alpha \geq i + z(\mathcal{Z}_c) \geq i + 1$ by 5 and Lemma 3.4.7.

Now we check that $\left|\mathcal{S}^{\mathcal{X}}_{>\alpha}\right| := \bigcup_{\beta > \alpha} r_\beta\left(\left|\mathcal{S}^{\mathcal{X}}_\beta\right|\right)$ is closed for all $\alpha \in \mathbf{\Gamma}$. If $\alpha = (d(\mathcal{X}), \infty)$, then $\left|\mathcal{S}^{\mathcal{X}}_{>\alpha}\right| = \varnothing$, which is closed. If $\alpha = ((d(\mathcal{X}), c), \alpha')$ with $c \in \mathbb{Q}_{>0}$, then

$$\left|\mathcal{S}^{\mathcal{X}}_{>\alpha}\right| = p\left(\operatorname{gr}^{-1}\left(\left|\mathcal{S}^{\mathcal{Z}_c}_{>\alpha'}\right|\right) \cup \bigcup_{c' > c} |\mathcal{S}_{c'}|\right) \cup |\mathcal{X}^{\max}|,$$

which is closed by induction and because p is proper. If $d_0^\alpha < d(\mathcal{X})$, then $\left|\mathcal{S}^{\mathcal{X}}_{>\alpha}\right| = \left|\mathcal{S}^{\mathcal{U}}_{>\alpha}\right| \cup \left|\pi^{-1}\pi(\mathcal{X}^{\max})\right|$, which is also closed by induction and because π is universally closed.

It is left to show properties 4 and 5. The former is true by construction. The latter is true for $i = 0$ because $d(\mathcal{X}) \geq z(\mathcal{X})$, and it is true if $d_0^\alpha < d(\mathcal{X})$ by induction, using the result for $\mathcal{U}$. The remaining case is when α is of the form $\alpha = ((d(\mathcal{X}), c), \alpha')$. Then $\mathcal{S}^{\mathcal{Z}_c}_{\alpha'} \neq \varnothing$ and thus $d^\alpha_{i+1} = d^{\alpha'}_i \geq i + z(\mathcal{Z}_c) \geq i + 1 + z(\mathcal{X})$ by induction and by Lemma 3.4.7. □

Definition 3.5.3 (The balancing stratification)**.** Let $\mathcal{X}$ be a normed noetherian good moduli stack with affine diagonal. The *balancing stratification* of $\mathcal{X}$ is the sequential stratification $(\mathcal{S}^{\mathcal{X}}_\alpha)_{\alpha \in \mathbf{\Gamma}}$ of $\mathcal{X}$ from Theorem 3.5.2.

Remark **3.5.4.** It is not hard to construct, using the convex-geometric picture introduced later (Corollary 5.2.18), that for every $\alpha \in \Gamma$ there is a normed good moduli stack $\mathcal{X}$ with $\mathcal{S}_\alpha^{\mathcal{X}} \neq \varnothing$. We can in fact take $\mathcal{X}$ to be of the form $\mathcal{X} = \mathbb{A}_k^n/\mathbb{G}_{m,k}^l$, with k any field.

The balancing stratification is well-behaved under certain pullbacks.

Proposition 3.5.5 (Compatibility with pullback)**.** *Let $f\colon \mathcal{X}' \to \mathcal{X}$ be a norm-preserving morphism between normed noetherian good moduli stacks with affine diagonal. Let $\pi\colon \mathcal{X} \to X$ and $\pi'\colon \mathcal{X}' \to X'$ be the good moduli spaces. Assume further that f is either*

1. *a closed immersion, or*
2. *it fits in a cartesian diagram*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{f} & \mathcal{X} \\ \downarrow & \ulcorner & \downarrow \\ X' & \xrightarrow{h} & X. \end{array}$$

Then the balancing stratification on $\mathcal{X}'$ is the pullback along f (Definition 3.3.4) of the balancing stratification on $\mathcal{X}$, that is, for all $\alpha \in \mathbf{\Gamma}$ we have $\mathcal{S}_\alpha^{\mathcal{X}'} = f^*\mathcal{S}_\alpha^{\mathcal{X}}$.

Proof. We prove the statement by induction on $N(\mathcal{X}) = d(\mathcal{X}) - z(\mathcal{X})$. The base case is $N(\mathcal{X}) = -\infty$, corresponding to the empty stack, for which the statement is obvious.

We have an upper semicontinuous function r on $|X|$ given by

$$\begin{aligned} r\colon |X| &\longrightarrow \mathbb{N} \\ p &\longmapsto d(\pi^{-1}(p)), \end{aligned}$$

where $d(-)$ denotes "maximal stabiliser dimension" (Definition 3.1.5). For $d \in \mathbb{N}$, let $X_{\leq d}$ be the open subspace of X with $|X_{\leq d}| = \{p \in |X| \mid r(p) \leq d\}$ and let $\mathcal{X}_{\leq d} = \pi^{-1}(X_{\leq d})$. In the case where $\mathcal{X}' = \mathcal{X}_{\leq d}$ and f is the inclusion $\mathcal{X}_{\leq d} \to \mathcal{X}$, the result follows from Item 2 in Theorem 3.5.2.

Note that in both cases we have $\mathcal{X}'_{\leq d} = f^{-1}(\mathcal{X}_{\leq d})$ for all $d \in \mathbb{N}$. To prove the statement for given $\alpha \in \mathbf{\Gamma}$, we may assume $d(\mathcal{X}) = d_0^\alpha$ by replacing $\mathcal{X}$ by $\mathcal{X}_{\leq d_0^\alpha}$ and $\mathcal{X}'$ by $\mathcal{X}'_{\leq d_0^\alpha} = f^{-1}(\mathcal{X}_{\leq d_0^\alpha})$. If $d(\mathcal{X}') < d(\mathcal{X})$, then $\mathcal{S}_\alpha^{\mathcal{X}'} = \varnothing = f^*(\mathcal{S}_\alpha^{\mathcal{X}})$. Thus we may assume $d := d(\mathcal{X}) = d(\mathcal{X}')$. We prove that $\mathcal{S}_\alpha^{\mathcal{X}'} = f^*\mathcal{S}_\alpha^{\mathcal{X}}$ in this case.

First, we have $(\mathcal{X}')^{\max} = f^{-1}(\mathcal{X}^{\max})$ by [27, Proposition C.5], since in both cases f is stabiliser-preserving (that is, the map $\mathcal{I}_{\mathcal{X}'} \to f^*\mathcal{I}_{\mathcal{X}}$ between inertia stacks is an isomorphism). Therefore $\mathcal{S}_{(d,\infty)}^{\mathcal{X}'} = f^*\mathcal{S}_{(d,\infty)}^{\mathcal{X}}$. We may thus assume $c := c_0^\alpha \in \mathbb{Q}_{>c}$ and write $\alpha = ((d,c),\beta)$ with $\beta \in \Gamma$. In both cases we have a diagram

$$\begin{array}{ccccc} \mathcal{Y}' & \xhookrightarrow{\iota} & X' \times_X \mathcal{Y} & \longrightarrow & \mathcal{Y} \\ & \searrow & \downarrow & & \downarrow \\ & & X' & \longrightarrow & X \end{array}$$

with ι a closed immersion. The Θ-stratification $(\mathcal{S}'_c)_{c\in\mathbb{Q}_{\geq 0}}$ on $\mathcal{Y}'$ is the pullback of the Θ-stratification $(\mathcal{S}_c)_{c\in\mathbb{Q}_{\geq 0}}$ on $\mathcal{Y}$ by Proposition 2.5.19. Let $c > 0$, let $\mathcal{Z}_c$ be the centre of $\mathcal{S}_c$, let $\mathcal{Z}'_c$ be the centre of $\mathcal{S}'_c$, and let $g\colon \mathcal{Z}'_c \to \mathcal{Z}_c$ be the natural map. By cartesianity of the square

$$\begin{array}{ccc} \mathcal{Z}'_c & \xrightarrow{g} & \mathcal{Z}_c \\ \downarrow & & \downarrow \\ \mathcal{Y}' & \xrightarrow{f} & \mathcal{Y} \end{array}$$

we see that g is a composition of a closed immersion and a base change of a map between the good moduli spaces of $\mathcal{Z}'_c$ and $\mathcal{Z}_c$. By induction, since $N(\mathcal{Z}_c) < N(\mathcal{X})$, we have $\mathcal{S}_\beta^{\mathcal{Z}'_c} = g^*(\mathcal{S}_\beta^{\mathcal{Z}_c})$. It follows from Item 3 in Theorem 3.5.2 that $\mathcal{S}_\alpha^{\mathcal{X}'} = f^*(\mathcal{S}_\alpha^{\mathcal{X}})$. □

Remark **3.5.6.** A crucial ingredient that makes Proposition 3.5.5 possible is that the closed substack structure of the maximal dimension stabiliser locus is compatible with pullbacks. It would not hold if we had used the reduced substack structure on $|\mathcal{X}^{\max}|$ instead.

We will often use the following form of Proposition 3.5.5.

COROLLARY 3.5.7 (Compatibility with fibres)**.** *Let $\mathcal{X}$ be a normed noetherian algebraic stack with affine diagonal and a good moduli space $\pi\colon \mathcal{X} \to X$, let k be a field and let $x\colon \operatorname{Spec} k \to \mathcal{X}$ be a point. Let $\mathcal{F} = \pi^{-1}\pi(x)$. Then the balancing stratification of $\mathcal{F}$ is the pullback along $\mathcal{F} \to \mathcal{X}$ of the balancing stratification of $\mathcal{X}$.* □

The balancing stratification defines, for every point x of the stack $\mathcal{X}$, a canonical sequential filtration of x.

DEFINITION 3.5.8 (Iterated balanced filtration)**.** Let $\mathcal{X}$ be a normed noetherian good moduli stack with affine diagonal, let k be a field and let $x \in \mathcal{X}(k)$ be a k-point. The *iterated balanced filtration* $\lambda_{\mathrm{ib}}(x) \in \mathbb{Q}^\infty\text{-}\operatorname{Filt}(\mathcal{X}, x)$ of x is the sequential filtration of x induced by the balancing stratification of $\mathcal{X}$ (Definition 3.3.2).

Remark **3.5.9.** By Corollary 3.5.7, in order to compute the iterated balanced filtration a point $x \in \mathcal{X}(k)$, it is enough to compute the balancing filtration of x inside the fibre $\mathcal{F} = \pi^{-1}\pi(x)$. If k is algebraically closed, then $\mathcal{F}$ is of the form $\operatorname{Spec} A/G$, where G is the stabiliser of the closed point of $\mathcal{F}$, by Corollary 2.1.5. Then $\operatorname{Spec} A$ can be embedded G-equivariantly inside a finite dimensional representation V of G, and by Proposition 3.5.5, we can compute the iterated balanced filtration of x in V/G. This seems to reduce the problem to the case of a stack of the form V/G. However, the choice of embedding $\mathcal{F} \to V/G$ is not canonical. Moreover, at each step of the blow-up procedure, new fibres of good moduli spaces have to be taken and embeddings into vector spaces need to be chosen, making things difficult to track. Nevertheless, we will develop a bookkeeping device, *chains of stacks*, in Chapter 4 that will allow us to give combinatorial descriptions of the iterated balanced filtration in two cases: the quotient of a vector space by a torus (Chapter 5) and moduli of objects in abelian categories (Chapter 7).

In the presence of Θ-stratifications whose centres have good moduli spaces, we can use the refined

Definition 3.5.10 (Refined Harder-Narasimhan filtration)**.** Let $\mathcal{X}$ be an algebraic stack over an algebraic space B, satisfying Assumption 2.2.3, and endowed with a linear form ℓ and a norm q that define a Θ-stratification $(\mathcal{S}_c)_{c\in\mathbb{Q}_{\geq 0}}$, where we use the direct convention (see Remark 2.5.17). Suppose that all the centres $\mathcal{Z}_c$ are quasi-compact with affine diagonal and have good moduli spaces, so that each $\mathcal{Z}_c$ has a well-defined balancing stratification. The *refined Harder-Narasimhan stratification* of $\mathcal{X}$ induced by ℓ and q is the sequential stratification

$$\left(\mathcal{S}^{\mathcal{X}}_{(c,\alpha)}\right)_{(c,\alpha)\in\mathbb{Q}_{\geq 0}\times\mathbf{\Gamma}} := \left(\mathrm{ind}^{\mathcal{X}}_{\mathcal{Z}_c}\left(\mathcal{S}^{\mathcal{Z}_c}_{\alpha}\right)\right)_{(c,\alpha)\in\mathbb{Q}_{\geq 0}\times\mathbf{\Gamma}}$$

of $\mathcal{X}$, indexed by the poset $\mathbb{Q}_{\geq 0}\times\mathbf{\Gamma}$ with lexicographic order. The *refined Harder-Narasimhan filtration* $\lambda_{\mathrm{rHN}}(x)$ of a field-valued point $x\in\mathcal{X}(k)$ is the sequential filtration $\lambda_{\mathrm{rHN}}(x)\in\mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X},x)$ induced by the refined Harder-Narasimhan stratification of $\mathcal{X}$ (Definition 3.3.2).

It is clear from the definition of induced sequential stratum (Definition 3.3.8) that $\left(\mathcal{S}^{\mathcal{X}}_{(c,\alpha)}\right)_{(c,\alpha)\in\mathbb{Q}_{\geq 0}\times\mathbf{\Gamma}}$ is indeed a sequential stratification of $\mathcal{X}$.

Remark **3.5.11.** More generally, we can consider a stack $\mathcal{X}$ as in Definition 3.5.10, endowed with a norm q and a Θ-stratification $(\mathcal{S}_i)_{i\in I}$ (not necessarily induced by some linear form and q) indexed by some poset I such that all the centres $\mathcal{Z}_i$ are quasi-compact and have affine diagonal and a good moduli space. Again, we have the canonical sequential stratification $\left(\mathrm{ind}^{\mathcal{X}}_{\mathcal{Z}_i}\left(\mathcal{S}^{\mathcal{Z}_i}_{\alpha}\right)\right)_{(i,\alpha)\in I\times\mathbf{\Gamma}}$ refining $(\mathcal{S}_i)_{i\in I}$, and a corresponding sequential filtration for every point of $\mathcal{X}$.

Remark **3.5.12** (Derived stacks)**.** The concept of good moduli space for derived Artin 1-stacks has recently been introduced in [1], and blow-ups along the substack of points with maximal stabiliser dimension in this context have been studied in [42]. On the other hand, Θ-stratifications are also available in derived geometry [35]. Therefore we expect that the theory of balancing stratifications extends to normed derived Artin stacks $\mathcal{X}$ with good moduli space. However, in view of Proposition 3.5.5 we anticipate that the iterated balanced filtration a point x of $\mathcal{X}$ that one gets from this theory only depends on the classical truncation of $\mathcal{X}$.

3.6 EXAMPLES

In moduli theory there are many natural instances of normed good moduli stacks, for which the balancing stratification is defined. We collect here a few examples.

3.6.1 STACKS PROPER OVER A GOOD MODULI STACK

Assume the framework of Theorem 2.6.4, that is, $\mathcal{X}$ is a normed noetherian good moduli stack with affine diagonal, $f: \mathcal{Y} \to \mathcal{X}$ is representable and proper and ℓ is an f-positive linear form on $\mathcal{Y}$. Then we have a Θ-stratification $(\mathcal{S}_c)_{c\in\mathbb{Q}_{\geq 0}}$ of $\mathcal{Y}$ and every centre $\mathcal{Z}_c$ is again a normed noetherian good moduli stack with affine diagonal. Therefore the norm on $\mathcal{X}$ and the linear form ℓ induce a refined Harder-Narasimhan stratification on $\mathcal{Y}$ (Definition 3.5.10) and a refined Harder-Narasimhan filtration for every point of $\mathcal{Y}$.

3.6.2 GEOMETRIC INVARIANT THEORY

Let k be a field, let G be a linearly reductive affine algebraic group over k admitting a split maximal torus T, with Weyl group W, and endowed with a norm on cocharacters (Definition 2.3.8). Let A be a finite type k-algebra and consider an action of G on $\operatorname{Spec} A$. The quotient stack $\mathcal{X} = \operatorname{Spec}(A)/G$ has a good moduli space $\mathcal{X} \to \operatorname{Spec}(A^G)$. Given any G-equivariant projective morphism $f: Y \to \operatorname{Spec} A$ and an ample linearisation on Y (that is, a line bundle on Y/G ample with respect to $h = f/G: Y/G \to \operatorname{Spec}(A)/G$), the previous example applied to h gives a refined Harder-Narasimhan stratification of Y/G, indexed by $\mathbb{Q}_{\geq 0} \times \mathbf{\Gamma}$. For every k-point $y \in Y(k)$, the refined Harder-Narasimhan filtration $\lambda_{\mathrm{rHN}}(y)$ of y can be seen as a sequence $\lambda_0, \dots, \lambda_n \in \Gamma^{\mathbb{Q}}(G)$ of commuting rational cocharacters of G, considered up to certain equivalence relation, by Remark 3.2.16.

This stratification was first defined by Kirwan [58] in the case where $k = \mathbb{C}$, $A = \mathbb{C}$ and $Y \to \operatorname{Spec} \mathbb{C}$ is smooth, building on the ideas introduced in [56]. The indexing set used in [58] is different from the one used here, and it depends on the quotient presentation of Y/G. The strata obtained in [58] are open and closed substacks of the strata defined here, which does not make a substantial difference. This further partition of the strata arises in two different ways. First, instead of the Θ-stratification $(\mathcal{S}_c)_{c\in\mathbb{Q}_{\geq 0}}$ of Y/G considered here, indexed by $\mathbb{Q}_{\geq 0}$, the stratification considered by Kirwan is indexed by $\Gamma^{\mathbb{Q}}(T)/W$ (see also [55]). The set $\Gamma^{\mathbb{Q}}(T)/W$ can be seen as parametrising certain unions of connected components on $\mathrm{Grad}_{\mathbb{Q}}(Y/G)$ by [36, Theorem 1.4.8], and the strata obtained in this way are closed and open substacks of the $\mathcal{S}_c$. The same kind of difference on indexing sets appears each time a blow-up is performed in the construction. Also, each time a locus of maximal dimension stabilisers is considered, for example $(Y^{\mathrm{ss}}/G)^{\max}$, Kirwan writes it as a disjoint union of loci of the form $G\left((Y^{\mathrm{ss}})^R\right)/G$, where R is a reductive subgroup of G of maximal dimension such that $(Y^{\mathrm{ss}})^R$ is nonempty. This further refines the indexing set and the stratification, but again only breaking down the strata into pieces that are closed and open.

Even in the case considered in [58], the fact that the balancing stratification has the structure of a sequential stratification is a novelty of our approach. The definition of the iterated balanced filtration (Definition 3.5.8) is new also in this case.

3.6.3 QUIVER REPRESENTATIONS

Let Q be a quiver with set of vertices Q_0, set of arrows Q_1 and source and target maps $s, t\colon Q_1 \to Q_0$. Let d be a dimension vector for Q and consider the moduli stack $\mathcal{Rep}(Q,d)$ of representations of Q with dimension vector d over an algebraically closed field k of characteristic 0. A *central charge* Z for Q is defined by a family $(a_i)_{i\in Q_0}$ with $a_i \in \mathbb{Q}\oplus i\mathbb{Q}_{>0} \subset \mathbb{C}$. For a finite dimensional representation E of Q we set $Z(E) = \sum_{i\in\mathbb{Q}} a_i \dim E_i$. This defines a linear form ℓ and a norm q on $\mathcal{Rep}(Q,d)$ as follows. A graded point $g\colon B\mathbb{G}_{m,k} \to \mathcal{Rep}(Q,d)$ corresponds to a representation E with dimension vector d and a direct sum decomposition $E = \bigoplus_{c\in\mathbb{Z}} E_c$. We then set

$$\ell(g) = \sum_{c\in\mathbb{Z}} -c\,\mathfrak{Re}\,Z(E_c) \tag{3.1}$$

$$q(g) = \sum_{c\in\mathbb{Z}} c^2\,\mathfrak{Im}\,Z(E_c). \tag{3.2}$$

The linear form ℓ comes from the rational line bundle on $\mathcal{Rep}(Q,d)$ given by the rational character $\prod_{i\in Q_0}\det_{\mathrm{GL}_{d_i}}^{\mathfrak{Re}\,a_i}$ of $\prod_{i\in Q_0}\mathrm{GL}_{d_i}$. By Section 3.6.1, the stack $\mathcal{Rep}(Q,d)$ has a refined Harder-Narasimhan stratification induced by ℓ and q, and every representation $E\in\mathcal{Rep}(Q,d)(k)$ has a well-defined refined Harder-Narasimhan filtration.

We remark that semistability with respect to ℓ is precisely a King stability condition [54]. We can also see ℓ-semistable representations as Bridgeland semistable representations of slope $\pi/2$. The iterated balanced filtration of a semistable representation E coincides with the iterated weight filtration (or HKKP filtration) of E defined by Haiden–Katzarkov–Kontsevich–Pandit in [33]. This fact is proven in Chapter 7. Hence the balancing stratification of $\mathcal{R}ep(Q,d)^{\mathrm{ss}}$ can be seen as a stratification by type of HKKP filtration.

3.6.4 VECTOR BUNDLES ON A CURVE

Let C be a smooth projective curve over $\mathbb{C}$ and consider the stack $\mathrm{Bun}(C)_{r,d}$ of vector bundles on C of rank r and degree d. The stack $\mathrm{Bun}(C)_{r,d}$ has a norm on graded points q given by the rank and a linear form ℓ given by the degree as follows. A graded point $g\colon B\mathbb{G}_{m,\mathbb{C}} \to \mathrm{Bun}(C)_{r,d}$ corresponds to a vector bundle E of degree d and rank r and a direct sum decomposition $E = \bigoplus_{c\in\mathbb{Z}} E_c$ as sum of subbundles. We set

$$q(g) = \sum_{c\in\mathbb{Z}} c^2\,\mathrm{rk}(E_c)$$

and

$$\ell(g) = \sum_{c\in\mathbb{Z}} c\,\deg(E_c).$$

The pair (ℓ, q) defines a Θ-stratification of $\mathrm{Bun}(C)_{r,d}$ all whose centres are quasi-compact and have good moduli spaces (see [44], [4] and [36]). Therefore the refined Harder-Narasimhan stratification (Definition 3.5.10) of $\mathrm{Bun}(C)_{r,d}$ is defined. This produces, for every vector bundle $E \in \mathrm{Bun}(C)_{r,d}(\mathbb{C})$, a sequential filtration of E that refines its Harder–Narasimhan stratification.

Let $\mu(E) = \mathrm{rk}(E)/\deg(E)$ denote the *slope* of the vector bundle E. There are two different notions of semistability on $\mathrm{Bun}(C)_{r,d}$. The usual one is given by μ: E is μ-semistable if for all subbundles $F \subset E$ we have $\mu(F) \leq \mu(E)$. The other is semistability with respect to the linear form ℓ, in the sense of Definition 2.5.12. The two notions do not agree unless $d = 0$. Indeed, semistability of E with respect to ℓ is equivalent to the Harder-Narasimhan filtration being trivial. If E is μ-semistable, then its Harder-Narasimhan filtration is $0 = E_0 < E_1 = E$ together with the label $\mu(E_1/E_0) = \mu(E)$. If $\mu(E)$ is not 0, then this filtration is not trivial, since our notion of filtration cares about the labels. Thus the semistable objects with respect to ℓ are the μ-semistable objects of slope 0. If $d \neq 0$, then the Θ-stratum $\mathrm{Bun}(C)^{\mathrm{ss}}_{r,d} \subset \mathrm{Bun}(C)_{r,d}$ of μ-semistable vector bundles, although being unstable with our conventions, is still special, since it is an open stratum, it is isomorphic to its centre via the associated graded map, and it admits a good moduli space.

In this setting, a coarser version of the balancing stratification of $\mathrm{Bun}(C)^{\mathrm{ss}}_{r,d}$ was defined and studied by Kirwan [57]. Each stratum in Kirwan's stratification is a connected component of a locally closed substack of the form $\bigcup_{\alpha\in\mathbf{\Gamma}} \mathcal{S}^{\mathcal{X}}_{((n,c),\alpha)}$, for $n \in \mathbb{N}$ and $c \in \mathbb{Q}_{>0} \cup \{\infty\}$, where $\mathcal{X} = \mathrm{Bun}(C)^{\mathrm{ss}}_{r,d}$. Therefore Kirwan's stratification can be thought of as the stratification by type of balanced filtration. Kirwan calls the filtrations associated to this stratification *balanced δ-filtrations of maximal triviality*.

In Chapter 7, we show that the iterated balanced filtration for a semistable vector bundle E coincides with the iterated HKKP filtration of the lattice of semistable subbundles of E of the same slope. Hence the (not iterated) HKKP filtration of E corresponds to Kirwan's balanced δ-filtration of maximal triviality of E.

3.6.5 BRIDGELAND SEMISTABLE OBJECTS

Let X be a projective scheme over an algebraically closed field k of characteristic 0, and consider a Bridgeland stability condition [15] given by the heart $\mathcal{C} \subset \mathrm{D}^{\mathrm{b}}(X)$ of a t-structure on the derived category of X and a central charge $Z\colon K_0^{\mathrm{num}}(\mathrm{D}^{\mathrm{b}}(X)) \to \mathbb{C}$, where $K_0^{\mathrm{num}}(\mathrm{D}^{\mathrm{b}}(X))$ is the numerical Grothendieck group. Let $\mathcal{A} = \mathrm{Ind}(\mathcal{C})$ be the ind-completion of $\mathcal{C}$. Under certain natural assumptions on $(\mathcal{C}, Z)$, good moduli stacks of Bridgeland semistable objects can be constructed as open substacks of $\mathcal{M}_{\mathcal{A}}$, the moduli stack of objects in $\mathcal{A}$ defined in [8, Section 7] following [9]. We assume that Z is algebraic in the sense that $Z\left(\mathrm{D}^{\mathrm{b}}(X)\right) \subset \mathbb{Q} \oplus i\mathbb{Q}$, that Z factors through a finite free quotient of $K_0^{\mathrm{num}}(\mathrm{D}^{\mathrm{b}}(X))$, that $\mathcal{C}$ satisfies the generic flatness condition and that certain boundedness conditions also hold (see [36, Theorem 6.5.3] and [8, Example 7.29] for details). Under these assumptions, $\mathcal{M}_{\mathcal{A}}$ is an algebraic stack with affine diagonal locally of finite type over k.

For a numerical class $v \in K_0^{\mathrm{num}}(\mathrm{D}^{\mathrm{b}}(X))$, there is an open and closed substack $\mathcal{M}_v \subset \mathcal{M}_{\mathcal{A}}$ of objects in class v and, by our boundedness hypothesis, there is a quasi-compact open substack $\mathcal{M}_v^{\mathrm{ss}}$ of Bridgeland semistable objects. From the general results in [8], it follows that $\mathcal{M}_v^{\mathrm{ss}}$ admits a (proper) good moduli space [8, Example 7.29]. The imaginary part of the central charge defines a norm q on graded points of $\mathcal{M}_{\mathcal{A}}$ as in the previous examples, and the real part defines a linear form ℓ. If $g\colon B\mathbb{G}_{m,k} \to \mathcal{M}_{\mathcal{A}}$ corresponds to $E = \bigoplus_{c\in\mathbb{Z}} E_c$ in $\mathcal{C}$, then we define

$$q(g) = \sum_{c\in\mathbb{Z}} c^2\, \mathfrak{Im}\, Z(E_c)$$

and

$$\ell(g) = \sum_{c\in\mathbb{Z}} -c\, \mathfrak{Re}\, Z(E_c)$$

as above (this is the norm used in [36, Chapter 6] to define the numerical invariant on $\mathcal{M}_v$, by [36, Lemma 6.4.8]). Therefore $\mathcal{M}_v^{\mathrm{ss}}$ is naturally a normed good moduli stack, noetherian and with affine diagonal, and thus the balancing stratification and the iterated balanced filtration of every point are defined.

For $E \in \mathcal{M}_v^{\mathrm{ss}}(k)$, the iterated balanced filtration of E coincides with the HKKP filtration of the lattice of semistable subobjects of E of the same slope. This follows from Theorem 1.6.1.

More generally, ℓ and q define a Harder-Narasimhan stratification of $\mathcal{M}_v$ all whose centres are quasi-compact and have a good moduli space, so the refined Harder-Narasimhan stratification of $\mathcal{M}_v$ (Definition 3.5.10) is defined. This produces a refined Harder-Narasimhan filtration for every point of $\mathcal{M}_v$. We warn the reader that with our conventions $\mathcal{M}_v^{\mathrm{ss}}$ is the minimal stratum for the Harder-Narasimhan stratification, and not in general the semistable locus with respect to ℓ, similarly to the case of vector bundles on a curve explained above.

3.6.6 K-SEMISTABLE FANO VARIETIES

Let k be a field of characteristic 0. It has recently been established that there is an algebraic stack $\mathcal{X}_{n,V}^{\mathrm{K}}$ of finite type over k with affine diagonal parametrising families of K-semistable Fano varieties over k of dimension n and volume V, and that $\mathcal{X}_{n,V}^{\mathrm{K}}$ admits a proper good moduli space [5, 13, 14] (see [76] for a book account of these results). The L^2-norm of a test configuration [13, Section 2.3] defines a norm on graded points of $\mathcal{X}_{n,V}^{\mathrm{K}}$ [76, Lemmas 2.36 and 8.44], and thus the balancing stratification of $\mathcal{X}_{n,V}^{\mathrm{K}}$ is defined.

The stack $\mathcal{X}_{n,V}$ of Fano varieties of dimension n and volume V has a Θ-stratification, defined in [13], whose semistable locus is $\mathcal{X}_{n,V}^{\mathrm{K}}$. This Θ-stratification is not induced by a linear form and a norm. Nevertheless, since $\mathcal{X}_{n,V}$ does have a norm on graded points, it may be possible to apply Remark 3.5.11 to define a refined Harder-Narasimhan stratification of $\mathcal{X}_{n,V}$. However, at the time this thesis is being written, it is not known whether the centres have good moduli spaces, apart from the K-semistable stratum.

It is plausible that the iterated balanced filtration of a smooth K-semistable Fano variety X over $\mathbb{C}$ is related to the asymptotics of the Calabi flow on X. This would provide a refinement of the results in [18, 19].

3.6.7 G-BUNDLES AND GAUGED MAPS

Let C be a smooth projective over $\mathbb{C}$ and let G be a connected reductive group. The notion of semistability for principal G-bundles over C was defined in [68]. A moduli space of semistable G-bundles on C was constructed in [69,70] using GIT. The stack of semistable G-bundles $\mathrm{Bun}_G(C)^{\mathrm{ss}}$ is thus an open substack of the stack $\mathrm{Bun}_G(C)$ of all G-bundles that admits a good moduli space. A choice of norm on cocharacters of G induces a norm on $\mathrm{Bun}_G(C)$ as follows. Let T be a maximal torus of G with Weyl group W, then we have identifications

$$\begin{aligned}\mathrm{Grad}(\mathrm{Bun}_G(C)) &= \underline{\mathrm{Hom}}(B\mathbb{G}_{m,\mathbb{C}}, \underline{\mathrm{Hom}}(C, BG)) = \underline{\mathrm{Hom}}(C, \underline{\mathrm{Hom}}(B\mathbb{G}_{m,\mathbb{C}}, BG)) \\ &= \bigsqcup_{\lambda\in\Gamma^{\mathbb{Z}}(T)/W} \underline{\mathrm{Hom}}(L(\lambda), C) = \bigsqcup_{\lambda\in\Gamma^{\mathbb{Z}}(T)/W} \mathrm{Bun}_{L(\lambda)}(C)\end{aligned}$$

by [36, Theorem 1.4.8]. Therefore there is a natural map

$$\pi_0(\mathrm{Grad}(\mathrm{Bun}_G(C))) \to \pi_0(\mathrm{Grad}(BG)) = \Gamma^{\mathbb{Z}}(T)/W,$$

along which we can pull back the norm on BG to get a norm on cocharacters of $\mathrm{Bun}_G(C)$. This gives $\mathrm{Bun}_G(C)^{\mathrm{ss}}$ a natural structure of normed good moduli stack, and therefore defines the iterated balanced filtration for any semistable G-bundle. We expect the Yang–Mills flow for G-bundles [10] to be related to the iterated balanced filtration, in analogy with [34], which deals with the case $G = \mathrm{GL}_{n,\mathbb{C}}$.

The moduli stack of G-bundles is a special case of the general framework of gauged maps studied in [37]. For a projective-over-affine scheme X over $\mathbb{C}$ endowed with an action of G and $n \geq 0$, Halpern-Leistner and Fernandez Herrero define a stack $\mathcal{M}_n^G(X)$ parametrising families of Kontsevich stable maps from C to the quotient stack X/G. Taking $X = \mathrm{Spec}\,\mathbb{C}$ and $n = 0$ recovers the moduli stack of G-bundles on C. For suitable numerical invariants μ, the semistable locus $\mathcal{M}_n^G(X)^{\mu\text{-ss}}$ admits a good moduli space. The numerical invariant μ depends on a choice of norm on cocharacters of G, that then gives a norm on graded points of $\mathcal{M}_n^G(X)$ similarly to the case of G-bundles [37, Definition 2.21]. Therefore $\mathcal{M}_n^G(X)^{\mu\text{-ss}}$ is a normed good moduli stack and the balancing stratification of $\mathcal{M}_n^G(X)^{\mu\text{-ss}}$ is defined. The construction in [37] works over a general noetherian base S of characteristic 0 with affine diagonal. In this setting one still gets a noetherian normed good moduli stack $\mathcal{M}_n^G(X)^{\mu\text{-ss}}$ with affine diagonal, and its balancing stratification is thus defined. More generally, $\mathcal{M}_n^G(X)$ has a Θ-stratification whose centres have good moduli spaces [37, Section 5.2], so the refined Harder-Narasimhan stratification of $\mathcal{M}_n^G(X)$ is defined.

CHAPTER 4

CHAINS OF STACKS

We introduce the formalism of *chains of stacks* (Definition 4.1.1) as a tool to compute the iterated balanced filtration. For every chain of stacks there is an associated sequential filtration (Definition 4.1.3). We give two different constructions of chains. The first, the *balancing chain* (Construction 4.2.1), is very close to the balancing stratification and it computes the iterated balanced filtration. The second, the *torsor chain* (Construction 4.3.1) is more closely related to combinatorial versions of the iterated balanced filtration. The main theorem of this chapter states that the torsor chain also computes the iterated balanced filtration (Theorem 4.3.4). This fact will be used to relate the iterated balanced filtration to convex geometry (Theorem 5.2.16 and Corollary 5.2.18) and to artinian lattices (Corollary 7.5.11).

4.1 CHAINS

Let k be a field. A *k-pointed stack* is an algebraic stack $\mathcal{X}$ together with a k-point $x\colon \operatorname{Spec} k \to \mathcal{X}$. k-Pointed stacks form a 2-category as follows. A morphism $(\mathcal{X}, x) \to (\mathcal{Y}, y)$ of k-pointed stacks is a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of stacks and a 2-isomorphism $\alpha\colon f \circ x \to y$. The composition of $(f, \alpha)\colon (\mathcal{X}, x) \to (\mathcal{Y}, y)$ and $(g, \beta)\colon (\mathcal{Y}, y) \to (\mathcal{Z}, z)$ is $(g \circ f, \beta \circ (1_g * \alpha))$, where $*$ denotes horizontal composition. If $(f, \alpha), (f', \alpha')\colon (\mathcal{X}, x) \to (\mathcal{Y}, y)$ are morphisms of k-pointed stacks, a 2-morphism $(f, \alpha) \to (f', \alpha')$ is a 2-morphism $\gamma\colon f \to f'$ such that

$$\begin{array}{ccc} x \circ f & \xrightarrow{1_x * \gamma} & x \circ f' \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\alpha'} & \\ & y & \end{array}$$

commutes.

Definition 4.1.1 (Chain). A *chain* $(\mathcal{X}_n, x_n, \gamma_n, u_n)_{n \in \mathbb{N}}$ of k-pointed stacks is data:

1. For each $n \in \mathbb{N}$, a k-pointed normed stack $(\mathcal{X}_n, x_n)$, where $\mathcal{X}_n$ is of finite presentation over $\operatorname{Spec} k$ with affine diagonal and such that $\mathcal{X}_n \to \operatorname{Spec} k$ is a good moduli space.
2. For each $n \in \mathbb{N}$, a $\mathbb{Q}$-filtration $\gamma_n \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}_n, x_n)$ of x_n.
3. For each $n \in \mathbb{N}$, a representable, separated and norm-preserving morphism

$$u_n \colon (\mathcal{X}_{n+1}, x_{n+1}) \to (\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}_n), \mathrm{gr}\,\gamma_n).$$

We say that the chain $(\mathcal{X}_n, x_n, \gamma_n, u_n)_{n\in\mathbb{N}}$ is *bounded* if there is $N \in \mathbb{N}$ such that, for all $n \geq N$, we have that $\gamma_n = 0$ and u_n induces an isomorphism between $\mathcal{X}_{n+1}$ and $\mathcal{X}_n$, seen as the closed and open substack of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}_n)$ of "trivial gradings".

A *morphism* $f \colon (\mathcal{X}'_n, x'_n, \gamma'_n, u'_n) \to (\mathcal{X}_n, x_n, \gamma_n, u_n)$ *of chains* consists of morphisms $f_n \colon (\mathcal{X}'_n, x'_n) \to (\mathcal{X}_n, x_n)$ of pointed stacks, together with isomorphisms $f(\gamma'_n) \to \gamma_n$ of filtrations and 2-commutative squares

$$\begin{array}{ccc} (\mathcal{X}'_{n+1}, x'_{n+1}) & \xrightarrow{u'_n} & (\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}'_n), \mathrm{gr}\,\gamma'_n) \\ \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle \mathrm{Grad}_{\mathbb{Q}}(f_n)} \\ (\mathcal{X}_{n+1}, x_{n+1}) & \xrightarrow{u_n} & (\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}_n), \mathrm{gr}\,\gamma_n) \end{array}$$

of pointed stacks.

Suppose that $(\mathcal{X}_n, x_n, \gamma_n, u_n)_{n\in\mathbb{N}}$ is a bounded chain. For $n \in \mathbb{N}$, we define a map $c_n \colon \mathcal{X}_{n+1} \to \mathrm{Grad}^{n+1}_{\mathbb{Q}}(\mathcal{X}_0)$ by

$$c_n = \mathrm{Grad}^n_{\mathbb{Q}}(u_0) \circ \mathrm{Grad}^{n-1}_{\mathbb{Q}}(u_1) \circ \cdots \circ u_n.$$

By Proposition 2.2.21, c_n induces an injection

$$\mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}_{n+1}, x_{n+1}) \to \mathbb{Q}\text{-}\mathrm{Filt}(\mathrm{Grad}^{n+1}_{\mathbb{Q}}(\mathcal{X}_0), c_n(x_{n+1})).$$

Define $\lambda_{n+1} \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathrm{Grad}^{n+1}_{\mathbb{Q}}(\mathcal{X}_0), c_n(x_{n+1}))$ to be the image of γ_{n+1} under this injection. Define also $\lambda_0 := \gamma_0 \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}_0, x_0)$.

Lemma 4.1.2. *There is a canonical isomorphism $c_n(x_{n+1}) \simeq \mathrm{gr}\,\lambda_n$, for all $n \in \mathbb{N}$.*

Proof. For $n = 0$, $c_0(x_1) = u_0(x_1) \simeq \mathrm{gr}\,\gamma_0 = \mathrm{gr}\,\lambda_0$ is given by u_0 as a pointed map. For $n > 0$, we have

$$\begin{aligned} \mathrm{gr}\,\lambda_n = \mathrm{gr}\,(\mathrm{Filt}_{\mathbb{Q}}(c_{n-1})(\gamma_n)) = \mathrm{Grad}_{\mathbb{Q}}(c_{n-1})(\mathrm{gr}\,\gamma_n) \simeq \\ \mathrm{Grad}_{\mathbb{Q}}(c_{n-1})\,(u_n(x_{n+1})) = c_n(x_{n+1}), \end{aligned}$$

as desired. □

Note that there is $N \in \mathbb{N}$ such that $\lambda_n = 0$ for $n \geq N$, because the chain is bounded. The lemma gives canonical isomorphisms

$$\mathbb{Q}\text{-}\operatorname{Filt}(\operatorname{Grad}_{\mathbb{Q}}^{n+1}(\mathcal{X}_0), c_n(x_{n+1})) \cong \mathbb{Q}\text{-}\operatorname{Filt}(\operatorname{Grad}_{\mathbb{Q}}^{n+1}(\mathcal{X}_0), \operatorname{gr}\lambda_n)$$

for all n. Therefore, by Remark 3.2.14, $(\lambda_n)_{n\in\mathbb{N}}$ defines an element of $\mathbb{Q}^\infty\text{-}\operatorname{Filt}(\mathcal{X}_0, x_0)$.

Definition 4.1.3 (Sequential filtration of a chain)**.** The $\mathbb{Q}^\infty$*-filtration associated to* the bounded chain $(\mathcal{X}_n, x_n, \gamma_n, u_n)_{n\in\mathbb{N}}$ is $(\lambda_n)_{n\in\mathbb{N}} \in \mathbb{Q}^\infty\text{-}\operatorname{Filt}(\mathcal{X}_0, x_0)$.

4.2 THE BALANCING CHAIN

We now construct a chain closely related to the balancing stratification.

Construction **4.2.1.** Let $\mathcal{X}$ be a normed noetherian algebraic stack with affine diagonal and a good moduli space $\pi\colon \mathcal{X} \to X$. Let $x\colon \operatorname{Spec} k \to \mathcal{X}$ be a k-point, with k a field. We define, a chain $(\mathcal{X}_n, x_n, \gamma_n, u_n)_{n\in\mathbb{N}}$ over k as follows.

We set $(\mathcal{X}_0, x_0) = (\pi^{-1}(\pi(x)), x)$, which has good moduli space $\operatorname{Spec} k$. For $n \in \mathbb{N}$, assume $\mathcal{X}_n$ and x_n are defined. We now define $\mathcal{X}_{n+1}$, x_{n+1}, γ_n and u_n in terms of $\mathcal{X}_n$ and x_n. We consider two cases:

Case 1. The point x_n is closed in $\mathcal{X}_n$. We then define $\mathcal{X}_{n+1} = \mathcal{X}_n$, $x_{n+1} = x_n$, $\gamma_n = 0$ the trivial filtration in $\mathbb{Q}\text{-}\operatorname{Filt}(\mathcal{X}_n, x_n)$, and

$$u_n\colon (\mathcal{X}_{n+1}, x_{n+1}) \to (\operatorname{Grad}_{\mathbb{Q}}(\mathcal{X}_n), \operatorname{gr}\gamma_n)$$

given by the "trivial grading" map.

Case 2. The point x_n is not closed in $\mathcal{X}_n$. Then we consider the blow-up $p\colon \mathcal{B} = \operatorname{Bl}_{\mathcal{X}_n^{\max}} \mathcal{X}_n \to \mathcal{X}_n$, where $\mathcal{X}_n^{\max}$ is the closed substack of points with maximal dimension stabiliser [27, Appendix C]. In fact, $|\mathcal{X}_n^{\max}|$ is a singleton consisting of the unique closed point of $|\mathcal{X}_n|$ [2, Proposition 9.1], which is different from x_n by assumption. Thus the point x_n lifts uniquely to a point of $\mathcal{B}$ that we still denote x_n.

By Theorem 2.6.4 and Example 2.6.8, $\mathcal{B}$ has a Θ-stratification induced by the natural p-ample line bundle on $\mathcal{B}$ and the norm. Let $\mathcal{S}$ be the locally closed Θ-stratum containing x_n and let $\mathcal{Z}$ be its centre (Definition 2.5.1). Let $\gamma_n \in \mathbb{Q}\text{-}\operatorname{Filt}(\mathcal{B}, x_n)$ be the Harder–Narasimhan filtration of x_n in $\mathcal{B}$ (Definition 2.5.6) and let $x_{n+1} = \operatorname{gr}\gamma_n \in \mathcal{Z}(k)$. We identify $\mathbb{Q}\text{-}\operatorname{Filt}(\mathcal{B}, x_n) = \mathbb{Q}\text{-}\operatorname{Filt}(\mathcal{X}_n, x_n)$ by Proposition 2.2.20, and under this identification γ_n is the balanced filtration of x_n in $\mathcal{X}_n$ (Definition 3.1.6 and Proposition 3.1.2). By Theorem 2.6.4, $\mathcal{Z}$ has a good moduli space $\pi_{\mathcal{Z}}\colon \mathcal{Z} \to Z$. We set $\mathcal{X}_{n+1} = \pi_{\mathcal{Z}}^{-1}(\pi_{\mathcal{Z}}(x_{n+1}))$. We define u_n to be the composition

$$u_n\colon \mathcal{X}_{n+1} \to \mathcal{Z} \hookrightarrow \operatorname{Grad}_{\mathbb{Q}}(\mathcal{B}) \xrightarrow{\operatorname{Grad}_{\mathbb{Q}}(p)} \operatorname{Grad}_{\mathbb{Q}}(\mathcal{X}_n),$$

which is representable and separated, since applying $\mathrm{Grad}_{\mathbb{Q}}$ preserves representability and separatedness, and the first two maps are immersions.

The stack $\mathcal{X}_{n+1}$ inherits a norm from $\mathcal{X}_n$ along the composition

$$\mathcal{X}_{n+1} \to \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}_n) \to \mathcal{X}_n.$$

By commutativity of

$$\begin{array}{ccc} \mathrm{Filt}_{\mathbb{Q}}(\mathcal{B}) & \longrightarrow & \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}_n) \\ {\scriptstyle \mathrm{gr}}\downarrow & & \downarrow{\scriptstyle \mathrm{gr}} \\ \mathrm{Grad}_{\mathbb{Q}}(\mathcal{B}) & \longrightarrow & \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}_n) \end{array}$$

we get a pointed morphism $u_n\colon (\mathcal{X}_{n+1}, x_{n+1}) \to (\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}_n), \mathrm{gr}\,\gamma_n)$.

Definition 4.2.2 (The balancing chain)**.** Let $\mathcal{X}$ be a normed noetherian good moduli stack with affine diagonal, let k be a field and let $x \in \mathcal{X}(k)$ be a k-point. The *balancing chain* of $(\mathcal{X}, x)$ is the chain $(\mathcal{X}_n, x_n, \gamma_n, u_n)_{n\in\mathbb{N}}$ of k-stacks constructed in Construction 4.2.1.

Remark **4.2.3.** The following properties of the balancing chain are clear from Theorem 3.1.3:

1. For every $n \in \mathbb{N}$, we have $\gamma_n = 0$ if and only if x_n is closed in $\mathcal{X}_n$.
2. For every $n \in \mathbb{N}$, we have that $\mathrm{ev}_0(\gamma_n)$ is the unique closed point of $\mathcal{X}_n$.

Lemma 4.2.4. *Assume the setup of Definition 4.2.2. For every $n \in \mathbb{N}$ such that x_n is not closed in $\mathcal{X}_n$, we have that $z(\mathcal{X}_n) \geq z(\mathcal{X}_0) + n$, where $z(-)$ denotes central rank (Definition 3.4.1).*

Note as well that $z(\mathcal{X}_0) \geq z(\mathcal{X})$.

Proof. Let $n \in \mathbb{N}$ and suppose that x_n is not closed in $\mathcal{X}_n$. Then $z(\mathcal{X}_{n+1}) > z(\mathcal{X}_n)$ by Lemma 3.4.7. The results follows by induction. □

Corollary 4.2.5. *Assuming the setup of Definition 4.2.2, the balancing chain of $(\mathcal{X}, x)$ is bounded.*

Proof. If it was not bounded, then for every $n \in \mathbb{N}$ we would have that x_n is not closed in $\mathcal{X}_n$ and thus that $z(\mathcal{X}_n) \geq n$ by Lemma 4.2.4. This would contradict the bound $z(\mathcal{X}_n) \leq d(\mathcal{X}_0)$, where $d(-)$ denotes maximal stabiliser dimension (Definition 3.1.5). □

Proposition 4.2.6. *Let $\mathcal{X}$ be a normed noetherian good moduli stack with affine diagonal, and let $x \in \mathcal{X}(k)$ be a field-valued point. Then the sequential filtration of x associated to the balancing chain of $(\mathcal{X}, x)$ (Definitions 4.1.3 and 4.2.2) equals the iterated balanced filtration of x (Definition 3.5.8).*

Proof. Let $(\mathcal{X}_n, x_n, \gamma_n, u_n)_{n\in\mathbb{N}}$ be the balancing chain of $(\mathcal{X}, x)$, and let

$$\lambda_{\mathrm{bc}}(x) \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}_0, x_0) = \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, x)$$

be its associated sequential filtration. Note that, by Corollary 3.5.7, the iterated balanced filtration $\lambda_{\mathrm{ib}}(x)$ of $(\mathcal{X}, x)$ equals that of $(\mathcal{X}_0, x)$.

We prove the statement by induction on $N(\mathcal{X}) = d(\mathcal{X}) - z(\mathcal{X})$. If $N(\mathcal{X}) = 0$, then x is closed in $\mathcal{X}_0$ and therefore $\lambda_{\mathrm{ib}}(x) = \lambda_{\mathrm{bc}}(x) = 0$. If x is not closed in $\mathcal{X}_0$ then, using the notation of Construction 4.2.1, $\gamma_0 \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}_0, x)$ is the balanced filtration of x (Definition 3.1.6). From Proposition 3.2.11, we get a map $\varphi\colon \mathbb{Q}^{\infty}\text{-}\mathrm{Filt}(\mathcal{X}_1, x_1) \hookrightarrow \mathbb{Q}^{\infty}\text{-}\mathrm{Filt}(\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}_0), \mathrm{gr}\,\gamma_0) \to \mathbb{Q}^{\infty}\text{-}\mathrm{Filt}(\mathcal{X}_0, x_0)$ that, in the notation of Remark 3.2.14, can be written as

$$(\lambda_0, \lambda_1, \ldots) \mapsto (\gamma_0, \lambda_0, \lambda_1, \ldots).$$

By construction of the balancing chain, we have $\lambda_{\mathrm{bc}}(x) = \varphi(\lambda_{\mathrm{bc}}(x_1))$, and by construction of the balancing stratification, we have $\lambda_{\mathrm{ib}}(x) = \varphi(\lambda_{\mathrm{ib}})(x_1)$. By Lemma 3.4.7, $N(\mathcal{X}_1) < N(\mathcal{X}_0)$. Therefore $\lambda_{\mathrm{ib}}(x_1) = \lambda_{\mathrm{bc}}(x_1)$ by induction, and hence $\lambda_{\mathrm{ib}}(x) = \lambda_{\mathrm{bc}}(x)$. □

Remark **4.2.7.** One could define chain similar to the balancing chain but where one replaces $\mathcal{X}_n$ with $\overline{\{x_n\}}$ (with reduced structure) at each step, and Proposition 4.2.6 would still be true by the same reasons.

4.3 THE TORSOR CHAIN

We introduce a second chain whose construction is similar to that of the balancing chain, but where at each step the exceptional divisor is replaced by the natural $\mathbb{G}_m$-torsor over it. Then we prove (Theorem 4.3.4) that this new chain also computes the iterated balanced filtration.

Construction **4.3.1.** Let $\mathcal{X}$ be a normed noetherian algebraic stack with affine diagonal and with a good moduli space $\pi\colon \mathcal{X} \to X$. Let k be a field and let $x \in \mathcal{X}(k)$ be a k-point. We construct a chain $(\mathcal{Y}_n, y_n, \eta_n, v_n)_{n\in\mathbb{N}}$ inductively as follows.

We set $(\mathcal{Y}_0, y_0) = (\pi(\pi^{-1}(x)), x)$. Suppose that $(\mathcal{Y}_n, y_n)$ is defined. We define η_n, v_n and $(\mathcal{Y}_{n+1}, y_{n+1})$ in terms of $(\mathcal{Y}_n, y_n)$.

Case 1. The point y_n is in $\mathcal{Y}_n^{\max}$. In that case, we set $\eta_n = 0$, $(\mathcal{Y}_{n+1}, y_{n+1}) = (\mathcal{Y}_n, y_n)$ and $v_n\colon (\mathcal{Y}_{n+1}, y_{n+1}) \to (\mathrm{Grad}_{\mathbb{Q}}(\mathcal{Y}_n), \mathrm{gr}\,\eta_n)$ the "trivial grading" map.

Case 2. The point y_n is not in $\mathcal{Y}_n^{\max}$. Then y_n lifts uniquely to $\mathcal{B} := \mathrm{Bl}_{\mathcal{Y}_n^{\max}} \mathcal{Y}_n$, which is canonically endowed with a linear form on graded points, coming from the exceptional divisor, and with the induced norm on graded points (Theorem 2.6.4 and Example 2.6.8). We let $\eta_n \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{Y}_n, y_n)$ be the HN filtration of y_n in $\mathcal{B}$. Let $\mathcal{S}$ be the locally closed Θ-stratum of $\mathcal{B}$ containing y_n, and let $\mathcal{Z}$ be its centre. Let $\mathcal{E}$ be the exceptional divisor and $\mathcal{N} \to \mathcal{E}$ the natural $\mathbb{G}_{m,k}$-torsor, that is, $\mathcal{N}$ is the complement of the zero section inside the total space of the normal cone to $\mathcal{Y}_n^{\max} \to \mathcal{Y}_n$. If $\mathcal{L}$ is the ideal sheaf of the exceptional divisor $\mathcal{E}$, then $\mathcal{N} = \mathbb{A}((\mathcal{L}|_{\mathcal{E}})^\vee)^*$, where $\mathbb{A}(-)^*$ denotes total space minus zero section.

Lemma 4.3.2. *The "forget the grading" map $h\colon \mathcal{Z} \to \mathcal{B}$ factors through $\mathcal{E} \to \mathcal{B}$. As a consequence, the open immersion $\mathcal{Z} \to \mathrm{Grad}_{\mathbb{Q}}(\mathcal{B})$ factors through the closed immersion $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{E}) \to \mathrm{Grad}_{\mathbb{Q}}(\mathcal{B})$. In particular, the induced map $\mathcal{Z} \to \mathrm{Grad}_{\mathbb{Q}}(\mathcal{E})$ is an open immersion.*

Proof. The centre $\mathcal{Z}$ carries a canonical rational $B\mathbb{G}_m$-action, which endows every quasi-coherent sheaf $\mathcal{M}$ on $\mathcal{Z}$ with a $\mathbb{Q}$-grading $\mathcal{M} = \bigoplus_{c\in\mathbb{Q}} \mathcal{M}_c$. To see how this grading originates, let us assume for simplicity that the $B\mathbb{G}_m$-action is integral. Then for any morphism $f\colon T \to \mathcal{Z}$ there is an associated map $g\colon T \times B\mathbb{G}_m \to \mathcal{Z}$. The pullback $g^*\mathcal{M}$ is a $\mathbb{G}_m$-equivariant sheaf on T, that is, a $\mathbb{Z}$-graded sheaf on T, and the underlying sheaf is $(T \to T \times B\mathbb{G}_m)^* g^*\mathcal{M} = f^*\mathcal{M}$. Thus $f^*\mathcal{M}$ has a canonical $\mathbb{Z}$-grading for every f, and this gives the $\mathbb{Z}$-grading for $\mathcal{M}$ itself.

Let $\mathcal{L}$ be the ideal sheaf of $\mathcal{E}$. Since $\mathcal{Z}$ is the centre of a stratum, unstable with respect to the linear form $\langle -, \mathcal{L}\rangle$, we must have that the $\mathbb{Q}$-grading on $h^*\mathcal{L}$ is concentrated in degree -1. Indeed, we know that for every map $\operatorname{Spec} l \to \mathcal{Z}$, with l a field, the pullback $(\operatorname{Spec} l \times B\mathbb{G}_m \to \mathcal{Z})^* h^*\mathcal{L}$ is concentrated in degree -1. Thus by Nakayama's lemma, $(h^*\mathcal{L})_c = 0$ for all $c \in \mathbb{Q} \setminus \{-1\}$.

On the other hand, the structure sheaf $\mathcal{O}_{\mathcal{Z}}$ is concentrated in degree 0. There is a map $\mathcal{L} \to \mathcal{O}_{\mathcal{B}}$ because $\mathcal{L}$ is an ideal sheaf. Pulling this map back along h, we get a homomorphism $h^*\mathcal{L} \to \mathcal{O}_{\mathcal{Z}}$ that must be 0 for degree reasons. Therefore h factors through $\mathcal{E} \to \mathcal{B}$. Since $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{E}) = \mathcal{E} \times_{\mathcal{B}} \mathrm{Grad}_{\mathbb{Q}}(\mathcal{B})$, the lemma follows. □

Let $\mathcal{M}$ be the fibre product

$$\begin{array}{ccc} \mathcal{M} & \longrightarrow & \mathcal{N} \\ \downarrow & \ulcorner & \downarrow \\ \mathcal{Z} & \longrightarrow & \mathcal{E}, \end{array}$$

and let y_{n+1} be any k-point of $\mathcal{M}$ lying above $\operatorname{gr}\eta_n$. Any two choices of y_{n+1} are related by a unique $\mathbb{G}_{m,k}$-torsor automorphism of $\mathcal{M} \to \mathcal{Z}$. Now $\mathcal{M}$ has a good moduli space $\pi_{\mathcal{M}}\colon \mathcal{M} \to M$ because $\mathcal{M} \to \mathcal{Z}$ is affine and $\mathcal{Z}$ has a good moduli space. Finally, we let $\mathcal{Y}_{n+1} := \pi_{\mathcal{M}}^{-1}(\pi_{\mathcal{M}}(y_{n+1}))$ and $v_n\colon (\mathcal{Y}_{n+1}, y_{n+1}) \to (\mathcal{M}, y_{n+1}) \to$

$(\mathcal{Z}, \operatorname{gr}\eta_n) \to (\operatorname{Grad}_{\mathbb{Q}}(\mathcal{X}_n), \operatorname{gr}\eta_n)$. We are abusively denoting η_n both the filtration of x_n in $\mathcal{X}_n$ and in $\mathcal{B}$, and by $\operatorname{gr}\eta_n$ the associated graded point in the two cases.

DEFINITION 4.3.3 (Torsor chain)**.** Let $\mathcal{X}$ be a normed noetherian good moduli stack with affine diagonal, let k be a field and let $x \in \mathcal{X}(k)$ be a k-point. The *torsor chain* of $(\mathcal{X}, x)$ is the chain $(\mathcal{Y}_n, y_n, \eta_n, v_n)_{n\in\mathbb{N}}$ of k-stacks from Construction 4.3.1.

THEOREM 4.3.4. *Let k be a field and let $(\mathcal{X}, x)$ be a k-pointed normed noetherian good moduli stack with affine diagonal. Then the torsor chain of $(\mathcal{X}, x)$ is bounded and its associated $\mathbb{Q}^\infty$-filtration (Definitions 4.1.3 and 4.3.3) equals the iterated balanced filtration of $(\mathcal{X}, x)$ (Definition 3.5.8).*

Proof. Let $(\mathcal{X}_n, x_n, \gamma_n, u_n)_{n\in\mathbb{N}}$ be the balancing chain of $(\mathcal{X}, x)$ and let $(\mathcal{Y}_n, y_n, \eta_n, v_n)_{n\in\mathbb{N}}$ be the torsor chain. The iterated balanced filtration is well-behaved under field extension by Proposition 3.5.5, and the torsor chain also commutes with base field extension by a similar argument. Therefore, we may assume that k is algebraically closed.

By Corollary 2.1.5, $\mathcal{X}_0 = R_0/G_0$, where R_0 is an affine scheme and G_0 is the stabiliser of a k-point p_0 of R_0 which is also the unique closed G_0-orbit of R_0. By Proposition 3.1.7, the norm on graded points of $\mathcal{X}_0$ is induced by a norm on cocharacters of G_0. The stack $\mathcal{X}_n$ can be written as $\mathcal{X}_n = R_n/G_n$, with R_n affine and G_n the stabiliser of a k-point p_n of R_n which is also the unique closed G_n-orbit; and the norm on $\mathcal{X}_n$ is induced by a norm on cocharacters of G_n.

We start by introducing some natural extra structure on the balancing chain. Let N the smallest natural number such that $\gamma_N = 0$. For every $n \leq N$, there are line bundles $\mathcal{L}_{n,1}, \cdots, \mathcal{L}_{n,n}$ on $\mathcal{X}_n$ constructed inductively as follows. If $n < N$, then $\mathcal{L}_{n+1,i} = (\mathcal{X}_{n+1} \to \mathcal{X}_n)^*\mathcal{L}_{n,i}$ for all $i = 1, \ldots, n$. The map $\mathcal{X}_{n+1} \to \mathcal{X}_n$ we are considering is the composition of $u_n \colon \mathcal{X}_{n+1} \to \operatorname{Grad}_{\mathbb{Q}}(\mathcal{X}_n)$ and the "forget the grading" map $\operatorname{Grad}_{\mathbb{Q}}(\mathcal{X}_n) \to \mathcal{X}_n$. Using the notation of Construction 4.2.1, we have a relatively ample line bundle $\mathcal{L}_{\mathcal{B}} = \mathcal{O}_{\mathcal{B}}(-\mathcal{E})$ on the blow-up $\mathcal{B} = \operatorname{Bl}_{\mathcal{X}_n^{\max}} \mathcal{X}_n$, and we let $\mathcal{L}_{n+1,n+1} = (\mathcal{X}_{n+1} \to \mathcal{B})^*\mathcal{L}_{\mathcal{B}}$, where the morphism considered is the composition $\mathcal{X}_{n+1} \to \mathcal{Z} \to \operatorname{Grad}_{\mathbb{Q}}(\mathcal{B}) \to \mathcal{B}$.

There is also a natural non-degenerate rational $B\mathbb{G}_m^n$-action on $\mathcal{X}_n$ for each $n \leq N$. Indeed, if this action has been constructed for $\mathcal{X}_n$, with $n < N$, we construct it for $\mathcal{X}_{n+1}$ as follows. Again, we use the notations of Case 2 in Construction 4.2.1. By Lemma 3.4.5, the rational $B\mathbb{G}_m^n$-action on $\mathcal{X}_n$ lifts canonically to a rational $B\mathbb{G}_m^n$-action on $\mathcal{B}$. Since $\mathcal{Z}$ is an open substack of $\operatorname{Grad}_{\mathbb{Q}}(\mathcal{B})$, it has a natural $B\mathbb{G}_m^n \times B\mathbb{G}_m$-action, and it is nondegenerate by the argument in the proof of Lemma 3.4.7. This action now restricts to the closed substack $\mathcal{X}_{n+1}$ of $\mathcal{Z}$.

Let T_n be a maximal torus of G_n. It follows from [36, Theorem 1.4.8] that the rational $B\mathbb{G}_m^n$-action on $\mathcal{X}_n$ is given by rational one-parameter subgroups $\beta_1, \dots, \beta_n$ of the centre $Z(G_n)$ that act trivially on R_n. Indeed, the $B\mathbb{G}_m^n$-action is given by an element $\beta \in \operatorname{Hom}(\mathbb{G}_{m,k}^n, T_n) \otimes_{\mathbb{Z}} \mathbb{Q}$ and a connected component C of the fixed point locus $R_n^{\beta,0}$ such that the induced "forget the grading" map $C/L_{G_n}(\beta) \subset \operatorname{Grad}_{\mathbb{Q}}(\mathcal{X}_n) \to \mathcal{X}_n$ is an isomorphism. Since R_n has a point fixed by G_n, it must be $L_{G_n}(\beta) = G_n$, and consequently $C = R_n$, from what we see that $(\beta_1, \dots, \beta_n) = \beta$ has the desired property.

The blow-up $\mathcal{B}_n = \operatorname{Bl}_{\mathcal{X}_n^{\max}} \mathcal{X}_n$can be written as $\mathcal{B}_n = B_n/G_n$, with B_n the blow-up of R_n along a closed subscheme $R_n^{\max}$ given by $\mathcal{X}_n^{\max}$. The centre of the locally closed Θ-stratum of $\mathcal{B}_n$ containing the lift of x_n to $\mathcal{B}_n$ is $\mathcal{Z}_n = Z_n/L_{G_n}(\beta_{n+1})$, where $\beta_{n+1} \in \Gamma^{\mathbb{Q}}(T_n)$ corresponds to $\operatorname{gr}\gamma_n$ and Z_n is an open subscheme of the fixed point locus $B_n^{\beta_{n+1},0}$. The group G_{n+1} is the stabiliser of a point of Z_n, so it is identified with a subgroup of $L_{G_n}(\beta_{n+1})$ containing $\beta_1, \dots, \beta_{n+1}$ in its centre. With this identifications we regard the β_i as independent of n.

Claim **4.3.5.** For all $n < N$, for all $1 \le j \le n$, we have the equalities $\langle \beta_j, \mathcal{L}_{n+1,n+1} \rangle = 0$ and $\langle \beta_{n+1}, \mathcal{L}_{n+1,n+1} \rangle = 1$. Therefore $\langle \beta_i, \mathcal{L}_{n,j} \rangle = \delta_{ij}$ for $n \le N$ and $i \le j \le n$.

The first equality follows from the fact that the $\beta_1, \dots, \beta_n$ act trivially on R_n and thus also on the normal cone to $R_n^{\max}$. The second follows from the definition of γ_n as the minimiser of $\|-\|$ subject to $\langle \gamma_n, \mathcal{L}_{n+1,n+1} \rangle = 1$, and the fact that $\operatorname{gr}\gamma_n$ is given by β_{n+1}.

Claim **4.3.6.** For all $n \le N$ and all $1 \le i \ne j \le n$, we have $(\beta_i, \beta_j) = 0$, where $(-,-)$ denotes the inner product in $\Gamma^{\mathbb{Q}}(T_n)$ defined by the norm on cocharacters of G_n.

We prove the claim by induction. Let $n < N$. By the Linear Recognition Theorem 2.6.9, a point $z \in (B_n)^{\beta_{n+1},0}$ belongs to Z_n if and only if it is semistable with respect to the shifted linear form, which in this case is

$$\ell_n = \langle -, \mathcal{L}_{\mathcal{B}_n} \rangle - \frac{1}{\|\beta_{n+1}\|^2}(\beta_{n+1}, -),$$

where $\mathcal{L}_{\mathcal{B}_n}$ is the relatively ample line bundle on $\mathcal{B}_n$. Since β_i for $i \le n$ all fix z, we must have $\ell_n(\beta_i) \le 0$ and $\ell_n(-\beta_i) \le 0$, but $\ell_n(\beta_i) = \frac{1}{\|\beta_{n+1}\|^2}(\beta_{n+1}, \beta_i)$ and $\ell_n(-\beta_i) = -\frac{1}{\|\beta_{n+1}\|^2}(\beta_{n+1}, \beta_i)$, so $(\beta_{n+1}, \beta_i) = 0$. This proves the claim.

Let us denote $\mathcal{V}_n = \mathbb{A}(\mathcal{L}_{1,n}^\vee)^* \times_{\mathcal{X}_n} \cdots \times_{\mathcal{X}_n} \mathbb{A}(\mathcal{L}_{n,n}^\vee)^*$, where $\mathbb{A}(-)^*$ denotes total space minus zero section. There is a natural quotient presentation $\mathcal{V}_n = V_n/G_n$, where $V_n = \mathbb{A}(\mathcal{L}_{n,1}^\vee|_{R_n})^* \times_{R_n} \cdots \times_{R_n} \mathbb{A}(\mathcal{L}_{n,n}^\vee|_{R_n})^*$, which is a $\mathbb{G}_m^n$-torsor over R_n, and in particular carries a $\mathbb{G}_{m,k}^n$-action. Let $T'_n = \operatorname{im}(\beta_1, \dots, \beta_n) \subset Z(G_n)$. Note that this is well-defined even if the β_i are rational cocharacters and not necessarily integral.

Claim **4.3.7.** The torus T'_n acts on V_n via a homomorphism $\delta_n \colon T'_n \to \mathbb{G}_{m,k}^n$. Moreover, δ_n is an isogeny.

Recall that the torus T'_n acts trivially on R_n. The $\mathbb{G}_m^n$-torsor $V_n/T'_n \to R_n/T'_n = R_n \times BT'_n$ is given by a map $r \colon R_n \times BT'_n \to B\mathbb{G}_{m,k}^n$, which is in turn given by a map $R_0 \to \underline{\operatorname{Hom}}(BT'_n, B\mathbb{G}_{m,k}^n)$. We have the equality

$$\underline{\operatorname{Hom}}(BT'_n, B\mathbb{G}_{m,k}^n) = \coprod_{\alpha \in \operatorname{Hom}(T'_n, \mathbb{G}_{m,k}^n)} B\mathbb{G}_m^n$$

because T'_n is a split torus and by [36, Theorem 1.4.8]. Therefore, since R_n is connected, r corresponds to a pair $(R_0 \xrightarrow{o} B\mathbb{G}_{m,k}^n, \delta_n)$, where o corresponds to the $\mathbb{G}_m^n$-torsor $V_n \to R_0$ and $\delta_n \in \operatorname{Hom}(T', \mathbb{G}_{m,k}^n)$. We recover r as the composition $R_0 \times BT' \xrightarrow{(o, B\delta_n)} B\mathbb{G}_{m,k}^n \times B\mathbb{G}_{m,k}^n \to B\mathbb{G}_{m,k}^n$, the last map being multiplication. The homomorphism δ_n induces a map $D(\delta_n)_{\mathbb{Q}} \colon \Gamma_{\mathbb{Q}}(\mathbb{G}_{m,k}^n) \to \Gamma_{\mathbb{Q}}(T'_n)$. If we take $\beta_1, \dots, \beta_n$ as a basis of $\Gamma_{\mathbb{Q}}(T'_n)$ and the standard basis of $\Gamma_{\mathbb{Q}}(\mathbb{G}_{m,k}^n)$, then $D(\delta_n)_{\mathbb{Q}}$ is given by the matrix $\langle \beta_i, \mathcal{L}_{n,j} \rangle$, which we have shown is upper triangular with 1's in the diagonal. Therefore $D(\delta_n)_{\mathbb{Q}}$ is an isomorphism and δ_n is an isogeny.

Claim **4.3.8.** Let $\overline{\mathcal{X}}_n = R_n/(G_n/T'_n)$ and let $D_n = \ker \delta_n$. Then $V_n/(G_n/D_n) \cong \overline{\mathcal{X}}_n$.

The group $(G_n/D_n)/(T'_n/D_n)$ acts on the stack $V_n/(T'_n/D_n)$ and there is a natural isomorphism

$$V_n/(G_n/D_n) \cong \big(V_n/(T'_n/D_n)\big) \,/\, \big((G_n/D_n)/(T'_n/D_n)\big),$$

by [71, Remark 2.4]. Since $T'_n/D_n = \mathbb{G}_{m,k}^n$ and $Y_n \to R_n$ is a $\mathbb{G}_m^n$-torsor, we have $V_n/(T'_n/D_n) = R_n$. Noting that $(G_n/D_n)/(T'_n/D_n) = G_n/T'_n$, we get the desired isomorphism.

As a consequence of the claim, we see that the good moduli space of $\mathcal{V}_n$ is $\operatorname{Spec} k$.

Claim **4.3.9.** Let $f_n \colon \mathcal{V}_n \to \mathcal{X}_n$ be the $\mathbb{G}_m^n$-torsor map. Then $f_n^{-1}(\mathcal{X}_n^{\max}) = \mathcal{V}_n^{\max}$ and $d(\mathcal{X}_n) = d(\mathcal{V}_n) + n$.

Let p'_n be a k-point of V_n mapping to p_n along $\mathcal{V}_n \to \mathcal{X}_n$. Necessarily $G_n p'_n$ is the unique closed orbit inside V_n. Let H_n be the stabiliser of p'_n. By [7, Theorem 10.4, (5) and (6)] together with the fact that the good moduli space of $\mathcal{V}_n$ is $\operatorname{Spec} k$, there is a locally closed H_n-equivariant subscheme S_n of V_n containing x'_n such that $S_n/H_n \to \mathcal{V}_n$ is an isomorphism. The map $S_n/(H_n/D_n) \to V_n/(G_n/D_n)$ must also be an isomorphism. Since S_n has a point fixed by H_n, the maximal dimension stabiliser locus is $\mathcal{V}_n^{\max} = S_n^{(H_n)_\circ}/H_n$, where $(H_n)_\circ$ is the reduced identity component of H_n (see [27, Appendix C]). Similarly, $(V_n/(G_n/D_n))^{\max} = (S_n)^{(H_n/D_n)_\circ}/(H_n/D_n)$. Since D_n acts trivially on S_n, we have $(S_n)^{(H_n/D_n)_\circ} = S_n^{(H_n)_\circ}$. This proves that, denoting $\rho_n\colon \mathcal{V}_n \to \overline{\mathcal{X}}_n$ the obvious map, we have $\mathcal{V}_n^{\max} = \rho_n^{-1}(\mathcal{X}_n^{\max})$. On the other hand, since R_n has a point fixed by G_n, we have $\mathcal{X}_n^{\max} = R_n^{(G_n)_\circ}/G_n$. Since T_n' acts trivially on R_n, $\overline{\mathcal{X}}_n^{\max} = R_n^{(G_n/T_n')_\circ}/(G_n/T_n') = R_n^{(G_n)_\circ}/(G_n/T_n')$ so, denoting $q_n\colon \mathcal{X}_n \to \overline{\mathcal{X}}_n$, we have $q_n^{-1}\left(\overline{\mathcal{X}}_n^{\max}\right) = \mathcal{X}_n^{\max}$. Hence $f_n^{-1}\left(\mathcal{X}_n^{\max}\right) = q_n^{-1}\left(\overline{\mathcal{X}}_n^{\max}\right) = \rho_n^{-1}\left(\overline{\mathcal{X}}_n^{\max}\right) = \mathcal{V}_n^{\max}$. For the last statement, just note that $d(\mathcal{V}_n) = d\left(\overline{\mathcal{X}}_n\right)$ and that $d(\mathcal{X}_n) = d\left(\overline{\mathcal{X}}_n\right) + n$. This proves the claim.

Suppose that $n < N$. Let x_n' be a k-point of V_n mapping to x_n along $\mathcal{V}_n \to \mathcal{X}_n$.

Claim **4.3.10.** The balanced filtration of $(\mathcal{V}_n, x_n')$ equals the balanced filtration of $(\mathcal{X}_n, x_n)$ under the injection $\mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{V}_n, x_n') \to \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}_n, x_n)$.

We identify the balanced filtration of $(\mathcal{X}_n, x_n)$ with $\beta_{n+1} \in \Gamma^{\mathbb{Q}}(G_n)$. Since β_{n+1} is orthogonal to the cocharacters in T_n', by Claim 4.3.6, and since $\mathcal{X}_n \to \overline{\mathcal{X}}_n$ preserves the substacks of maximal stabiliser dimensions, the image $\overline{\beta}_{n+1} \in \Gamma^{\mathbb{Q}}(G_n/T_n')$ of β_{n+1} inside G_n/T_n' is the balanced filtration of $(\overline{\mathcal{X}}_n, x_n)$. Since $\rho_n\colon \mathcal{V}_n \to \overline{\mathcal{X}}_n$ is a gerbe banded by the finite group D_n, $\mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{V}_n, x_n') = \mathbb{Q}\text{-}\mathrm{Filt}(\overline{\mathcal{X}}_n, x_n)$, and this equality identifies the balanced filtrations of $\mathcal{V}_n$ and $\overline{\mathcal{X}}_n$ because $\rho_n^{-1}(\overline{\mathcal{X}}_n^{\max}) = \mathcal{V}_n^{\max}$. Therefore β_{n+1} is also the balanced filtration of $(\mathcal{V}_n, x_n')$.

Let $\mathcal{C}_n = \mathrm{Bl}_{\mathcal{V}_n^{\max}} \mathcal{V}_n = C_n/G_n$. Since $\mathcal{V}_n \to \mathcal{X}_n$ is flat, being a $\mathbb{G}_m^n$-torsor, and since $f_n^{-1}(\mathcal{X}_n^{\max}) = \mathcal{V}_n^{\max}$, we have that the blow-ups form a cartesian diagram

$$\begin{array}{ccc} \mathcal{C}_n & \longrightarrow & \mathcal{V}_n \\ \downarrow & \ulcorner & \downarrow f_n \\ \mathcal{B}_n & \longrightarrow & \mathcal{X}_n. \end{array}$$

Let $\mathcal{Z}_n'$ be the centre of the locally closed Θ-stratum of $\mathcal{C}_n$ containing the lift of x_n' to $\mathcal{C}_n$.

Claim **4.3.11.** There is a natural cartesian square

$$\begin{array}{ccc} \mathcal{Z}_n' & \longrightarrow & \mathcal{V}_n \\ \downarrow & \ulcorner & \downarrow f_n \\ \mathcal{Z}_n & \longrightarrow & \mathcal{X}_n. \end{array}$$

We denote $\beta_{n+1} = \lambda$ for simplicity. As mentioned above, the centre $\mathcal{Z}_n$ is the semistable locus inside $\mathscr{B}_n^\lambda := B_n^{\lambda,0}/L_{G_n}(\lambda)$ for the shifted linear form

$$\langle -, \mathscr{L}\rangle - \frac{1}{\|\lambda\|^2}(\lambda, -),$$

where here $\mathscr{L} := \mathscr{L}_{\mathscr{B}_n}|_{\mathscr{B}_n^\lambda}$ is the standard relatively ample line bundle on $\mathscr{B}_n$, pulled back to $\mathscr{B}_n^\lambda$. Similarly, $\mathcal{Z}'_n$ is the semistable locus inside $\mathscr{C}_n^\lambda := C_n^{\lambda,0}/L_{G_n}(\lambda)$ for the form

$$\langle -, \mathscr{L}|_{\mathscr{C}_n^\lambda}\rangle - \frac{1}{\|\lambda\|^2}(\lambda, -),$$

since $\mathscr{L}|_{\mathscr{C}_n^\lambda}$ is the standard relatively ample line bundle on $\mathscr{C}_n$. First, note that the natural square

$$\begin{array}{ccc} \mathscr{C}_n^\lambda & \longrightarrow & \mathscr{C}_n \\ \big\downarrow & \ulcorner & \big\downarrow \\ \mathscr{B}_n^\lambda & \longrightarrow & \mathscr{B}_n \end{array}$$

is cartesian. This follows from cartesianity of the two squares

$$\begin{array}{ccc} \mathscr{C}_n^\lambda & \longrightarrow & \mathscr{C}_n \\ \big\downarrow & \ulcorner & \big\downarrow \\ \overline{\mathscr{C}}_n^\lambda & \longrightarrow & \overline{\mathscr{C}}_n \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathscr{B}_n^\lambda & \longrightarrow & \mathscr{B}_n \\ \big\downarrow & \ulcorner & \big\downarrow \\ \overline{\mathscr{B}}_n^\lambda & \longrightarrow & \overline{\mathscr{B}}_n, \end{array} \tag{4.1}$$

where $\overline{\mathscr{C}}_n = C_n/(G_n/D_n)$, $\overline{\mathscr{C}}_n^\lambda = C_n^{\lambda,0}/L_{G_n/D_n}(\lambda)$, $\overline{\mathscr{B}}_n = B_n/(G_n/T'_n)$ and $\overline{\mathscr{B}}_n^\lambda = B_n^{\lambda,0}/L_{G_n/T'_n}(\lambda)$, together with the isomorphisms $\overline{\mathscr{C}}_n \cong \overline{\mathscr{B}}_n$ and $\overline{\mathscr{C}}_n^\lambda \cong \overline{\mathscr{B}}_n^\lambda$. From the compatibility of the standard relatively ample line bundles on $\mathscr{C}_n$, $\mathscr{B}_n$ and $\overline{\mathscr{C}}_n = \overline{\mathscr{B}}_n$ under pullbacks and from the shape of the shifted linear forms, it follows that $(\overline{\mathscr{C}}_n^\lambda)^{\mathrm{ss}} \times_{\overline{\mathscr{C}}_n} \mathscr{C}_n = (\mathscr{C}_n^\lambda)^{\mathrm{ss}}$ and $\overline{\mathscr{B}}_n^\lambda \times_{\overline{\mathscr{B}}_n} \mathscr{B}_n = (\mathscr{B}_n^\lambda)^{\mathrm{ss}}$, and therefore $\mathcal{Z}'_n = (\mathscr{C}_n^\lambda)^{\mathrm{ss}} = (\mathscr{B}_n^\lambda)^{\mathrm{ss}} \times_{\mathscr{B}_n} \mathscr{C}_n = \mathcal{Z}_n \times_{\mathcal{X}_n} \mathcal{V}_n$, as desired. This proves the claim.

We now show by induction that $(\mathcal{V}_n, x'_n) \cong (\mathcal{Y}_n, y_n)$. For $n = 0$ there is nothing to prove. If $n < N$ and $(\mathcal{V}_n, x'_n) \cong (\mathcal{Y}_n, y_n)$, then $\mathcal{Y}_{n+1}$ is constructed as follows. We take the standard relatively ample line bundle $\mathscr{L}_{\mathscr{C}_n}$ on $\mathscr{C}_n$ and let $\mathcal{M} = \mathbb{A}(\mathscr{L}^\vee_{\mathscr{C}_n}|_{\mathcal{Z}'_n})^*$. We choose a point y_{n+1} in $\mathcal{M}$ mapping to $z := \lim_{t\to 0}\lambda(t)x'_n \in \mathcal{Z}'_n$, and we let $(\mathcal{Y}_{n+1}, y_{n+1})$ be the fibre of the good moduli space of $\mathcal{N}$ containing y_{n+1}. Note that by the previous claim,

$$\mathcal{Z}'_n = \mathbb{A}(\mathscr{L}^\vee_{n,1}|_{\mathcal{Z}_n})^* \times_{\mathcal{Z}_n} \cdots \times_{\mathcal{Z}_n} \mathbb{A}(\mathscr{L}^\vee_{n,n}|_{\mathcal{Z}_n})^*,$$

so actually $\mathcal{M} = \mathbb{A}(\mathscr{L}^\vee_{n,1}|_{\mathcal{Z}_n})^* \times_{\mathcal{Z}_n} \cdots \times_{\mathcal{Z}_n} \mathbb{A}(\mathscr{L}^\vee_{n,n}|_{\mathcal{Z}_n})^* \times_{\mathcal{Z}_n} \mathbb{A}(\mathscr{L}^\vee_{\mathscr{B}_n}|_{\mathcal{Z}_n})^*$. The stack $\mathcal{X}_{n+1}$ is the fibre of the good moduli space of $\mathcal{Z}_n$ containing $\lim_{t\to 0}\lambda(t)x_n$, and from the definitions we have $\mathcal{V}_{n+1} = \mathcal{X}_{n+1} \times_{\mathcal{Z}_n} \mathcal{M}$. Since we have seen that $\mathcal{V}_{n+1}$ has for moduli space $\operatorname{Spec} k$, it must be the fibre of the good moduli space of $\mathcal{M}$. Therefore $\mathcal{V}_{n+1} \cong \mathcal{Y}_{n+1}$. By either choosing the lifts x'_{n+1} appropriately or applying a torsor automorphism of $\mathcal{Y}_n$, we can arrange that $y_{n+1} = x'_{n+1}$.

In particular, from these isomorphisms it follows that y_N is closed in $\mathcal{Y}_N$, so the torsor chain is bounded. We have seen that the balanced filtration of $\mathcal{V}_n$ maps to the balanced filtration of $\mathcal{X}_n$. Therefore the maps $\mathcal{Y}_n \to \mathcal{X}_n$ give a morphism of chains (the compatibility of link morphisms $\mathcal{Y}_{n+1} \to \operatorname{Grad}_{\mathbb{Q}}(\mathcal{Y}_n)$ and $\mathcal{X}_{n+1} \to \operatorname{Grad}_{\mathbb{Q}}(\mathcal{X}_n)$ naturally follows from the construction). Since $\mathcal{Y}_0 = \mathcal{X}_0$, the sequential filtration of the torsor chain equals the sequential filtration of the balancing chain, as we wanted to show. □

CHAPTER 5

THE ITERATED BALANCED FILTRATION AND CONVEX GEOMETRY

Despite its seemingly convoluted definition in terms of repeated blow-ups and Θ-stratifications, we illustrate in this chapter how the iterated balanced filtration can be explicitly computed in terms of convex geometry and convex optimisation for stacks of the form $V/\mathbb{G}_{m,k}^n$ where V is a vector space. Even if we are only interested in actions by a torus $\mathbb{G}_{m,k}^n$, we will need to deal more generally with actions by diagonalisable groups, that is, groups of the form $\mathbb{G}_{m,k}^{n_0} \times \mu_{n_1,k} \times \cdots \mu_{n_l,k}$, where $\mu_{n_i,k}$ is denotes the group of n_ith roots of unity over k. The reason is that these groups will naturally appear as stabilisers. We recall some facts about diagonalisable groups at the beginning of Section 5.2.

Suppose one is interested in the iterated balanced filtration of a geometric point $x\colon \operatorname{Spec} k \to \mathcal{X}$ in a normed good moduli stack $\mathcal{X}$. By taking the fibre of the good moduli space at x, we may assume that $\mathcal{X} \to \operatorname{Spec} k$ is the good moduli space. If y is the unique closed k-point of $\mathcal{X}$, and G is the stabiliser of y, then $\mathcal{X}$ is of the form $\mathcal{X} = \operatorname{Spec} A/G$ by Corollary 2.1.5. Since closed immersions have no effect in the iterated balanced filtration, we may assume after embedding $\operatorname{Spec} A$ in a representation of G that $\operatorname{Spec} A = V$ is a finite dimensional vector space on which G acts linearly, and the point x is given by some vector $x \in V$. From now, we assume that G is diagonalisable (for example, $G = \mathbb{G}_{m,k}^n$ is a split torus). Then the structure of G-representation V is determined by a direct sum decomposition $V = \bigoplus_{\chi \in \Gamma_{\mathbb{Z}}(G)} V_\chi$, where G acts on V_χ by the character χ. The *state* of x (named after [53]) is the finite set $\Xi = \{\chi \in \Gamma_{\mathbb{Z}}(G) \mid p_\chi(x) \neq 0\}$, where $p_\chi\colon V \to V_\chi$ denotes the projection. Writing $V = V' \oplus V_0$ and considering the closed immersion $V'/G \cong (V' \times \{p_0(x)\})/G \to V/G$, we may assume that $0 \notin \Xi$. We may simplify the situation further by considering $\mathbb{A}_k^\Xi$ to be the product of $\#(\Xi)$ many copies of $\mathbb{A}_k^1$, endowed with the action of G via the characters in Ξ. There is a G-equivariant linear closed immersion $\mathbb{A}^{\Xi} \to V$ sending the point $(1, \dots, 1)$ to x, so we may replace V/G and x with $\mathbb{A}^{\Xi}/G$ and $(1, \dots, 1)$. This is the pointed stack *associated* to the state Ξ (Definition 5.2.1).

We will be able to determine the balanced filtration of x in terms of its state Ξ. To this aim, we develop a theory of polarised states (where, in addition to the set Ξ, we have the data of a character $\alpha\colon G \to \mathbb{G}_{m,k}$) purely in combinatorial terms. We introduce a notion of filtration (Definition 5.1.3) and $\mathbb{Q}^{\infty}$-filtration (Definition 5.1.5) for states, and an analogue of the balancing chain and of the iterated balanced filtration (Definition 5.1.19). We then define a functor (Definition 5.2.1) that associates a pointed stack to every state, and we prove that the iterated balanced filtration of the pointed stack coincides with that of the state (Corollary 5.2.18). Computing the iterated balanced filtration of a state boils down to a simple convex optimisation problem.

5.1 POLARISED STATES

We develop here a combinatorial analogue of the theory of iterated balanced filtrations. For the rest of this section, we fix a subring A of $\mathbb{R}$ with field of fractions K, and we assume that A is a principal ideal domain. We endow A and K with the order induced from $\mathbb{R}$. The two most important cases for us are $A = \mathbb{Z}$ and $K = \mathbb{Q}$, and $A = K = \mathbb{R}$. The first is important for the comparison between stacks and states, while the second allows to formulate a more general conjecture on asymptotics of gradient flows (Conjecture 5.3.5)

Definition 5.1.1 (Polarised state). A *polarised state* $\boldsymbol{\Xi}$ over A is a triple $\boldsymbol{\Xi} = (M, \Xi, \alpha)$ where

1. M is a finite type A-module,
2. $\Xi \subset M \setminus \{0\}$ is a finite subset (the *state*), and
3. $\alpha \in M_K$ (the *polarisation*).

We denote $M_K = M \otimes_A K$, $M^{\vee} = \mathrm{Hom}(M, A)$ and $M_K^{\vee} = M^{\vee} \otimes_A K$. A *normed polarised state* is a polarised state $\boldsymbol{\Xi} = (M, \Xi, \alpha)$ together with a K-rational inner product on $M_K^{\vee}$.

For the rest of this section, all polarised states considered will be defined over A. Let $\boldsymbol{\Xi} = (M, \Xi, \alpha)$ be a polarised state. We denote $\langle -, - \rangle\colon M^{\vee} \times M \to \mathbb{Z}$ the duality pairing. For $\lambda \in M_K^{\vee}$, we let $H_\lambda = \{\beta \in M_K \mid \langle \lambda, \beta \rangle \geq 0\}$ be the half-space defined by λ and $\partial H_\lambda = \{\beta \in M_K \mid \langle \lambda, \beta \rangle = 0\}$ the hyperplane defined by λ. We will need a few basic notions about convex geometry for which a sufficient source is [30]. A *cone* in M_K is a subset of the form $H_{\lambda_1} \cap \cdots \cap H_{\lambda_n}$ for $\lambda_1, \ldots, \lambda_n \in M_K^\vee$ or, equivalently, of the form $(K_{\geq 0})\chi_1 + \cdots + (K_{\geq 0})\chi_k$ for $\chi_1, \ldots, \chi_k \in M_K$. We include the degenerate cases $\{0\}$ and M_K. If C is a cone, a *face* of C is a subset of the form $C \cap \partial H_\lambda$, where $\lambda \in M_K^\vee$ is such that $C \subset H_\lambda$.

Definition 5.1.2 (Semistable and polystable polarised states)**.** We say that the polarised state $\boldsymbol{\Xi}$ is *semistable* if α is in cone (Ξ), the convex cone generated by Ξ inside M_K. We say that $\boldsymbol{\Xi}$ is *polystable* if it is semistable and the smallest face of cone (Ξ) containing α is cone (Ξ) itself (that is, α is in the relative interior of cone (Ξ)).

We are abusing notation by denoting the image of Ξ inside M_K also by Ξ. By cone (Ξ) we will always mean a subset of M_K.

Definition 5.1.3 (Filtrations of a state)**.** Suppose that $\boldsymbol{\Xi}$ is semistable. The set of *K-rational filtrations* (or *K-filtrations*) of $\boldsymbol{\Xi}$ is

$$K\text{-}\mathrm{Filt}(\boldsymbol{\Xi}) := \{\lambda \in M_K^\vee \mid \langle \lambda, \Xi\rangle \geq 0 \text{ and } \langle \lambda, \alpha\rangle = 0\}.$$

We are using the notation $\langle \lambda, \Xi\rangle \geq 0$ to mean that $\langle \lambda, \chi\rangle \geq 0$ holds for all $\chi \in \Xi$.

Definition 5.1.4 (Associated graded state)**.** Suppose that $\boldsymbol{\Xi}$ is semistable and let $\lambda \in K\text{-}\mathrm{Filt}(\boldsymbol{\Xi})$. The *associated graded state* $\mathrm{Grad}_\lambda(\boldsymbol{\Xi})$ is the semistable polarised state $\mathrm{Grad}_\lambda(\boldsymbol{\Xi}) = (M, \Xi_{\lambda,0}, \alpha)$, where $\Xi_{\lambda,0} := \Xi \cap \partial H_\lambda$.

Proof that $\mathrm{Grad}_\lambda(\boldsymbol{\Xi})$ *is semistable.* Since $\Xi \subset H_\lambda$, we have the equality $\mathrm{cone}(\Xi) \cap \partial H_\lambda = \mathrm{cone}(\Xi \cap \partial H_\lambda)$. Therefore $\alpha \in \mathrm{cone}(\Xi_{\lambda,0})$. □

Definition 5.1.5 (Sequential filtrations of a state)**.** Let $\boldsymbol{\Xi} = (M, \Xi, \alpha)$ be a semistable polarised state. The set $K^\infty\text{-}\mathrm{Filt}(\boldsymbol{\Xi})$ of *sequential filtrations* (or *K^∞-filtrations*) of $\boldsymbol{\Xi}$ is the set of those sequences $(\lambda_n)_{n\in\mathbb{N}}$ in $M_K^\vee$ satisfying

1. $\lambda_n = 0$ for $n \gg 0$, and
2. $\lambda_0 \in K\text{-}\mathrm{Filt}(\boldsymbol{\Xi})$ and, for all $n \in \mathbb{N}_{>0}$, we have

$$\lambda_n \in K\text{-}\mathrm{Filt}\left(\mathrm{Grad}_{\lambda_{n-1}}\left(\cdots \mathrm{Grad}_{\lambda_1}\left(\mathrm{Grad}_{\lambda_0}(\boldsymbol{\Xi})\right)\cdots\right)\right).$$

Definition 5.1.6 (Morphism between states)**.** A *morphism* $\varphi\colon \boldsymbol{\Xi}_1 = (M_1, \Xi_1, \alpha_1) \to \boldsymbol{\Xi}_2 = (M_2, \Xi_2, \alpha_2)$ between semistable polarised states is a surjective homomorphism $\varphi\colon M_2 \to M_1$ such that

1. for all $\chi \in \Xi_2$, either $\varphi(\chi) \in \Xi_1$ or $\varphi(\chi) = 0$; and
2. $\alpha_2 \in \mathrm{cone}\,(\Xi_2 \cap \ker \varphi)$.

If Ξ_1 and Ξ_2 are normed, we say that φ is a morphism between normed semistable polarised states if the inner product on $(M_1)_K^\vee$ is the restriction of that on $(M_2)_K^\vee$ along the inclusion $(M_1)_K^\vee \hookrightarrow (M_2)_K^\vee$ defined by φ.

With the obvious composition and identity, semistable (normed) polarised states form a category.

Lemma 5.1.7. *Let* $\varphi\colon \Xi_1 = (M_1, \Xi_1, \alpha_1) \to \Xi_2 = (M_2, \Xi_2, \alpha_2)$ *be a morphism between semistable polarised states. Then the injection* $\varphi_K^\vee\colon (M_1)_K^\vee \to (M_2)_K^\vee$ *induces a map* $K\text{-}\mathrm{Filt}(\Xi_1) \to K\text{-}\mathrm{Filt}(\Xi_2)$ *between sets of* K*-rational filtrations.*

Proof. If $\lambda \in K\text{-}\mathrm{Filt}(\Xi_1)$ and $\chi \in \Xi_2$, then $\langle \varphi_K^\vee(\lambda), \chi\rangle = \langle \lambda, \varphi(\chi)\rangle \geq 0$, since either $\varphi(\chi) = 0$ or $\varphi(\chi) \in \Xi_1$. Likewise, $\langle \varphi_K^\vee(\lambda), \alpha_2\rangle = \langle \lambda, \varphi(\alpha_2)\rangle = 0$, since $\varphi(\alpha_2) = 0$ in $(M_1)_K$. Therefore $\varphi_K^\vee(\lambda) \in K\text{-}\mathrm{Filt}(\Xi_2)$. □

In the situation of Lemma 5.1.7, we will use the simpler notation $\varphi(\lambda) := \varphi_K^\vee(\lambda)$.

Proposition 5.1.8. *Let* $\varphi\colon \Xi_1 = (M_1, \Xi_1, \alpha_1) \to \Xi_2 = (M_2, \Xi_2, \alpha_2)$ *be a morphism between semistable polarised states and let* $\lambda \in K\text{-}\mathrm{Filt}(\Xi_1)$*. The homomorphism* $\varphi\colon M_2 \to M_1$ *induces a map* $\mathrm{Grad}_\lambda(\varphi)\colon \mathrm{Grad}_\lambda(\Xi_1) \to \mathrm{Grad}_{\varphi(\lambda)}(\Xi_2)$ *between associated graded states.*

Proof. The first condition to be checked is that for all $\chi \in (\Xi_2)_{\lambda,0}$, we have $\varphi(\chi) = 0$ or $\varphi(\chi) \in (\Xi_1)_{\lambda,0}$. If $\varphi(\chi)$ is not 0, then $\varphi(\chi) \in \Xi_1$, and since $\langle \lambda, \varphi(\chi)\rangle = \langle \varphi(\lambda), \chi\rangle = 0$, we have indeed $\varphi(\chi) \in (\Xi_1)_{\lambda,0}$. For the second condition, we have

$$\begin{aligned}\alpha_2 \in \operatorname{cone}(\Xi_2 \cap \ker\varphi) \cap \partial H_{\varphi(\lambda)} &= \operatorname{cone}\left(\Xi_2 \cap \ker\varphi \cap \partial H_{\varphi(\lambda)}\right) \\ &= \operatorname{cone}\left((\Xi_2)_{\varphi(\lambda),0} \cap \ker\varphi\right),\end{aligned}$$

as desired. □

Definition 5.1.9 (Chain of states)**.** A *chain of normed semistable polarised states* is a sequence $(\Xi_n, \lambda_n, u_n)_{n\in\mathbb{N}}$ where

1. for each $n \in \mathbb{N}$, Ξ_n is a normed semistable polarised state;
2. $\lambda_n \in K\text{-}\mathrm{Filt}(\Xi_n)$ for each $n \in \mathbb{N}$;
3. $u_n\colon \Xi_{n+1} \to \mathrm{Grad}_{\lambda_n}(\Xi_n)$ is a morphism of normed semistable polarised states; and
4. for all $n \gg 0$, $\lambda_n = 0$ and u_n is an isomorphism (boundedness).

Suppose that $(\Xi_n, \lambda_n, u_n)_{n\in\mathbb{N}}$ is a chain of normed semistable polarised states. We denote $c_n\colon \Xi_{n+1} \to \mathrm{Grad}_{\lambda_n}(\cdots \mathrm{Grad}_{\lambda_1}(\mathrm{Grad}_{\lambda_0}(\Xi_0))\cdots)$ the map defined by

$$c_n = u_n \circ \mathrm{Grad}_{\lambda_n}(u_{n-1}) \circ \cdots \circ \mathrm{Grad}_{\lambda_n}(\cdots \mathrm{Grad}_{\lambda_1}(\mathrm{Grad}_{\lambda_0}(u_0))\cdots).$$

We are abusively denoting by λ_n the image of λ_n along the relevant maps between sets of filtrations, but note as well that all these maps are injective. The sequence $(c_{n-1}(\lambda_n))_{n\in\mathbb{N}}$ (where $c_{-1}(\lambda_0) := \lambda_0$) is a K^∞-filtration of $\boldsymbol{\Xi}_0$.

Definition 5.1.10 (Associated sequential filtration)**.** The *sequential filtration associated to the chain* $(\boldsymbol{\Xi}_n, \lambda_n, u_n)_{n\in\mathbb{N}}$ is $(c_{n-1}(\lambda_n))_{n\in\mathbb{N}} \in K^\infty\text{-}\operatorname{Filt}(\boldsymbol{\Xi}_0)$. We will simply denote it $(\lambda_n)_{n\in\mathbb{N}}$.

Definition 5.1.11 (Slice of a state)**.** Let $\boldsymbol{\Xi} = (M, \Xi, \alpha)$ be a semistable polarised state. Let F be the smallest face of $\operatorname{cone}(\Xi)$ containing α, let R be the submodule of M generated by $F \cap \Xi$, let $M' = M/R$, let $q\colon M \to M'$ be the quotient map, and set $\Xi' = q(\Xi \setminus F)$. We define the *slice* of $\boldsymbol{\Xi}$ to be the state $\boldsymbol{\Xi}' = (M', \Xi', 0)$ together with the morphism $q\colon \boldsymbol{\Xi}' \to \boldsymbol{\Xi}$ given by $q\colon M \to M'$. If $\boldsymbol{\Xi}$ is normed, we regard $\boldsymbol{\Xi}'$ as a normed polarised state, where $(M')^\vee_K$ inherits an inner product along the inclusion $q^\vee_K\colon (M')^\vee_K \to M^\vee_K$.

Note that $\boldsymbol{\Xi}'$ is semistable, since 0 belongs to any cone. We are again abusing notation: the expression $F \cap \Xi$ denotes the subset of Ξ consisting of the elements χ whose image in M_K is contained in F.

Proposition 5.1.12. *Let $\boldsymbol{\Xi} = (M, \Xi, \alpha)$ be a semistable polarised state and let $q\colon \boldsymbol{\Xi}' \to \boldsymbol{\Xi}$ be its slice. Then the map $K\text{-}\operatorname{Filt}(\boldsymbol{\Xi}') \to K\text{-}\operatorname{Filt}(\boldsymbol{\Xi})$ induced by q is a bijection.*

Proof. We use notation from Definition 5.1.11. Let $\lambda \in K\text{-}\operatorname{Filt}(\boldsymbol{\Xi})$. Then $\operatorname{cone}(\Xi) \cap \partial H_\lambda$ is a face of $\operatorname{cone}(\Xi)$ containing α, so $F \subset \operatorname{cone}(\Xi) \cap \partial H_\lambda$. In particular, $\langle \lambda, \chi \rangle = 0$ for all $\chi \in F \cap \Xi$. Therefore λ has a preimage λ' along $q^\vee_K\colon (M')^\vee_K \hookrightarrow M^\vee_K$, and $\langle \lambda', q(\chi) \rangle = \langle \lambda, \chi \rangle \geq 0$ for all $\chi \in \Xi \setminus F$, so $\lambda' \in K\text{-}\operatorname{Filt}(\boldsymbol{\Xi}')$. Hence the map $K\text{-}\operatorname{Filt}(\boldsymbol{\Xi}') \to K\text{-}\operatorname{Filt}(\boldsymbol{\Xi})$ is surjective. Since it is also injective, it is a bijection. □

Definition 5.1.13 (Complementedness of a filtration)**.** Let $\boldsymbol{\Xi} = (M, \Xi, \alpha)$ be a semistable polarised state and let $\boldsymbol{\Xi}' = (M', \Xi', 0)$ be its slice. Let $\lambda \in K\text{-}\operatorname{Filt}(\Xi) = K\text{-}\operatorname{Filt}(\Xi')$. We define the *complementedness* $\langle \lambda, \mathfrak{l} \rangle \in K_{\geq 0} \cup \{\infty\}$ of λ to be

$$\langle \lambda, \mathfrak{l} \rangle = \inf_{\chi \in \Xi'} \langle \lambda, \chi \rangle.$$

Proposition 5.1.14. *Let $\boldsymbol{\Xi} = (M, \Xi, \alpha)$ be a semistable normed polarised state. There is a unique element $\lambda \in K\text{-}\operatorname{Filt}(\boldsymbol{\Xi})$ such that*

1. $\langle \lambda, \mathfrak{l} \rangle \geq 1$, *and*
2. *for all $\gamma \in K\text{-}\operatorname{Filt}(\boldsymbol{\Xi})$ such that $\langle \gamma, \mathfrak{l} \rangle \geq 1$, we have $\|\lambda\| \leq \|\gamma\|$.*

Proof. We may assume $\boldsymbol{\Xi} = \boldsymbol{\Xi}'$. If $\Xi = \varnothing$, then $\lambda = 0$ is the unique K-rational filtration satisfying the conditions. Otherwise, $P = \{\gamma \in M_{\mathbb{R}}^{\vee} \mid \langle \gamma, \chi \rangle \geq 1,\ \forall \chi \in \Xi\}$ is a nonempty closed convex set (actually, an intersection of translated half-spaces) inside $M_{\mathbb{R}}^{\vee} = M_K^{\vee} \otimes_K \mathbb{R}$, so there is a unique element $\lambda \in P$ minimising the norm $\|-\|$. To see that $\lambda \in M_K^{\vee}$, note that λ lies in the relative interior of a face F of P. The affine space F generates is of the form $V_{\mathbb{R}} + v$, where V is a vector subspace of $M_K^{\vee}$, $V_{\mathbb{R}} = V \otimes_K \mathbb{R}$ and $v \in M_K^{\vee}$. Since λ is also the closest point to the origin in $V_{\mathbb{R}} + v$, we must have that $\lambda = v - p(v)$, where p is the orthogonal projection $p\colon M_{\mathbb{R}}^{\vee} \to V_{\mathbb{R}}$. Since the inner product on $M_K^{\vee}$ is K-rational, p is defined over K, and thus λ is K-rational. □

Definition 5.1.15 (Balanced filtration of a state)**.** Let $\boldsymbol{\Xi}$ be a semistable normed polarised state. The *balanced filtration* $\lambda_{\mathrm{b}}(\boldsymbol{\Xi})$ of $\boldsymbol{\Xi}$ is the unique $\lambda \in K\text{-}\mathrm{Filt}(\boldsymbol{\Xi})$ satisfying the conditions of Proposition 5.1.14.

Remark **5.1.16.** For a normed semistable polarised state $\boldsymbol{\Xi} = (M, \Xi, \alpha)$, we have $\Xi' = \varnothing$ if and only if $\boldsymbol{\Xi}$ is polystable, if and only if the balanced filtration $\lambda_{\mathrm{b}}(\boldsymbol{\Xi}) = 0$.

Definition 5.1.17. Let $\boldsymbol{\Xi}$ be a normed semistable polarised state, and let $\lambda \in K\text{-}\mathrm{Filt}(\boldsymbol{\Xi})$ be such that $\langle \lambda, \mathfrak{l} \rangle \geq 1$. We define a normed polarised state $\Lambda_\lambda(\boldsymbol{\Xi})$ as follows. Let $\boldsymbol{\Xi}' = (M', \Xi', 0)$ be the slice of $\boldsymbol{\Xi}$, let

$$(\Xi')_{\lambda,1} = \{\chi \in M' \mid \langle \lambda, \chi \rangle = 1\} \subset M',$$

and let $\lambda^{\vee}$ be the unique element of $(M')_K$ satisfying $\langle \gamma, \lambda^{\vee} \rangle = (\gamma, \lambda)$ for all $\gamma \in (M')_K^{\vee}$, where $(-,-)$ denotes the inner product on $(M')_K^{\vee}$. Finally, we set

$$\Lambda_\lambda(\boldsymbol{\Xi}) = (M', (\Xi')_{\lambda,1}, \lambda^{\vee}).$$

The following theorem is an analogue for states of Theorem 2.6.9 in the case of algebraic stacks and [33, Theorem 4.9] in the case of artinian lattices.

Theorem 5.1.18 (Recognition of the balanced filtration for states)**.** *Let $\boldsymbol{\Xi}$ be a normed semistable polarised state. Then the balanced filtration of $\boldsymbol{\Xi}$ is the unique filtration $\lambda \in K\text{-}\mathrm{Filt}(\boldsymbol{\Xi})$ satisfying*

1. $\langle \lambda, \mathfrak{l} \rangle \geq 1$ *and*
2. *the polarised state $\Lambda_\lambda(\boldsymbol{\Xi})$ is semistable.*

Proof. We use the notation of the proof of Proposition 5.1.14. By definition, the balanced filtration λ is the unique element in P minimising the function $1/2\|-\|^2$, whose differential at a point $\gamma \in (M')^\vee_{\mathbb{R}}$ is precisely $\gamma^\vee \in (M')_{\mathbb{R}}$. By the Karush–Kuhn–Tucker conditions[1] [59], for $\gamma \in P$ we have $\gamma = \lambda$ if and only if there are numbers $u_\chi \geq 0$ for each $\chi \in \Xi'$ such that $\gamma^\vee = \sum_{\chi \in \Xi'} u_\chi \chi$ and $u_w = 0$ if $\langle \gamma, \chi \rangle > 1$. But this precisely means that $\gamma^\vee \in \operatorname{cone}\big((\Xi')_{\gamma,1}\big)$, i.e. that $\Lambda_\gamma(\boldsymbol{\Xi})$ is semistable. □

If λ is the balanced filtration of $\boldsymbol{\Xi}$, then $\mathrm{id}_{M'}$ defines a morphism $\Lambda_\lambda(\boldsymbol{\Xi}) \to \mathrm{Grad}_\lambda(\boldsymbol{\Xi}')$ (note that $(\Xi')_{\lambda,0} = \varnothing$), and the quotient map $M \to M'$ defines a morphism $\mathrm{Grad}_\lambda(\boldsymbol{\Xi}') \to \mathrm{Grad}_\lambda(\boldsymbol{\Xi})$. Therefore $\Lambda_\lambda(\boldsymbol{\Xi})$ is equipped with a canonical map $\Lambda_\lambda(\boldsymbol{\Xi}) \to \mathrm{Grad}_\lambda(\boldsymbol{\Xi})$.

Definition 5.1.19 (Balancing chain of a state and the iterated balanced filtration)**.** Let $\boldsymbol{\Xi}$ be a normed semistable polarised state. The *balancing chain* of $\boldsymbol{\Xi}$ is the chain $(\boldsymbol{\Xi}_n, \lambda_n, u_n)_{n \in \mathbb{N}}$ of normed semistable polarised states defined inductively as follows:

1. $\boldsymbol{\Xi}_0 := \boldsymbol{\Xi}'$;
2. for every $n \in \mathbb{N}$, λ_n is the balanced filtration of $\boldsymbol{\Xi}_n$, $\boldsymbol{\Xi}_{n+1} := (\Lambda_{\lambda_n}(\boldsymbol{\Xi}_n))'$, and $u_n \colon \boldsymbol{\Xi}_{n+1} \to \mathrm{Grad}_{\lambda_n}(\boldsymbol{\Xi}_n)$ is the composition of $(\Lambda_{\lambda_n}(\boldsymbol{\Xi}_n))' \to \Lambda_{\lambda_n}(\boldsymbol{\Xi}_n)$ and the canonical map $\Lambda_{\lambda_n}(\boldsymbol{\Xi}_n) \to \mathrm{Grad}_{\lambda_n}(\boldsymbol{\Xi}_n)$.

The *iterated balanced filtration* of $\boldsymbol{\Xi}$ is the sequential filtration $\lambda_{\mathrm{ib}}(\boldsymbol{\Xi}) \in K^\infty\text{-}\mathrm{Filt}(\boldsymbol{\Xi})$ associated to the balancing chain of $\boldsymbol{\Xi}$.

Proof that the balancing chain is well-defined. We need to check the boundedness condition in Definition 5.1.9. We observe that, denoting $\boldsymbol{\Xi} = (M, \Xi, \alpha)$, if $\alpha \neq 0$ and $\boldsymbol{\Xi}$ is not polystable, then $\#(\Xi') < \#(\Xi)$. Therefore, eventually $\lambda_n = 0$, and hence $\Xi'_n = \varnothing$ by Remark 5.1.16, from where the chain stabilises. □

Remark **5.1.20** (Torsion). The torsion of M does not affect the iterated balanced filtration, but we allow for any finite-type A-module M in the definition of polarised state in order to make the correspondence with stacks cleaner. Even if we are only interested in the case of a torus action, other diagonalisable groups that are not tori will show up as stabilisers of points, and their group of characters can be any finite-type abelian group.

[1] Recall that, when minimising a convex differentiable function $f \colon \mathbb{R}^n \to \mathbb{R}$ on a convex set $R \subset \mathbb{R}^n$ defined by inequalities $g_i(x) \geq 0$, the Karush–Kuhn–Tucker conditions say that for $x \in U$ to be a minimum of f, it is necessary and sufficient that the gradient $\nabla f(x)$ belongs to the span of the $\nabla g_i(x)$ with $g_i(x) = 0$.

5.2 FROM STATES TO GOOD MODULI STACKS

We now define a functor from states to pointed stacks, and prove that both theories of iterated balanced filtrations coincide. We fix a field k for the rest of this section. All polarised states considered in this section will be defined over $\mathbb{Z}$ (i.e. $A = \mathbb{Z}$).

We recall that a diagonalisable group G over k is an algebraic group of the form $G = \operatorname{Spec} k[M]$ for a finite-type abelian group M, where the multiplication is given by the map $k[M] \to k[M] \otimes_k k[M] \colon m \mapsto m \otimes m$ for $m \in M$. The group G is said to be *Cartier dual* to M, and we write $G = D(M)$. The group of characters of G is naturally identified with M, $\Gamma_{\mathbb{Z}}(G) \cong M$, and the Cartier duality functor $M \mapsto D(M)$ gives an anti-equivalence of categories between the category of finite-type abelian groups and the category of diagonalisable groups, with inverse $\Gamma_{\mathbb{Z}}(-)$. We have $D(\mathbb{Z}) = \mathbb{G}_{m,k}$, $D(\mathbb{Z}/nZ) = \mu_{n,k}$ (the group of nth roots of unity) and $D(M_1 \times M_2) = D(M_1) \times D(M_2)$. Hence, by the classification of finite-type abelian groups, any diagonalisable group is isomorphic to $\mathbb{G}_{m,k}^{n_0} \times \mu_{n_1,k} \times \cdots \times \mu_{n_l,k}$ for some $n_0, l \in \mathbb{N}$ and $n_1, \ldots, n_l \in \mathbb{Z}_{>1}$. For detailed account of diagonalisable algebraic groups, we refer the reader to [64, §12.c,d].

DEFINITION 5.2.1 (Stack associated to a state)**.** Let $\boldsymbol{\Xi} = (M, \Xi, \alpha)$ be a (normed) polarised state. We define a k-pointed (normed) good moduli stack $(\mathcal{X}_{\boldsymbol{\Xi}}, x_{\boldsymbol{\Xi}})$ over k and a line bundle $\mathcal{L}_{\boldsymbol{\Xi}}$ on $\mathcal{X}_{\boldsymbol{\Xi}}$ as follows.

First, we denote $G_{\boldsymbol{\Xi}} = D(M)$ the diagonalisable algebraic group over k Cartier dual to M. The group of characters $\Gamma_{\mathbb{Z}}(G_{\boldsymbol{\Xi}})$ of $G_{\boldsymbol{\Xi}}$ is identified with M. Let $\mathbb{A}_k^{\Xi}$ be the product of $\#(\Xi)$ many copies of $\mathbb{A}_k^1$. Consider the action of $G_{\boldsymbol{\Xi}}$ on $\mathbb{A}_k^{\Xi}$ through the characters in Ξ, that is, $g \cdot (x_\chi)_{\chi \in \Xi} = (\chi(g) x_\chi)_{\chi \in \Xi}$ for g in $G_{\boldsymbol{\Xi}}$ and $(x_\chi)_{\chi\in\Xi}$ in $\mathbb{A}_k^{\Xi}$. Let $\mathcal{X}_{\boldsymbol{\Xi}} := \mathbb{A}_k^{\Xi}/G_{\boldsymbol{\Xi}}$ and $x_{\boldsymbol{\Xi}} = (1, \cdots, 1) \in \mathbb{A}_k^{\Xi}(k)$. We also denote $x_{\boldsymbol{\Xi}}$ the composition $\operatorname{Spec} k \xrightarrow{x_{\boldsymbol{\Xi}}} \mathbb{A}_k^{\Xi} \to \mathcal{X}_{\boldsymbol{\Xi}}$. We identify $M_{\mathbb{Q}} = \operatorname{Pic}(BG_{\boldsymbol{\Xi}}) \otimes_{\mathbb{Z}} \mathbb{Q}$, and thus the polarisation α defines a rational line bundle $\mathcal{L}_{\boldsymbol{\Xi}} := \mathcal{O}_{\mathcal{X}_{\boldsymbol{\Xi}}}(\alpha) = (\mathcal{X}_{\boldsymbol{\Xi}} \to BG_{\boldsymbol{\Xi}})^*\alpha$ on $\mathcal{X}_{\boldsymbol{\Xi}}$. We denote $\ell_{\boldsymbol{\Xi}}$ the linear form on $\mathcal{X}_{\boldsymbol{\Xi}}$ associated to $\mathcal{L}_{\boldsymbol{\Xi}}$. If $\boldsymbol{\Xi}$ is normed, the data of the inner product on $M_{\mathbb{Q}}^\vee$ is equivalent to that of a norm on cocharacters of $G_{\boldsymbol{\Xi}}$. It thus defines a norm on graded points of $BG_{\boldsymbol{\Xi}}$ and also a norm on $\mathcal{X}_{\boldsymbol{\Xi}}$ by pullback along $\mathcal{X}_{\boldsymbol{\Xi}} \to BG_{\boldsymbol{\Xi}}$.

We fix a polarised state $\boldsymbol{\Xi} = (M, \Xi, \alpha)$, and denote $\boldsymbol{\Xi}^\circ = (M, \Xi, 0)$ the associated "unpolarised" state. We abbreviate $(\mathcal{X}, x) = (\mathcal{X}_{\boldsymbol{\Xi}}, x_{\boldsymbol{\Xi}})$ and $G = G_{\boldsymbol{\Xi}}$.

PROPOSITION 5.2.2. *There is a canonical bijection*

$$\mathbb{Q}\text{-}\operatorname{Filt}(\mathcal{X}, x) = \mathbb{Q}\text{-}\operatorname{Filt}(\boldsymbol{\Xi}^\circ).$$

Proof. By Remark 2.2.19, the set $\mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, x)$ of rational filtrations of x in $\mathcal{X}$ is identified with the set of those rational cocharacters $\lambda \in M_{\mathbb{Q}}^{\vee} = \Gamma^{\mathbb{Q}}(G)$ such that $\lim_{t\to 0} \lambda(t)x$ exists in $\mathbb{A}_k^{\Xi}$. Since $\lambda(t)x = \left(t^{\langle\lambda,\chi\rangle}\right)_{\chi\in\Xi}$, the limit exists precisely when $\langle\lambda,\chi\rangle \geq 0$ for all $\chi \in \Xi$. □

For $y = (y_\chi)_{\chi\in\Xi} \in \mathbb{A}^{\Xi}(k)$, we define the *state* of y to be the set $\Xi_y = \{\chi \in \Xi \mid y_\chi \neq 0\}$. If we let $\boldsymbol{\Xi}_y = (M, \Xi_y, \alpha)$, then we have a closed immersion $f\colon \mathcal{X}_{\boldsymbol{\Xi}_y} \to \mathcal{X}$ given by

$$f\left((z_\chi)_{\chi\in\Xi_y}\right)_\chi = \begin{cases} z_\chi y_\chi, & \chi \in \Xi_y \\ 0, & \text{else,} \end{cases}$$

and f maps $x_{\boldsymbol{\Xi}_y} = (1, \cdots, 1)$ to y.

Proposition 5.2.3. *Let $\lambda \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, x)$. Then the state of $y = \lim_{t\to 0} \lambda(t)x$ is $\Xi_y = \Xi \cap \partial H_\lambda$.*

Proof. Again, $\lambda(t)x = (t^{\langle\lambda,\chi\rangle})_{\chi\in\Xi}$ and thus

$$y_\chi = \begin{cases} 0, & \langle\lambda,\chi\rangle > 0; \\ 1, & \langle\lambda,\chi\rangle = 0; \end{cases}$$

which implies the claim. □

The state also determines the stabiliser of x.

Proposition 5.2.4. *Let K be the subgroup of M generated by the elements of Ξ, and let $C = M/K$. Let $S = D(C)$ be the Cartier dual of C, which is equipped with an injection $S \to G$. Then S is the stabiliser group of x.*

Proof. The group G acts on $\mathbb{A}^{\Xi}$ via the characters $\chi\colon G \to \mathbb{G}_{m,k}$, $\chi \in \Xi$, each of which can be seen as the Cartier dual of the map $\mathbb{Z} \to \Gamma_{\mathbb{Z}}(G)\colon 1 \mapsto \chi$. If $\Xi = \{\chi_1, \ldots, \chi_n\}$, then the stabiliser of x is the kernel of $(\chi_1, \ldots, \chi_n)\colon G \to \mathbb{G}_m^n$, and its group of characters is, by Cartier duality, the cokernel of the map $\mathbb{Z}^n \to M\colon e_i \mapsto \chi_i$, which is C. □

Proposition 5.2.5. *The point x is semistable in $\mathcal{X}$ for the linear form $\ell_{\boldsymbol{\Xi}}$ if and only if the polarised state $\boldsymbol{\Xi}$ is semistable.*

Proof. It follows from Proposition 5.2.2, together with the fact that $\langle\lambda, \mathcal{O}_{\mathcal{X}}(\alpha)\rangle = -\langle\lambda,\alpha\rangle$ with our sign conventions (Remark 2.4.4), that x is semistable if and only if for all $\lambda \in M_{\mathbb{Q}}^{\vee}$ such that $\langle\lambda, \Xi\rangle \geq 0$ (that is, $\Xi \subset H_\lambda$) we have $\langle\lambda,\alpha\rangle \geq 0$ (that is,

$\alpha \in H_\lambda$). The result now follows from this and the fact that cone (Ξ) is the intersection of those half-spaces H_λ such that $\Xi \subset H_\lambda$ (where we are again abusing notation by identifying Ξ with its image inside $M_{\mathbb{Q}}$). □

PROPOSITION 5.2.6. *Suppose that x is semistable (equivalently, Ξ is semistable) and let $\lambda \in \mathbb{Q}\text{-Filt}(\mathcal{X}, x)$. Then the limit $y = \lim_{t\to 0} \lambda(t)x$ is semistable if and only if $\langle \lambda, \alpha\rangle = 0$.*

Proof. We have equalities $\partial H_\lambda \cap \text{cone}\,(\Xi) = \text{cone}\,(\partial H_\lambda \cap \Xi) = \text{cone}\,(\Xi_y)$, the second of which follows from Proposition 5.2.3. The result follows from Proposition 5.2.5 applied to $\mathcal{X}_{\Xi_y}$, which is a closed substack of $\mathcal{X}$ containing y. □

Note that $\mathcal{X}$ admits a good moduli space, and ℓ_Ξ is trivially $\text{id}_{\mathcal{X}}$-positive (Definition 2.6.1). Therefore Theorem 2.6.4 implies that the semistable locus $\mathcal{X}^{\text{ss}} = (\mathbb{A}_k^\Xi)^{\text{ss}}/G$ with respect to ℓ_Ξ has a good moduli space $\pi\colon \mathcal{X}^{\text{ss}} \to X$.

PROPOSITION 5.2.7. *There is a canonical bijection $\mathbb{Q}\text{-Filt}(\Xi) \cong \mathbb{Q}\text{-Filt}(\mathcal{X}^{ss}, x)$.*

Proof. $\mathbb{Q}\text{-Filt}(\mathcal{X}^{\text{ss}}, x)$ is the subset of those $\lambda \in \mathbb{Q}\text{-Filt}(\mathcal{X}, x)$ such that $\lim_{t\to 0}\lambda(t)x$ is semistable, while $\mathbb{Q}\text{-Filt}(\Xi)$ is the subset of those $\lambda \in \mathbb{Q}\text{-Filt}(\Xi^\circ)$ such that $\langle\lambda, \alpha\rangle = 0$. Thus the result follows from Propositions 5.2.2 and 5.2.6. □

We can also characterise polystability in terms of the state:

PROPOSITION 5.2.8. *The point x is polystable inside $\mathcal{X}^{ss}$ if and only if the state Ξ is polystable.*

Proof. The point x being polystable is equivalent to it being semistable and, for all $\lambda \in M_{\mathbb{Q}}^\vee$ such that $\langle\lambda, \Xi\rangle \geq 0$ and $y = \lim_{t\to 0}\lambda(t)x$ is semistable, having $x = y$. By Propositions 5.2.3, 5.2.5 and 5.2.6, this condition is equivalent to having that Ξ is semistable and, for all $\lambda \in M_{\mathbb{Q}}^\vee$, having that the conditions $\Xi \subset H_\lambda$ and $\langle\lambda, \alpha\rangle = 0$ imply that $\Xi \subset \partial H_\lambda$. This means that the smallest face of cone (Ξ) containing x is cone (Ξ) itself, that is, that Ξ is polystable. □

From now, we do not fix a particular polarised state Ξ, and we stop abbreviating $(\mathcal{X}, x) = (\mathcal{X}_\Xi, x_\Xi)$ and $G = G_\Xi$.

DEFINITION 5.2.9. Let $\varphi\colon \Xi_1 = (M_1, \Xi_1, \alpha_1) \to \Xi_2 = (M_2, \Xi_2, \alpha_2)$ be a morphism between semistable polarised states. We define pointed morphisms $f_\varphi\colon (\mathcal{X}_{\Xi_1}, x_{\Xi_1}) \to (\mathcal{X}_{\Xi_2}, x_{\Xi_2})$ and $f_\varphi^{\text{ss}}\colon (\mathcal{X}_{\Xi_1}^{\text{ss}}, x_{\Xi_1}) \to (\mathcal{X}_{\Xi_2}^{\text{ss}}, x_{\Xi_2})$ as follows.

First, the homomorphism $\varphi\colon M_2 \to M_1$ defines a Cartier dual group homomorphism $D(\varphi)\colon G_{\Xi_1} \to G_{\Xi_2}$. The k-algebra homomorphism

$$k[t_\chi;\ \chi \in \Xi_2] \to k[t_\psi;\ \psi \in \Xi_1]\colon \chi \mapsto \begin{cases} t_{\varphi(\chi)}, & \varphi(\chi) \in \Xi_1, \\ 1, & \varphi(\chi) = 0. \end{cases}$$

defines, after taking Spec, a $D(\varphi)$-equivariant map

$$\mathbb{A}_k^{\Xi_1} \to \mathbb{A}_k^{\Xi_2}\colon (y_\psi)_{\psi\in\Xi_1} \mapsto \left(\begin{cases} y_{\varphi(\chi)}, & \varphi(\chi) \in \Xi_1, \\ 1, & \varphi(\chi) = 0. \end{cases}\right)_{\chi\in\Xi_2}$$

sending $(1,\dots,1)$ to $(1,\dots,1)$, and thus a pointed morphism of stacks $f_\varphi\colon (\mathcal{X}_{\Xi_1}, x_{\Xi_1}) \to (\mathcal{X}_{\Xi_2}, x_{\Xi_2})$. For all geometric points $y = (y_\psi)_{\psi\in\Xi_1} \in \mathbb{A}_k^{\Xi_1}\left(\overline{k}\right)$, the state of $f_\varphi(y)$ contains all elements of $\Xi_2 \cap \ker\varphi$ and thus $f_\varphi(y)$ is semistable. Therefore f_φ restricts to a morphism $f_\varphi^{\mathrm{ss}}\colon (\mathcal{X}_{\Xi_1}^{\mathrm{ss}}, x_{\Xi_1}) \to (\mathcal{X}_{\Xi_2}^{\mathrm{ss}}, x_{\Xi_2})$.

The assignments $\varphi \mapsto f_\varphi$ and $\varphi \mapsto f_\varphi^{\mathrm{ss}}$ respect composition. If φ is a morphism of *normed* semistable polarised states, then f_φ and f_φ^{ss} are normed morphisms of stacks. Therefore, the assignments $\Xi \mapsto (\mathcal{X}_\Xi^{\mathrm{ss}}, x_\Xi)$ and $\varphi \mapsto f_\varphi^{\mathrm{ss}}$ define a functor from the category of normed semistable polarised states to the category of k-pointed normed good moduli stacks with affine diagonal and finitely presented over k.

PROPOSITION 5.2.10. *For Ξ a semistable polarised state and $\lambda \in \mathbb{Q}\text{-}\mathrm{Filt}(\Xi)$, there is a natural pointed isomorphism $\left(\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}_\Xi^{ss})_{\mathrm{gr}\,\lambda}, \mathrm{gr}\,\lambda\right) \cong \left(\mathcal{X}_{\mathrm{Grad}_\lambda(\Xi)}^{ss}, x_{\mathrm{Grad}_\lambda(\Xi)}\right)$. Here, $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}_\Xi^{ss})_{\mathrm{gr}\,\lambda}$ is the connected component of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}_\Xi^{ss})$ containing $\mathrm{gr}\,\lambda$.*

Proof. By [36, Theorem 1.4.8], $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}_\Xi^{\mathrm{ss}})_{\mathrm{gr}\,\lambda} = \left(\mathbb{A}_k^\Xi\right)^{\lambda,0,\mathrm{ss}}/G_\Xi$, where $\left(\mathbb{A}_k^\Xi\right)^{\lambda,0}$ is the fixed point locus for the (rational) $\mathbb{G}_{m,k}$ action on $\mathbb{A}_k^\Xi$ given by λ and $\left(\mathbb{A}_k^\Xi\right)^{\lambda,0,\mathrm{ss}}$ is the semistable locus. On the other hand, looking at the weights one gets the equality $\left(\mathbb{A}_k^\Xi\right)^{\lambda,0} = \mathbb{A}_k^{\Xi_{\lambda,0}}$, and both $\mathrm{gr}\,\lambda$ and $x_{\mathrm{Grad}_\lambda(\Xi)}$ are the point $(1,\cdots,1)$, giving the desired isomorphism of pointed stacks. □

DEFINITION 5.2.11. Let $(\Xi_n, \lambda_n, u_n)_{n\in\mathbb{N}}$ be a chain of normed semistable polarised states. Applying the functor $\Xi \mapsto (\mathcal{X}_\Xi^{\mathrm{ss}}, x_\Xi)$ gives an *associated chain of k-stacks* $(\mathcal{X}_{\Xi_n}^{\mathrm{ss}}, x_{\Xi_n}, \lambda_n, h_n)$, where each h_n is the composition of

$$f_{u_n}^{\mathrm{ss}}\colon \left(\mathcal{X}_{\Xi_{n+1}}^{\mathrm{ss}}, x_{\Xi_{n+1}}\right) \to \left(\mathcal{X}_{\mathrm{Grad}_\lambda(\Xi_n)}^{\mathrm{ss}}, x_{\mathrm{Grad}_\lambda(\Xi_n)}\right),$$

the isomorphism $\left(\mathcal{X}_{\mathrm{Grad}_\lambda(\Xi_n)}^{\mathrm{ss}}, x_{\mathrm{Grad}_\lambda(\Xi_n)}\right) \xrightarrow{\sim} \left(\mathrm{Grad}_{\mathbb{Q}}\left(\mathcal{X}_{\Xi_n}^{\mathrm{ss}}\right)_{\mathrm{gr}\,\lambda}, \mathrm{gr}\,\lambda\right)$ from Proposition 5.2.10, and the open and closed immersion

$$\left(\mathrm{Grad}_{\mathbb{Q}}\left(\mathcal{X}_{\Xi_n}^{\mathrm{ss}}\right)_{\mathrm{gr}\,\lambda}, \mathrm{gr}\,\lambda\right) \to \left(\mathrm{Grad}_{\mathbb{Q}}\left(\mathcal{X}_{\Xi_n}^{\mathrm{ss}}\right), \mathrm{gr}\,\lambda\right).$$

Proposition 5.2.12. *Let Ξ be a semistable polarised state. Then there is a canonical bijection $\mathbb{Q}^\infty\text{-}\operatorname{Filt}(\Xi) \cong \mathbb{Q}^\infty\text{-}\operatorname{Filt}(\mathcal{X}^{ss}_\Xi, x_\Xi)$.*

Proof. The bijection follows from the description of $\mathbb{Q}^\infty\text{-}\operatorname{Filt}(\mathcal{X}^{\mathrm{ss}}_\Xi, x_\Xi)$ in Remark 3.2.14 and an iterated application of Propositions 5.2.7 and 5.2.10. □

Proposition 5.2.13. *Let Ξ be a semistable polarised state, and let $\Xi' \to \Xi$ be its slice. Then the associated morphism $\mathcal{X}^{ss}_{\Xi'} = \mathcal{X}_{\Xi'} \to \mathcal{X}^{ss}_\Xi$ identifies $\mathcal{X}_{\Xi'}$ with the fibre of the good moduli space $\mathcal{X}^{ss}_\Xi \to X^{ss}_\Xi$ containing x_Ξ.*

Proof.

We use the notations of Definition 5.1.11. Let $\chi_1, \ldots, \chi_n$ be the different elements of Ξ and assume, after reordering, that $\{\chi_1, \ldots, \chi_l\} = F \cap \Xi$ (where F is the smallest face of $\operatorname{cone}(\Xi)$ containing α), the equality to be interpreted inside $M_\mathbb{Q}$ (that is, modulo torsion). The state of Ξ' is $\Xi' = q(\{\chi_{l+1}, \ldots, \chi_n\}) \subset M'$, where $q\colon M \to M'$ is the quotient map.

We remark that for any $\lambda \in M^\vee_\mathbb{Q}$ such that $\Xi \subset H_\lambda$ and $\partial H_\lambda \cap \operatorname{cone}(\Xi) = F$, the limit $y = \lim_{t\to 0} \lambda(t)x$ is polystable. Note also that y does not depend on the choice of λ. By Proposition 5.2.4 applied to Ξ_y, the stabiliser of y is $H := G_{\Xi'}$.

We identify $\mathbb{A}^\Xi_k = \mathbb{A}^n_k$, the action of $G = G_\Xi$ on $\mathbb{A}^n_k$ being via the characters $\chi_1, \ldots, \chi_n$. The G-equivariant open subscheme $\mathbb{G}^l_{m,k} \times \mathbb{A}^{n-l}_k \subset \left(\mathbb{A}^n_k\right)^{\mathrm{ss}}$ is saturated with respect to the good moduli space $\mathcal{X}^{\mathrm{ss}}_\Xi = \left(\mathbb{A}^n_k\right)^{\mathrm{ss}}/G \to \left(\mathbb{A}^n_k\right)^{\mathrm{ss}} /\!\!/ G = X^{\mathrm{ss}}_\Xi$, and thus the fibre of $\mathcal{X}^{\mathrm{ss}}_\Xi \to X^{\mathrm{ss}}_\Xi$ containing $x_\Xi = (1, \ldots, 1)$ equals the fibre of the good moduli space $\left(\mathbb{G}^l_{m,k} \times \mathbb{A}^{n-l}_k\right)/G \to \left(\mathbb{G}^l_{m,k} \times \mathbb{A}^{n-l}_k\right) /\!\!/ G$ containing $(1, \ldots, 1)$. Indeed, for every $\bar{k}$-point (a, b) in $\mathbb{G}^l_{m,k} \times \mathbb{A}^{n-l}_k$, $G_{\bar{k}}(a, 0)$ is the associated polystable orbit, and conversely if a semistable $\bar{k}$-point z in $\mathbb{A}^n_k$ has associated polystable orbit of the form $G_{\bar{k}}(a, 0)$, then it should lie in $\mathbb{G}^l_{m,k} \times \mathbb{A}^{n-l}_k$.

Consider the $(H \to G)$-equivariant map

$$h\colon \mathbb{A}^{n-l}_k \to \mathbb{G}^l_{m,k} \times \mathbb{A}^{n-l}_k \colon (z_1, \ldots, z_{n-l}) \mapsto (1, \ldots, 1, z_1, \ldots, z_{n-l}),$$

and the associated morphism $\mathbb{A}^{n-l}_k/H \to \left(\mathbb{G}^l_{m,k} \times \mathbb{A}^{n-l}_k\right)/G$, which is the restriction on the codomain of the map $f^{\mathrm{ss}}_q\colon \mathcal{X}^{\mathrm{ss}}_{\Xi'} \to \mathcal{X}^{\mathrm{ss}}_\Xi$. Let I be the image of the homomorphism $G \to \mathbb{G}^l_{m,k}$ given by $\chi_1, \ldots, \chi_k$. From an explicit computation of the ring of invariants it follows that $\left(\mathbb{G}^l_{m,k} \times \mathbb{A}^{n-l}_k\right)/G \to \left(\mathbb{G}^l_{m,k} \times \mathbb{A}^{n-l}_k\right) /\!\!/ G = \mathbb{G}^l_{m,k}/I$ is the good moduli space, and its fibre over $e \in \mathbb{G}^l_{m,k}/I(k)$ is $\left(I \times \mathbb{A}^{n-l}_k\right)/G$. To conclude, we note the isomorphism

$$\left(I \times \mathbb{A}^{n-l}_k\right)/G \cong \left((G/H) \times \mathbb{A}^{n-l}_k\right)/G \cong \mathbb{A}^{n-l}_k/H$$

induced by h. □

From Proposition 5.2.13 we get bijections

$$\mathbb{Q}\text{-}\mathrm{Filt}(\Xi') \cong \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}_{\Xi'}, x_{\Xi'}) \cong \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}_{\Xi}, x_{\Xi}) \cong \mathbb{Q}\text{-}\mathrm{Filt}(\Xi)$$

between sets of filtrations. This is consistent with Proposition 5.1.12.

Proposition 5.2.14. *Let Ξ be a semistable polarised state and let $\lambda \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}_{\Xi}, x_{\Xi})$. Then the Kempf number $\langle \lambda, \mathcal{X}_{\Xi'}^{\max}\rangle$ equals the complementedness of λ:*

$$\langle \lambda, \mathcal{X}_{\Xi'}^{\max}\rangle = \langle \lambda, \mathfrak{l}\rangle.$$

Proof. Write $\mathcal{F} = \mathcal{X}_{\Xi'} = \mathbb{A}^m_k/H$ and $x = (1, \dots, 1)$, using notation from the proof of Proposition 5.2.13 and where $m = n - l$. The maximal stabiliser locus is $\mathcal{F}^{\max} = \left(\mathbb{A}^m_k\right)^{H_\circ}/H = \{0\}/H$, where $H_\circ = (H^\circ)_{\mathrm{red}}$ is the reduced identity component [27, Proposition C.5]. Let $r = \langle \lambda, \mathcal{F}^{\max}\rangle$. There is a cartesian square

$$\begin{array}{ccc} \mathrm{Spec}\,(k[t]/(t^r)) & \longrightarrow & \mathbb{A}^1_k \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \lambda(t)x} \\ \{0\} & \longrightarrow & \mathbb{A}^m_k \end{array} \qquad \begin{array}{c} t \\ \downarrow \\ \left(t^{\langle \lambda, \chi\rangle}\right)_{\chi \in \Xi'}. \end{array}$$

Taking global sections, we get a cocartesian square

$$\begin{array}{ccc} k[t]/(t^r) & \longleftarrow & k[t] \\ \uparrow & \ulcorner & \uparrow \\ k & \longleftarrow & k[t_\chi, \chi \in \Xi'] \end{array} \qquad \begin{array}{c} t^{\langle \lambda, \chi\rangle} \\ \uparrow \\ t_\chi. \end{array}$$

Therefore $k[t]/(t^r) = k[t]/(t^{\langle \lambda, \chi\rangle}, \chi \in \Xi')$ and thus $r = \inf\{\langle \lambda, \chi\rangle \mid \chi \in \Xi'\} = \langle \lambda, \mathfrak{l}\rangle$. □

Proposition 5.2.15. *Let Ξ be a normed semistable polarised state. Then the balanced filtration of $(\mathcal{X}^{ss}_{\Xi}, x_{\Xi})$ (Definition 3.1.6) equals, under the bijection $\mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}^{ss}_{\Xi}, x_{\Xi}) \cong \mathbb{Q}\text{-}\mathrm{Filt}(\Xi)$, the balanced filtration of Ξ (Definition 5.1.15).*

Proof. This follows directly from Propositions 5.2.13 and 5.2.14. □

Theorem 5.2.16. *Let Ξ be a normed semistable polarised state and let $(\Xi_n, \lambda_n, u_n)_{n\in\mathbb{N}}$ be its balancing chain (Definition 5.1.19). Then the chain of stacks associated to $(\Xi_n, \lambda_n, u_n)_{n\in\mathbb{N}}$ (Definition 5.2.11) is isomorphic to the torsor chain of $(\mathcal{X}^{ss}_{\Xi}, x_{\Xi})$ (Definition 4.3.3).*

Proof. Let $(\mathcal{Y}_n, y_n, \eta_n, v_n)_{n\in\mathbb{N}}$ be the torsor chain of $(\mathcal{X}^{ss}_{\Xi}, x_{\Xi})$. We will provide, for all $m \in \mathbb{N}$, isomorphisms

$$i_m \colon (\mathcal{X}^{ss}_{\Xi_m}, x_{\Xi_m}) \cong (\mathcal{Y}_m, y_m)$$

such that,

1. under the identification $\mathbb{Q}\text{-}\mathrm{Filt}(\Xi_m) = \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{Y}_m, y_m)$, we have $\eta_m = \lambda_m$; and
2. the square

$$\begin{array}{ccc} (\mathcal{X}^{ss}_{\Xi_{m+1}}, x_{\Xi_{m+1}}) & \xrightarrow{f^{ss}_{um}} & (\mathcal{X}^{ss}_{\mathrm{Grad}_{\lambda_m}(\Xi_m)}, x_{\mathrm{Grad}_{\lambda_m}(\Xi_m)}) \\ {\scriptstyle i_{m+1}}\downarrow & & \downarrow \\ (\mathcal{Y}_{m+1}, y_{m+1}) & \xrightarrow{v_n} & (\mathrm{Grad}_{\mathbb{Q}}(\mathcal{Y}_m)_{\mathrm{gr}\,\eta_m}, \mathrm{gr}\,\eta_m) \end{array} \tag{5.1}$$

commutes, where the arrow on the right comes from Proposition 5.2.10.

For $n = 0$, we have that y_0 is the fibre of the good moduli space of $\mathcal{X}^{ss}_{\Xi}$ containing x_Ξ, and that $\Xi_0 = \Xi'$. Therefore $(\mathcal{Y}_0, y_0) = (\mathcal{X}^{ss}_{\Xi_0}, x_{\Xi_0})$, by Proposition 5.2.13. Let $n \in \mathbb{N}$ and suppose that isomorphisms i_m as above have been provided in such a way that conditions (1) and (2) above hold for all $m < n$. Since i_n is an isomorphism, we have the equality $\eta_n = \lambda_n$ by Proposition 5.2.15, so condition (1) also holds for $m = n$.

If $\lambda_n = 0$, then Ξ_n and $(\mathcal{Y}_n, y_n)$ are polystable. Therefore $\Xi_{n+1} = \Xi_n$ and $(\mathcal{Y}_{n+1}, y_{n+1}) = (\mathcal{Y}_n, y_n)$, so there is nothing to prove.

Assume $\lambda_n \neq 0$. We freely use the notation of Case 2 in Construction 4.3.1. Let $\chi_1, \dots, \chi_l$ be the different elements in Ξ_n, and let $G = G_{\Xi_n}$. We denote $V = k^l$ the G-representation given by the characters $\chi_1, \dots, \chi_l$, so that $\mathbb{A}^{\Xi_0}_k = \mathbb{A}(V)$ (where $\mathbb{A}$ denotes *total space*). We have $\mathcal{Y}_n = \mathbb{A}(V)/G$, and $\mathcal{Y}^{\max}_n = \{0\}/G$. Therefore the relevant blow-up is $\mathcal{B} = (\mathrm{Bl}_0\,\mathbb{A}(V))\,/G$, the exceptional divisor is $\mathcal{E} = \mathbb{P}(V)/G$, and the $\mathbb{G}_m$-torsor over it is $\mathcal{N} = (\mathbb{A}(V) \setminus \{0\})\,/G$. We denote y_n also the unique lift of $y_n = (1, \dots, 1)$ to $\mathrm{Bl}_0\,\mathbb{A}(V)$. The rational one-parameter subgroup λ_n has the property that $\langle \lambda_n, \chi_j \rangle \geq 1$ for all j and that equality holds for at least one j. Therefore, if we set $z = \lim_{t \to 0} \lambda_n(t) y_n$, the limit taken inside $\mathrm{Bl}_0\,\mathbb{A}(V)$, and if we write $z = [z_1, \dots, z_l]$ in projective coordinates, noting that z lies on the exceptional divisor $\mathbb{P}(V)$, then we have $z_j = 0$ if $\langle \lambda_n, \chi_j \rangle > 1$ and $z_j = 1$ if $\langle \lambda_n, \chi_j \rangle = 1$. We denote $z^* = (z_1, \dots, z_l)$ this lift of z to $\mathbb{A}(V)$.

The limit z lifts to the connected component $\overline{\mathcal{Z}}$ of $\mathrm{Grad}(\mathcal{B})$ containing $\mathrm{gr}\,\eta_n$. If we set $V_1 = \bigoplus_{\langle \lambda, \chi \rangle = 1} V_\chi$, where V_χ is the subrepresentation of V where G acts via the character χ, then $\overline{\mathcal{Z}} = \mathbb{P}(V_1)/G$. Thus $\mathrm{gr}\,\eta_n$ is identified with $z \in \mathbb{P}(V_1)(k)$, and the lift $z^* \in \mathbb{A}(V_1)$ to $\mathbb{A}(V_1)$ can be written in coordinates as $z = (1, \dots, 1)$ when seen inside V_1. The centre $\mathcal{Z}$ of the locally closed Θ-stratum of $\mathcal{B}$ containing y_n is the semistable locus for the shifted linear form ℓ_c, by the Linear Recognition Theorem 2.6.9.

In this case, $c = \dfrac{1}{\|\lambda_n\|^2}$, and the shifted linear form is

$$\ell_c = \ell - \frac{1}{\|\lambda_n\|^2}\langle \lambda_n^\vee, -\rangle,$$

where ℓ is the linear form associated to the ample line bundle $\mathcal{O}_{\mathbb{P}(V_1)/G}(1)$. Let $x \in \mathbb{P}(V_1)(\bar{k})$ and let $x^* \in \mathbb{A}(V_1)(\bar{k})$ be a lift of x to $\mathbb{A}(V_1)$ with state Ξ_x. The point x is semistable for ℓ_c if for all $\gamma \in \Gamma^{\mathbb{Q}}(G)$ we have

$$0 \geq \ell_c(\gamma) = \min\langle \gamma, \Xi_x\rangle - \frac{1}{\|\lambda_x\|^2}(\gamma, \lambda_n),$$

which holds if and only if $0 \in \operatorname{conv}\left(\Xi_x - \dfrac{\lambda_n^\vee}{\|\lambda_n\|^2}\right)$. Since all elements of Ξ_x are in the hyperplane $\langle \lambda_n, -\rangle = 1$, the condition that 0 is in the convex hull of $\Xi_x - \dfrac{\lambda_n^\vee}{\|\lambda_n\|^2}$ is equivalent to the condition that $\lambda_n^\vee$ is in the cone generated by Ξ_x, by Lemma 5.2.17 below. This is in turn equivalent to the lift x^* of x to $\mathbb{A}(V_1)$ being semistable for the linear form given by $\lambda_n^\vee$, by Proposition 5.2.5. Therefore we have a cartesian square

$$\begin{array}{ccc}
\mathcal{M} = \mathbb{A}(V_1)^{\mathrm{ss}(\lambda_n^\vee)}/G & \longrightarrow & \mathcal{N} = (\mathbb{A}(V)\setminus 0)/G \\
\big\downarrow & \ulcorner & \big\downarrow \\
\mathcal{Z} = \mathbb{P}(V_1)^{\mathrm{ss}(\ell_c)}/G & \longrightarrow & \mathcal{E} = \mathbb{P}(V)/G,
\end{array}$$

using the notations of Construction 4.3.1. Just from the definitions, we see that $\mathcal{M} = \mathcal{X}^{\mathrm{ss}}_{\Lambda_{\lambda_n}(\Xi_n)}$. We choose $y_{n+1} = z^*$ as the preimage of z along $\mathcal{M} \to \mathcal{Z}$ needed for the construction of the torsor chain. The stack $\mathcal{Y}_{n+1}$ is by definition the fibre of the good moduli space of $\mathcal{M}$, and hence by Proposition 5.2.13 we have the desired isomorphism $i_{n+1}\colon \mathcal{X}^{\mathrm{ss}}_{\Xi_{n+1}} = \mathcal{X}^{\mathrm{ss}}_{\Lambda_{\lambda_n}(\Xi_n)'} \cong \mathcal{Y}_{n+1}$, which sends $x_{\Xi_{n+1}}$ to y_{n+1} by our choice of z^*. Both $\mathcal{X}^{\mathrm{ss}}_{\mathrm{Grad}_{\lambda_n}(\Xi_n)}$ and $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{Y}_n)_{\mathrm{gr}\,\eta_n}$ are naturally identified with BG, so the square (5.1) commutes for $m = n$. Since the torsor chain is bounded, repeating this process we eventually reach the case $\eta_n = 0$, getting the desired isomorphism of chains. □

In the proof of Theorem 5.2.16 we used the following fact in convex geometry.

Lemma 5.2.17. *Let N be a finite dimensional $\mathbb{Q}$-vector space endowed with a rational inner product $(-,-)$. Let $\Xi \subset N$ be a nonempty finite set and let $\gamma \in N$ be an element such that $(\gamma, \chi) = 1$ for all $\chi \in \Xi$. Then we have $0 \in \operatorname{conv}\left(\Xi - \dfrac{\gamma}{\|\gamma\|^2}\right)$ if and only if $\gamma \in \operatorname{cone}(\Xi)$.*

Proof. Each $\chi \in \Xi$ can be written as $\chi = \dfrac{\gamma}{\|\gamma\|^2} + \beta_\chi$ with $(\gamma, \beta_\chi) = 0$. Note that the condition $0 \in \operatorname{conv}\left(\Xi - \dfrac{\gamma}{\|\gamma\|^2}\right)$ is equivalent to $0 \in \operatorname{cone}\left(\Xi - \dfrac{\gamma}{\|\gamma\|^2}\right)$. If this is satisfied, then $0 = \sum_\chi c_\chi \beta_\chi$ with $c_\chi \geq 0$. After rescaling we may assume that $\sum c_\chi = \|\gamma\|^2$. Then

$$\gamma = \left(\sum c_\chi\right) \frac{\gamma}{\|\gamma\|^2} + \sum_\chi c_\chi \beta_\chi = \sum c_\chi \chi,$$

and thus $\gamma \in \operatorname{cone}(\Xi)$.

Conversely, if $\gamma = \sum_\chi c_\chi \chi$ with the $c_\chi \geq 0$, then after applying $(\gamma, -)$ to both sides we get $\|\gamma\|^2 = \sum_\chi c_\chi$, so

$$\gamma = \sum_\chi c_\chi \left(\frac{\gamma}{\|\gamma\|} + \beta_\chi\right) = \gamma + \sum_\chi c_\chi \beta_\chi$$

and thus $0 = \sum c_\chi \beta_\chi$ is in cone $\left(\Xi - \frac{\gamma}{\|\gamma\|^2}\right)$. □

Corollary 5.2.18. *Let Ξ be a normed semistable polarised state. Then the iterated balanced filtration (Definition 5.1.19) of Ξ equals, under the bijection $\mathbb{Q}^\infty\text{-}\mathrm{Filt}(\Xi) \cong \mathbb{Q}^\infty\text{-}\mathrm{Filt}(\mathcal{X}^{ss}_\Xi, x_\Xi)$, the iterated balanced filtration of $(\mathcal{X}^{ss}_\Xi, x_\Xi)$ (Definition 3.5.8).*

Proof. By Theorem 5.2.16, the iterated balanced filtration of Ξ equals the sequential filtration associated to the torsor chain of $(\mathcal{X}^{ss}_\Xi, x_\Xi)$. The results then follows from Theorem 4.3.4. □

Example **5.2.19.** Consider the normed polarised state

$$\Xi = \left(\mathbb{Z}^2, \{(1,0),(1,1)\}, 0\right),$$

where $(\mathbb{Z}^2)^\vee = \mathbb{Z}^2$ has the standard inner product ($(1,0),(0,1)$ is an orthonormal base). The associated stack over $\mathbb{C}$ is $\mathcal{X}_\Xi = \mathbb{C}^2/(\mathbb{C}^\times)^2$, where $(\mathbb{C}^\times)^2$ acts by $(t_1,t_2)(v_1,v_2) = (t_1v_1, t_1t_2v_2)$. The linearisation $\mathcal{L}_\Xi$ is trivial, so every point is semistable.

Let us compute the iterated balanced filtration of the state Ξ. We have that Ξ is its own slice $\Xi' = \Xi$, and the balanced filtration $\lambda_0 = (a,b)$ of Ξ is the minimiser of a^2+b^2 subject to the condition that $\langle \lambda_0, (1,0)\rangle = a \geq 1$ and $\langle \lambda_0, (1,1)\rangle = a+b \geq 1$. Therefore $\lambda_0 = (1,0)$. The iterated state is

$$\Lambda_{\lambda_0}(\Xi) = \left(\mathbb{Z}^2, \{(1,0),(1,1)\}, (1,0)\right),$$

whose slice is $\Xi_1 = \Lambda_{\lambda_0}(\Xi)' = (\mathbb{Z}(0,1), \{(0,1)\}, 0)$, and the balanced filtration of Ξ_1 is $\lambda_1 = (0,1)$. Since $\Lambda_{\lambda_1}(\Xi_1)$ is polystable, the balancing chain of Ξ terminates here, and we conclude that the iterated balanced filtration of Ξ is $\lambda_0 = (1,0), \lambda_1 = (0,1)$. By Corollary 5.2.18, we deduce that the iterated balanced filtration of $x_\Xi = (1,1) \in \mathbb{C}^2/(\mathbb{C}^\times)^2$ is the sequence $(1,0),(0,1)$ in $\mathbb{Q}^2 = \Gamma^{\mathbb{Q}}((\mathbb{C}^\times)^2)$.

We now analyse Conjecture 1.7.1 in this case. We endow $\mathbb{C}^2$ with the standard hermitian metric. The associated Kempf–Ness potential for the point $(1,1) \in \mathbb{C}^2$ is

$$f : \mathbb{R}^2 \to \mathbb{R} : x \mapsto e^{x \cdot \binom{1}{0}} + e^{x \cdot \binom{1}{1}}$$

up to the addition of a constant, where we are identifying $\mathbb{R}^2 \xrightarrow{i \cdot -} \mathrm{Lie}((S^1)^2) \xrightarrow[\sim]{exp(\frac{1}{2i}-)} (\mathbb{C}^\times)^2/(S^1)^2$, the unit circle S^1 being the maximal compact subgroup of $\mathbb{C}^\times$. In this case, the exponential map is a global isometry between $\mathrm{Lie}((S^1)^2)$ and $(\mathbb{C}^\times)^2/(S^1)^2$. The gradient is

$$\nabla f(x) = e^{x_1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{x_1 + x_2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The equation for $h : (0, \infty) \to \mathbb{R}^2$ to be a flow line for $-\nabla f$ is

$$h'(t) = -e^{h_1(t)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - e^{h_1(t) + h_2(t)} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We write $h(t) = -\log(t)\binom{1}{0} - \log\log(t)\binom{0}{1} + z(u)$, where $u = \log\log(t)$. The equation for z becomes

$$z'(u) = \begin{pmatrix} e^u(1 - e^{z_1(u)}) - e^{z_1(u)+z_2(u)} \\ 1 - e^{z_1(u)+z_2(u)} \end{pmatrix}.$$

For $N_1 > 0$ real, we have that

1. if $z_2 = N_1$ and $z_1 \in (-N_1, N_1)$, then $z_2' < 0$;
2. if $z_2 = -N_1$ and $z_1 \in (-N_1, N_1)$, then $z_2' > 0$.

For $N_2 > 0$ big enough so that $-1 \leq \log(1 - e^{-N_2})$ and $u \geq 2$, we have that

1. if $z_1 = -N_2$ and $z_2 \in (-N_2 - 1, N_2 + 1)$, then $z_1' > 0$;
2. if $z_1 = N_2$ and $z_2 \in (-N_2 - 1, N_2 + 1)$, then $z_1' < 0$.

Therefore, an appropriate choice of N_1 and N_2 gives a rectangle that z cannot leave, because z' points inwards at the boundary. Therefore z is bounded when $t \gg 0$. We have verified Conjecture 1.7.1 in this example.

5.3 THE REFINED HARDER–NARASIMHAN FILTRATION OF A POLARISED STATE

In Section 5.1 we have defined the iterated balanced filtration of a normed semistable polarised state $\boldsymbol{\Xi} = (M, \Xi, \alpha)$. We now explain how to produce, in the case where $\boldsymbol{\Xi}$ may be unstable, a canonical sequential filtration of $\boldsymbol{\Xi}$, its *refined Harder–Narasimhan filtration* (Definition 5.3.3). We start by defining its *Harder–Narasimhan filtration* (Definition 5.3.2).

We fix a principal ideal domain A contained in $\mathbb{R}$, and we denote K the field of fractions of A. Let $\boldsymbol{\Xi} = (M, \Xi, \alpha)$ be a normed polarised state over A. We denote $\boldsymbol{\Xi}^\circ = (M, \Xi, 0)$ the unpolarised state.

Theorem 5.3.1 (Existence and uniqueness of Harder-Narasimhan filtrations for states)**.** *For a filtration* $\lambda \in K\text{-}\mathrm{Filt}(\boldsymbol{\Xi}^\circ)$*, the following are equivalent:*

1. $\mathrm{Grad}_\lambda(\boldsymbol{\Xi}^\circ)$ *is semistable for the polarisation* $\alpha + \lambda^\vee$.
2. *The filtration* λ *satisfies* $-\langle \lambda, \alpha\rangle \geq \|\lambda\|^2$ *and, for every* $\mu \in K\text{-}\mathrm{Filt}(\boldsymbol{\Xi}^\circ)$ *such that* $-\langle \mu, \alpha\rangle \geq \|\mu\|^2$*, we have* $\|\mu\|^2 \leq \|\lambda\|^2$.
3. *The filtration* λ *maximises the function* $-\langle \lambda, \alpha\rangle - \frac{1}{2}\|\lambda\|^2$ *on the set* $K\text{-}\mathrm{Filt}(\boldsymbol{\Xi}^\circ)$.

Moreover, there exists a unique filtration $\lambda \in K\text{-}\mathrm{Filt}(\boldsymbol{\Xi}^\circ)$ *satisfying these conditions.*

Proof. We have that $K\text{-}\mathrm{Filt}(\Xi^\circ)$ is a convex polyhedral cone inside $M_K^\vee$, and the function

$$f: M_K^\vee \to K: \lambda \mapsto -\langle \lambda, \alpha\rangle - \frac{1}{2}\|\lambda\|^2$$

on $M_K^\vee$ is strictly concave. Therefore, if $K = \mathbb{R}$ then there is a unique $\lambda \in \mathbb{R}\text{-}\mathrm{Filt}(\boldsymbol{\Xi}^\circ)$ maximising this function. The differential of f at a point $\lambda \in M_\mathbb{R}^\vee$ is $-\alpha - \lambda^\vee$. By the Karush–Kuhn–Tucker conditions, a point $\lambda \in \mathbb{R}\text{-}\mathrm{Filt}(\boldsymbol{\Xi}^\circ)$ maximises f if and only if there are real numbers $u_\chi \geq 0$ such that

$$\alpha + \lambda^\vee = \sum_{\chi \in \Xi} u_\chi \cdot \chi \tag{5.2}$$

and $u_\chi = 0$ if $\langle \lambda, \chi\rangle > 0$.

If K is not $\mathbb{R}$, we want to see that the maximiser λ of f on $\mathbb{R}\text{-}\mathrm{Filt}(\boldsymbol{\Xi}^\circ)$ actually belongs to $K\text{-}\mathrm{Filt}(\boldsymbol{\Xi}^\circ)$. For this, note that, since only the $\chi \in \Xi$ satisfying $\langle \lambda, \chi\rangle = 0$ are involved in the formula (5.2), λ is also the maximiser of f on the vector subspace $V_\mathbb{R} = V \otimes_K \mathbb{R}$ of $M_\mathbb{R}^\vee$, where

$$V = \bigcap_{\chi \in \Xi^{\lambda,0}} \partial H_\chi.$$

The differential of $f|_{V_\mathbb{R}}$ at λ is $-\alpha|_{V_\mathbb{R}} - (\lambda, -): V_\mathbb{R} \to \mathbb{R}$, and since λ is a maximiser, we have $-\alpha|_{V_\mathbb{R}} = (\lambda, -)$, so $\lambda = p(\alpha)$, where $p: M_\mathbb{R}^\vee \to V_\mathbb{R}$ is the orthogonal projection. Since the inner product is defined over K, the projection p is also defined over K and thus $p(\alpha)$ is in $M_K^\vee$. Therefore, $\lambda \in K\text{-}\mathrm{Filt}(\boldsymbol{\Xi}^\circ)$ is K-rational.

Now note that (5.2) is equivalent to $\alpha + \lambda^\vee \in \mathrm{cone}\left(\Xi^{\lambda,0}\right)$, that is, to $\mathrm{Grad}_\lambda(\boldsymbol{\Xi}^\circ)$ being semistable for the polarisation $\alpha + \lambda^\vee$. Thus 3 holds if and only if 1 holds.

Take any $0 \neq \lambda \in K\text{-}\mathrm{Filt}(\boldsymbol{\Xi}^\circ)$ and consider the function

$$h: K_{\geq 0} \to \mathbb{R}: c \mapsto f(c\lambda).$$

The function h has a unique maximum at $c_\lambda = -\frac{\langle \lambda, \alpha \rangle}{\|\lambda^2\|}$, so the maximum of f on $K\text{-}\mathrm{Filt}(\Xi^\circ)$ has to occur at a filtration λ with $c_\lambda = 1$ or at $\lambda = 0$. For the λ satisfying this condition, we have $f(\lambda) = \frac{1}{2}\|\lambda\|^2$. On the other hand, for those $\lambda \neq 0$ such that $-\langle \lambda, \alpha \rangle \geq \|\lambda\|^2$, replacing λ by $c_\lambda \lambda$ increases $\|-\|^2$ while making the condition $-\langle \lambda, \alpha \rangle = \|\lambda\|^2$ hold. Therefore 2 is equivalent to 3. □

Definition 5.3.2 (Harder–Narasimhan filtration of a state)**.** The *Harder–Narasimhan filtration* of the normed polarised state Ξ is the unique filtration $\lambda \in K\text{-}\mathrm{Filt}(\Xi^\circ)$ satisfying the equivalent conditions of Theorem 5.3.1.

Theorem 5.3.1 allows us to define the refined Harder–Narasimhan filtration of a normed polarised state.

Definition 5.3.3 (Refined Harder–Narasimhan filtration of a state)**.** The *refined Harder–Narasimhan filtration* of the normed polarised state Ξ is the sequential filtration $(\lambda_n)_{n\in\mathbb{N}} \in K^\infty\text{-}\mathrm{Filt}(\Xi^\circ)$ of Ξ° defined by the following conditions:

1. λ_0 is the Harder–Narasimhan filtration of Ξ;
2. $(\lambda_{n+1})_{n\in\mathbb{N}}$ is the iterated balanced filtration of the normed semistable polarised state $(M, \Xi^{\lambda_0,0}, \alpha + \lambda^\vee)$.

It is straightforward to check that the refined Harder–Narasimhan filtration is indeed an element of $K^\infty\text{-}\mathrm{Filt}(\Xi^\circ)$.

Proposition 5.3.4. *Suppose that $A = \mathbb{Z}$. Then, under the bijection*

$$\mathbb{Q}\text{-}\mathrm{Filt}(\Xi^\circ) = \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}_\Xi, x_\Xi)$$

of Proposition 5.2.2, the Harder–Narasimhan filtration of the normed polarised state Ξ (Definition 5.3.2) equals the Harder–Narasimhan filtration of the point x_Ξ in the normed pointed stack $\mathcal{X}_\Xi$ equipped with the linear form ℓ_Ξ with the direct convention (Remark 2.5.17). Moreover, under the bijection

$$\mathbb{Q}^\infty\text{-}\mathrm{Filt}(\Xi^\circ) = \mathbb{Q}^\infty\text{-}\mathrm{Filt}(\mathcal{X}_\Xi, x_\Xi),$$

of Proposition 5.2.12, the refined Harder–Narasimhan filtration of the normed polarised state Ξ (Definition 5.3.3) coincides with the refined Harder–Narasimhan filtration of the point x_Ξ in $\mathcal{X}_\Xi$ (Definition 3.5.10).

Note that Definition 3.5.10 applies to $\mathcal{X}_\Xi$ because it has a good moduli space.

Proof. The first statement follows readily from the fact that $\langle \lambda, \ell_\Xi \rangle = -\langle \lambda, \alpha \rangle$ and that the bijection between sets of filtrations preserves the norm, by definition.

Let $\lambda \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}_\Xi, x_\Xi)$ be the Harder-Narasimhan filtration, and let $\Xi_1 = (M, \Xi^{\lambda_0, 0}, \alpha + \lambda^\vee)$. Note that the centre of the Harder-Narasimhan stratum of $\mathcal{X}_\Xi$ containing x_Ξ is $\mathcal{X}^{\mathrm{ss}}_{\Xi_1}$. This follows from the recognition theorem Theorem 5.1.18 (but note that we are now working with the direct convention for HN filtrations). Moreover x_{Ξ_1} corresponds to the associated graded point of λ. The agreement of refined Harder-Narasimhan filtrations thus follows from Corollary 5.2.18. □

We finish this chapter with a conjecture on asymptotics of gradient flows that implies Conjecture 1.7.1 in the case of a torus action. We now assume that $A = \mathbb{R}$. Let $(c_\chi)_{\chi \in \Xi}$ be a family of numbers $c_\chi \in \mathbb{R}_{>0}$. We define the associated *Kempf–Ness potential* to be the function

$$p \colon M^\vee \to \mathbb{R} \colon \gamma \mapsto -\langle \gamma, \alpha \rangle + \sum_{\chi \in \Xi} c_\chi e^{\langle \gamma, \chi \rangle}.$$

Conjecture 5.3.5. *Let $\lambda_0, \ldots, \lambda_n \in M^\vee$ be the refined Harder-Narasimhan filtration of Ξ. Let $h \colon (0, \infty) \to M^\vee_{\mathbb{R}}$ be a negative gradient flow line for the Kempf–Ness potential p. Then the function*

$$t \mapsto h(t) + t\lambda_0 + \log(t)\lambda_1 + \log\log(t)\lambda_2 + \cdots \log^{\circ n}(t)\lambda_n$$

is bounded when $t \gg 0$.

The author has verified this conjecture in several examples.

Remark **5.3.6.** We may similarly upgrade Conjecture 1.7.1 to a statement involving the full refined Harder–Narasimhan filtration in the unstable case, and not just the iterated balanced filtration in the semistable case.

CHAPTER 6

MODULAR LATTICES

Modular lattices appear naturally when studying moduli of objects in an abelian category $\mathcal{A}$. For any object E of $\mathcal{A}$, the partially ordered set L_E of subobjects of E is a modular lattice. For any modular lattice L of finite length endowed with a *norm*, Haiden–Katzarkov–Kontsevich–Pandit in [33] define a canonical $\mathbb{R}$-filtration of L, the *HKKP filtration*. The construction can be applied iteratively to define a canonical sequential refinement of the Harder-Narasimhan filtration of a lattice endowed with the necessary stability data.

In this chapter, we revisit the theory of the HKKP filtration for modular lattices. Our main result is a new characterisation of the filtration (Theorem 6.6.26). Our methods naturally imply rationality of the HKKP filtration. These results will be used in Chapter 7 to compare the iterated HKKP filtration for lattices and the iterated balanced filtration for stacks.

6.1 GENERALITIES ABOUT LATTICES

We recall from [12] and [33, Section 4] some basic definitions about lattices.

A *lattice* is a poset $(L, \leq)$ such that every two elements $a, b \in L$ have a greatest lower bound $a \wedge b$, called the *meet* of a and b, and a least upper bound $a \vee b$, called the *join* of a and b. If L is a lattice, a *sublattice* is a subset L' of L closed under taking meet and join. If L' is a sublattice of L, then L' is regarded as a lattice with the poset structure inherited from L. If $a \leq b$ are elements of L, then the *interval* $[a, b]$ is the sublattice of L consisting of those $x \in L$ with $a \leq x \leq b$. When considering a sublattice L' of a lattice L and elements $a < b$ in L', we will use the notation $[a, b]$ to denote the interval in the bigger lattice L, and $[a, b] \cap L'$ to denote the interval in L'. A *morphism* $f : L_1 \to L_2$ of lattices is a map of sets preserving the poset structure, meets and joins. Sublattices L' of L are in correspondence with injective lattice morphism $L' \hookrightarrow L$.

Definition 6.1.1. A lattice L is said to be

1. *modular* if, for all $x, a, b \in L$ with $a \leq b$, we have
$$(x \vee a) \wedge b = (x \wedge b) \vee a; \tag{6.1}$$
2. *bound* if it has a minimum, denoted 0, and a maximum, denoted 1;
3. *finite length* if there exists $N \in \mathbb{N}$ such that, for all chains $a_0 < a_1 < \cdots < a_n$ of elements of L, we have $n \leq N$;
4. *artinian* if it is modular, of finite length, and nonempty;
5. *complemented* if it is bound and for all $a \in L$ there is $b \in L$ such that $a \vee b = 1$ and $a \wedge b = 0$;
6. *distributive* if, for all $a, b, c \in L$, we have $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, or, equivalently, if for all $a, b, c \in L$ we have $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$;
7. *complete* if every subset of L has a least upper bound and a greatest lower bound;
8. *a finite boolean algebra* if it is artinian, distributive and complemented.

In an arbitrary lattice L, there are two ways to project L onto an interval $[a, b]$, namely the maps $x \mapsto (x \vee a) \wedge b$ and $x \mapsto (x \wedge b) \vee a$. The modular law (6.1) precisely says that these two maps agree, and thus that there is a canonical projection $L \to [a, b]$ for every interval in L. The poset of normal subgroups of an abstract group is a modular lattice, and so is the poset L of subobjects of an object E in an abelian category. In the latter example, L being complemented is equivalent to E being semisimple. Distributive lattices and, in particular, finite boolean algebras, are modular. Any sublattice of the lattice of subsets 2^S of a given set S is distributive, while the lattice of subspaces of a given vector space V is modular but not distributive if $\dim V \geq 2$. Finite length lattices are complete and bound, and we use the notations 0 and 1 for the minimum and maximum of the lattice. When considering sublattices L' of a finite length lattice L, we reserve $0, 1$ to denote the minimum and maximum of the bigger lattice L. We say that a sublattice L' of L is *total* if $0, 1 \in L'$.

If L is a modular lattice, we denote $\sim$ the equivalence relation on intervals generated by $[a, a \vee b] \sim [a \wedge b, b]$ for $a, b \in L$. The modular law implies that two equivalent intervals of L are isomorphic as lattices. Indeed, the map $[a, a \vee b] \to [a \wedge b, b]\colon x \mapsto x \wedge b$ is a poset isomorphism with inverse $y \mapsto y \vee a$.

Artinian lattices provide a natural common framework for the celebrated Jordan–Hölder theorem in different settings, like that of finite groups or of finite length abelian categories.

Theorem 6.1.2 (Jordan–Hölder–Dedekind)**.** *Let L be an artinian lattice, and let $0 = a_0 < a_1 < \cdots < a_n = 1$ and $0 = b_0 < b_1 < \cdots < b_m = 1$ be two maximal chains in L. Then $n = m$ and there is a permutation σ of $\{1, \dots n\}$ such that $[a_{i-1}, a_i] \sim [b_{\sigma(i)-1}, b_{\sigma(i)}]$ for $i \in \{1, \dots n\}$.*

For a proof, see [46, Section 8.3]. We define the *length* of an artinian lattice L to be the length of any maximal chain in L.

For a bound lattice L and an element $x \in L$, we say that x is an *atom* if $0 < x$ and there is no $y \in L$ with $0 < y < x$. If L is a finite boolean algebra, then the set S of atoms of L is finite, and L is canonically isomorphic to the poset of subsets of S via the map $2^S \to L\colon U \mapsto \bigvee_{x \in U} x$. Since we can consider the substraction $U \setminus V$ of subsets of S, we have a well-defined substraction operation $x \setminus y$ for elements x, y of L.

There is a lattice analogue of the Grothendieck group of an abelian category, which we recall from [33].

Definition 6.1.3 (Grothendieck group of a modular lattice)**.** Let L be a modular lattice. The *Grothendieck group* $\mathrm{K}(L)$ of L is the abelian group generated by symbols $\overline{[a, b]}$ for every interval $[a, b]$ in L and subject to the relations

1. $\overline{[a, b]} + \overline{[b, c]} = \overline{[a, c]}$ for $a \leq b \leq c$ in L, and
2. $\overline{[x \wedge y, y]} = \overline{[x, x \vee y]}$ for $x, y \in L$.

If L is an artinian lattice and $0 = a_0 < a_1 < \cdots < a_n = 1$ is a maximal chain, Theorem 6.1.2 implies that $\mathrm{K}(L)$ is generated by the intervals $[a_{i-1}, a_i]$.

6.2 BACKGROUND ON THE HKKP FILTRATION AND STATEMENT OF MAIN RESULT

We now summarise the results in [33] on canonical filtrations of normed artinian lattices and introduce the main result of this chapter.

Let L be an artinian lattice. An $\mathbb{R}$*-filtration* F of L is a chain

$$0 = a_0 < \cdots < a_n = 1 \tag{6.2}$$

of elements in L together with a sequence

$$c_1 > \cdots > c_n \tag{6.3}$$

of real numbers. For $c \in \mathbb{R}$, we denote $F_{\geq c} = \bigvee_{c_i \geq c} a_i$ and $F_{>c} = \bigvee_{c' > c} F_{\geq c'}$. We denote $\mathbb{R}\text{-}\mathrm{Filt}(L)$ the set of all $\mathbb{R}$-filtrations of L.

Remark **6.2.1.** In [33], the authors define an $\mathbb{R}$-filtration to be a chain (6.2) in L labelled by an *increasing* sequence $b_1 < \cdots < b_n$ of real numbers. To move between their convention and our convention, one just needs to set $c_i = -b_i$.

Haiden–Katzarkov–Kontsevich–Pandit define an $\mathbb{R}$-filtration F to be *paracomplemented* if, for all $c \in \mathbb{R}$, the interval $[F_{\geq c+1}, F_{\geq c}]$ is complemented. Equivalently, if F is given by chains (6.2) and (6.3), F is paracomplemented if, for all $1 \leq i < j \leq n$ such that $c_i - c_j < 1$, we have that the interval $[a_{i-1}, a_j]$ is complemented. The set of paracomplemented $\mathbb{R}$-filtrations of L is denoted $\mathscr{B}_{\mathbb{R}}(L)$.

If F is a paracomplemented $\mathbb{R}$-filtration of L, Haiden–Katzarkov–Kontsevich–Pandit define a new artinian lattice $\Lambda(F)$ by setting

$$\Lambda(F) = \left\{ (x_c)_{c\in\mathbb{R}} \in \prod_{c\in\mathbb{R}} [F_{>c}, F_{\geq c}] \;\middle|\; \forall c \in \mathbb{R},\ [x_{c+1}, x_c] \text{ is complemented} \right\}.$$

It is a nontrivial result that $\Lambda(F)$ is a sublattice of $\prod_{c\in\mathbb{R}} [F_{>c}, F_{\geq c}]$ [33, Proposition 4.5].

An $\mathbb{R}$-valued *norm* X on the lattice L is a homomorphism $X\colon \mathrm{K}(L) \to \mathbb{R}$ such that for all $a < b$ in L we have $X([a,b]) > 0$. We fix one such norm X. If $F \in \mathscr{B}_{\mathbb{R}}(L)$, and $a = (a_c)_{c\in\mathbb{R}} \in \Lambda(F)$, then we define the quantity

$$F^\vee([0,a]) = \sum_{c\in\mathbb{R}} c\,X([F_{>c}, a_c]).$$

Haiden–Katzarkov–Kontsevich–Pandit define $\Lambda(F)$ to be *semistable with respect to* $-F^\vee$ (or *semistable of phase* 0 in their terminology) if $F^\vee(L) = 0$ and $F^\vee([0,a]) \leq 0$ for all $a \in \Lambda(L)$.

The main lattice-theoretic result of [33] is:

Theorem 6.2.2 (Haiden–Katzarkov–Kontsevich–Pandit)**.** *There is a unique paracomplemented $\mathbb{R}$-filtration $F \in \mathscr{B}_{\mathbb{R}}(L)$ such that $\Lambda(F)$ is semistable with respect to $-F^\vee$.*

This is [33]. The authors call the unique filtration in Theorem 6.2.2 the *weight filtration* of (L, X). We will use the name *HKKP filtration* instead.

Using the HKKP filtration iteratively (as recalled in Section 6.7), Haiden–Katzarkov–Kontsevich–Pandit construct a canonical $\mathbb{R}^\infty$-filtration of L, and then they prove in [34] that this $\mathbb{R}^\infty$-filtration describes the asymptotics of the Yang-Mills flow of a vector bundle on a curve as in Theorem 1.1.1. They prove a similar result for quiver representations in [33, Theorem 5.11]. These results on asymptotics on gradient flows are the main reason for the authors to pursue Theorem 6.2.2.

The norm X allows us to define the *norm squared* $\|F\|^2$ of a filtration $F \in \mathbb{R}\text{-Filt}(L)$ by

$$\|F\|^2 = \sum_{c\in\mathbb{R}} c^2 X([F_{>c}, F_{\geq c}]).$$

The main result of this chapter is then:

THEOREM 6.2.3. *The HKKP filtration of (L, X) is the unique paracomplemented $\mathbb{R}$-filtration $F \in \mathcal{B}_{\mathbb{R}}(L)$ minimising the norm squared $\|F\|^2$. Moreover, if $X([a,b]) \in \mathbb{Q}$ for all $a < b$ in L, then the HKKP filtration is a $\mathbb{Q}$-filtration (meaning that the labels c_i in (6.3) are all rational numbers).*

Theorem 6.2.3 is a direct consequence of Theorem 6.6.26 below, where we also give a new, simpler proof of Theorem 6.2.2, from which the rationality statement is automatic. The main novelty of our approach with respect to that of [33] is the systematic study of maximal distributive sublattices of L.

6.3 DISTRIBUTIVE ARTINIAN LATTICES AND DIRECTED ACYCLIC GRAPHS

In this section we will show that every distributive artinian lattice is canonically isomorphic to the poset of closed subgraphs of a directed acyclic graph (Theorem 6.3.5). We start with the following well-known property of distributive lattices.

LEMMA 6.3.1. *Let D be a bound distributive lattice and let $a \in D$. Then the map $f: D \to [0,a] \times [a,1]: x \mapsto (x \wedge a, x \vee a)$ is a lattice injection.*

Proof. Suppose $x, y \in D$ are such that $x \wedge a = y \wedge a$ and $x \vee a = y \vee a$. Then

$$x = x \wedge (x \vee a) = x \wedge (y \vee a) = (x \wedge y) \vee (x \wedge a) = x \wedge y.$$

Thus $x \leq y$. By symmetry, $x = y$. The map f preserves meets and joins by distributivity. □

It follows from the lemma that complements in bound distributive lattices are unique if they exist. Indeed, $b \in D$ is the complement of $a \in D$ precisely if $f(b) = (0, 1)$.

A *directed acyclic graph* Q is a pair (Q_0, Q_1) where Q_0 is a finite set, the set of *vertices*, and Q_1, the set of *arrows*, is a subset of $Q_0 \times Q_0$. The two projections $Q_0 \times Q_0 \to Q_0$ define source and target maps $s, t: Q_1 \to Q_0$. We require that Q has no oriented cycles, that is, there is no sequence of arrows $\alpha_1, \ldots, \alpha_n \in Q_1$ with $t(\alpha_i) = s(\alpha_{i+1})$ for all $i = 1, \ldots, n-1$ and $t(\alpha_n) = s(\alpha_1)$.

Finite posets are in correspondence with those directed acyclic graphs Q such that, for all $a, b, c \in Q_0$, if there are arrows $a \to b$ and $b \to c$, then there is also an arrow $a \to c$. We will say that directed acyclic graphs satisfying this property are *posetal*.

For any directed acyclic graph Q, a subset $R \subset Q_0$ is a *closed subgraph* if, for all $\alpha \in Q_1$, we have the implication $s(\alpha) \in R \implies t(\alpha) \in R$. The lattice of closed subgraphs of Q, with the order given by inclusion, is a distributive artinian lattice of length the cardinality of Q_0. In fact, all distributive artinian lattices arise in this way.

PROPOSITION 6.3.2. *Let D be a distributive artinian lattice and let $0 = a_0 < \cdots < a_n = 1$ be a maximal chain in D. We identify the product lattice $\prod_{i=1}^{n}[a_{i-1}, a_i]$ with the lattice 2^S of subsets of $S = \{1, \ldots, n\}$. Then the map $f : D \to 2^S : x \mapsto ((x \vee a_{i-1}) \wedge a_i)_i$ is injective and preserves meets and joins. Moreover, there is a directed acyclic graph Q with set of vertices $Q_0 = S$ such that $f(D)$ is the lattice of closed subgraphs of Q. There is a unique such Q that is posetal.*

Proof. Injectivity of f follows from Lemma 6.3.1 by induction on the length of D. Now, $f(D)$ is the collection of open sets of a topology on S. Since S is finite, for each $i \in S$ there is a smallest open set U_i containing i. We let

$$Q_1 = \{(i, j) \mid j \in U_i \setminus \{i\}\},$$

and denote Q the graph with set of vertices S and set of arrows Q_1.

Since $f(a_i) = \{1, \ldots, i\}$ is an open set containing i, we have $U_i \subset \{1, \ldots, i\}$, so if there is an arrow $i \to j$ then we have $j \leq i$. It follows that Q is acyclic. If $j \in U_i$, then $U_j \subset U_i$, so Q is posetal. A subset $R \subset S$ is a closed subgraph if and only if for all $i \in R$, we have $U_i \subset R$. That is, closed subgraphs are the open sets, which are the elements of $f(D)$.

Suppose that there is another posetal directed acyclic graph Q' with set of vertices S such that $f(D)$ is the set of closed subgraphs of Q'. Any closed subgraph of Q' containing $i \in S$ must also contain any element of U_i, which, Q' being posetal, implies the existence of an arrow $i \to j$ for every $j \in U_i$. On the other hand, since U_i is a closed subgraph of Q', there are no arrows $i \to j$ with j not in U_i. Therefore $Q' = Q$. □

Proposition 6.3.2 gives a complete classification of distributive artinian lattices as lattices of closed subgraphs of directed acyclic graphs. However, the way Proposition 6.3.2 is formulated, the quiver Q obtained seems to depend on the choice of maximal chain $0 = a_0 < \cdots < a_n = 1$. To formulate a choice-free version, we introduce a definition.

Definition 6.3.3 (Vertex of a distributive artinian lattice)**.** Let D be a distributive artinian lattice. An *vertex* of D is an element v of the Grothendieck group $\mathrm{K}(D)$ of the form $v = \overline{[a,b]}$, with $[a,b]$ a length-one interval of D. We denote $Q_{D,0}$ the set of vertices of D. We define the *finite boolean algebra associated to* D to be the lattice $2^{Q_{D,0}}$ of subsets of $Q_{D,0}$, and we denote $T_D = 2^{Q_{D,0}}$.

Theorem 6.1.2 implies that $Q_{D,0}$ is finite, of cardinality at most the length of D, so T_D is a finite boolean algebra.

Proposition 6.3.4. *Let D be a distributive artinian lattice. The set $Q_{D,0}$ of vertices of D is finite and $\mathrm{K}(D)$ is a free abelian group with basis $Q_{D,0}$.*

Proof. By Proposition 6.3.2 we may assume that D is the lattice of closed subgraphs of a directed acyclic graph Q, and thus we regard D as a sublattice $D \subset 2^{Q_0}$.

First, note that two intervals $[a,b]$ and $[a',b']$ in D are equivalent if and only if $b \backslash a = b' \backslash a'$ (in 2^{Q_0}). Indeed, for $x, y \in D$ it is clear that $(x \vee y) \backslash x = y \backslash (x \wedge y)$. Since the equivalence relation on intervals is generated by relations of the form $[x, x \vee y] \sim [x \wedge y, y]$, this observation gives one direction. Conversely, if $[a,b]$ and $[a',b']$ are intervals satisfying $b \setminus a = b' \setminus a'$, then $b \vee (a \vee a') = b \wedge b'$ and $b \wedge (a \vee a') = a$, so $[a,b] \sim [a \vee a', b \vee b']$. By symmetry, $[a',b'] \sim [a \vee a', b \vee b']$ too, so that $[a,b] \sim [a',b']$.

From this observation it follows that the map $f\colon \mathrm{K}(D) \to \mathbb{Z}^{\oplus Q_1}$ given by $\overline{[a,b]} \mapsto \sum_{i \in b \backslash a} 1 \cdot i$ is well-defined. Choose a maximal chain $0 = a_0 < a_1 < \cdots < a_n = 1$, giving generators $\overline{[a_{k-1}, a_k]}$ of $\mathrm{K}(D)$. For any $k \in \{1, \ldots, n\}$, there is a unique $i_k \in Q_0$ with $a_k \setminus a_{k-1} = \{i_k\}$, and the map $\{1, \ldots, n\} \to Q_0\colon k \mapsto i_k$ is a bijection. The map $1 \cdot i_k \mapsto \overline{[a_{k-1}, a_k]}$ is an inverse of f, so f is an isomorphism. It follows that $\overline{[a_0, a_1]}, \ldots, \overline{[a_{n-1}, a_n]}$ are the different elements of the set of vertices $Q_{D,0}$ of D, and that they form a basis of $\mathrm{K}(D)$. ☐

Let D be a distributive artinian lattice. The canonical isomorphism $\mathrm{K}(D) = \mathbb{Z}^{\oplus Q_{D,0}}$ gives $\mathrm{K}(D)$ the structure of a (non-artinian) distributive lattice, as a product of copies of $\mathbb{Z}$ with the standard order. The associated boolean algebra T_D embeds canonically into $\mathrm{K}(D)$ via the map $T_D \to \mathrm{K}(D)\colon U \mapsto \sum_{v \in U} v$, where U is regarded as a subset of $Q_{D,0}$. We are ready to state a version of Proposition 6.3.2 where the obtained directed acyclic graph is canonical.

Theorem 6.3.5. *Let D be a distributive artinian lattice. The map $D \to \mathrm{K}(D)\colon a \mapsto \overline{[0,a]}$ induces a lattice injection $D \hookrightarrow T_D$. Moreover, there is a unique posetal directed acyclic graph Q_D with set of vertices $Q_{D,0}$ such that D is identified with the lattice of closed subgraphs of Q_D via the map $D \hookrightarrow T_D = 2^{Q_{D,0}}$.*

Proof. It is clear that $D \to \mathrm{K}(D)$ factors through T_D. Choose a maximal chain $0 = a_0 < a_1 < \cdots < a_n = 1$ and let $S = \{1, \ldots, n\}$. The map $S \to Q_{0,D}\colon k \mapsto \overline{[a_{k-1}, a_k]}$ is a bijection. The induced isomorphism $T_D \cong 2^S$ identifies the map $D \to T_D$ in the statement with the map f in Proposition 6.3.2. The statements thus follows from Proposition 6.3.2. □

Remark **6.3.6.** The canonical inclusion $D \hookrightarrow T_D$ induces an isomorphism $\mathrm{K}(D) = \mathrm{K}(T_D)$ of Grothendieck groups.

Remark **6.3.7.** If $\iota\colon D' \hookrightarrow D$ is a lattice injection, then the assignment $[a, b] \mapsto [\iota(a), \iota(b)]$ on intervals gives a homomorphism $\mathrm{K}(D') \to \mathrm{K}(D)$ of Grothendieck groups that restricts to a lattice injection $T_{D'} \hookrightarrow T_D$ between associated boolean algebras. Moreover, if D' is a total sublattice of D, then $T_{D'}$ is a total sublattice of T_D.

Remark **6.3.8.** The associated boolean algebra of D can be characterised categorically as follows. The canonical map $D \hookrightarrow T_D$ is a lattice injection of D into a finite boolean algebra, and for any other such lattice injection $D \hookrightarrow T$, there is a unique lattice injection $T_D \hookrightarrow T$ such that the composition $D \hookrightarrow T_D \hookrightarrow T$ equals the given map $D \hookrightarrow T$. This characterises T_D up to unique isomorphism preserving $D \hookrightarrow T_D$. Then the set $Q_{D,0}$ of vertices of D can be defined as the set atoms of T_D, and Theorem 6.3.5 shows that, under the injection $D \hookrightarrow T_D$, D is identified with the lattice of closed subgraphs of a unique posetal directed acyclic graph with set of vertices $Q_{D,0}$.

Definition 6.3.9 (Associated directed acyclic graph)**.** Let D be a distributive artinian lattice. The *associated directed acyclic graph* of D is the graph Q_D from Theorem 6.3.5.

Theorem 6.3.5 can be seen as a reformulation of Birkhoff duality convenient for our purposes. The usual formulation [12, Theorem III.3.3] uses the notion of join-irreducible elements. Recall that an element a of a bound lattice L is said to be *join-irreducible* if $a \neq 0$ and whenever $a = b \vee c$ we have $a = b$ or $a = c$.

Corollary 6.3.10 (Birkhoff duality)**.** *Let D be a distributive artinian lattice and let $\mathcal{J}$ be the set of join-irreducible elements of D, then the map*

$$g\colon D \to 2^{\mathcal{J}}\colon a \mapsto \{b \in \mathcal{J} \mid b \leq a\}$$

is a lattice injection with image the set of those subsets U of $\mathcal{J}$ such that, for all $a, b \in \mathcal{J}$, if $a \leq b$ and $b \in U$ then $a \in U$. Moreover, the cardinality of $\mathcal{J}$ equals the length of D.

Proof. We use the notation from the proof of Proposition 6.3.2. We identify D with the sublattice $f(D)$ of 2^S. It is clear that the U_i are the join-irreducible elements of $f(D)$. Moreover, the bijection $S \to \mathcal{J}: i \mapsto U_i$ yields a commuting triangle

$$\begin{array}{ccccc} D & & \xrightarrow{\quad f \quad} & & 2^S \\ & {\scriptstyle g}\searrow & & \swarrow{\scriptstyle \sim} & \\ & & 2^{\mathcal{J}} & & \end{array}$$

The result follows. □

Birkhoff duality thus provides a correspondence between distributive lattices D of length n and posets $\mathcal{J}$ of cardinality n. The set $\mathcal{J}$ of join-irreducibles is in bijection with the set $Q_{D,0}$ of vertices. However, the isomorphism $\mathrm{K}(D) \cong \mathbb{Z}^{\oplus \mathcal{J}}$ induced by g does not identify an element $a \in \mathcal{J}$ with the class of the interval $[0, a]$, so from the point of view of Grothendieck groups it is more natural to express Birkhoff duality in terms of vertices, rather than join-irreducibles.

6.4 MAXIMAL DISTRIBUTIVE SUBLATTICES OF ARTINIAN LATTICES

Maximal distributive sublattices of artinian lattices are the lattice analogue of maximal tori of reductive algebraic groups. We study here their basic properties. Our main tool will be the following theorem of Lengvárszky [61, Corollary 5].

Theorem 6.4.1 (Lengvárszky)**.** *Let L be a modular lattice, let D be a distributive sublattice of L, let $a < b$ in D and let K be a distributive sublattice of $[a, b]$ containing $[a, b] \cap D$. Then the sublattice of L generated by D and K is distributive.*

Remark **6.4.2.** A useful special case of Theorem 6.4.1 is when $[a, b] \cap D = \{a, b\}$ and $K = \{a, x, b\}$ for some $x \in [a, b]$ (note that K is distributive in this case). This is also [60, Lemma 1].

For the rest of the section we fix an artinian lattice L.

Proposition 6.4.3. *Every distributive sublattice D of L is contained in a maximal distributive sublattice.*

Proof. This follows from Zorn's lemma, but we present an alternative argument. By applying Remark 6.4.2 repeatedly, we may assume that D contains a chain that is maximal in L, say of length n. Then, by Theorem 6.3.5, there are only finitely many distributive lattices of length n, and they are all finite, so any chain of distributive sublattices of L containing D stabilises. □

In the proof of Proposition 6.4.3 we also deduced the following fact, which is [60, Corollary 2].

PROPOSITION 6.4.4. *Let D be a maximal distributive sublattice of L. Then D and L have the same length.* □

PROPOSITION 6.4.5. *Let D be a maximal distributive sublattice of L and let $a \leq b$ in D. Then $[a, b] \cap D$ is a maximal distributive sublattice of $[a, b]$.*

Proof. Let X be a maximal distributive sublattice of $[a, b]$ containing $[a, b] \cap D$. By Theorem 6.4.1, the sublattice D' of L generated by D and X is distributive. Thus $D = D'$ and $X = D \cap [a, b]$. □

PROPOSITION 6.4.6. *Let D be a maximal distributive sublattice of L. Then L is complemented if and only if D is complemented.*

Note that in that case D is a finite boolean algebra. For the proof, we will need the following useful fact [33, Lemma 4.7].

LEMMA 6.4.7. *If $x \in L$ has a complement in L and both $[0, x]$ and $[x, 1]$ are complemented, then L is complemented.* □

Proof of Proposition 6.4.6. Suppose D is complemented, and take $0 = a_0 < \cdots < a_l = 1$ a maximal chain in D. By repeatedly applying Lemma 6.4.7, since each $[a_i, a_{i+1}]$ is complemented and each a_i has a complement in L, it follows that L is complemented.

Conversely, suppose that L is complemented, and let l be its length. For $l = 0, 1$ the result is trivial. If $l = 2$, take $0 < a < 1$ in D. If $D \setminus \{0, a, 1\} \neq \varnothing$, then any $b \in D \setminus \{0, a, 1\}$ is a complement of a. If $D = \{0, a, 1\}$ and b is a complement of a in L, then $\{0, a, b, 1\}$ is a distributive sublattice of L strictly containing D, a contradiction.

Suppose that $l > 2$ and let $0 < a_1 < a_2 < 1$ in D. Any interval in a complemented lattice is also complemented [12, Theorem 1.9.14], so in particular $[0, a_2]$ and $[a_2, 1]$ are complemented. By induction on the length, $[0, a_2] \cap D$ and $[a_2, 1] \cap D$ are complemented. Therefore, by Lemma 6.4.7, D is complemented if and only if a_2 has a complement in D. Let b be the complement of a_2 in $[a_1, 1] \cap D$, which is complemented by induction hypothesis. Let c be the complement of a_1 in $[0, b] \cap D$, also complemented by induction. Regarding L as a category where there is an arrow

$x \to y$ if $x \leq y$, we have two bicartesian squares

$$\begin{array}{ccccccc}
 & & & & 1 & & \\
 & & & \nearrow & & \nwarrow & \\
 & & b & & & & a_2 \\
 & \nearrow & & \nwarrow & & \nearrow & \\
c & & & & a_1 & & \\
 & \nwarrow & & \nearrow & & & \\
 & & 0 & & & &
\end{array}$$

so the concatenation is also bicartesian, that is, c is a complement of a_2 in D. □

Corollary 6.4.8. *Let D be a maximal distributive sublattice of L and let $a \leq b$ in D. Then $[a, b] \cap D$ is complemented if and only if $[a, b]$ is complemented.*

Proof. This follows directly from propositions Propositions 6.4.5 and 6.4.6. □

6.5 THE DEGENERATION FAN OF AN ARTINIAN LATTICE

In this section we start by defining the notion of *grading* of an artinian lattice L by a set (Definition 6.5.1), and then use it to define the concept of *filtration* of L by a poset P (Definition 6.5.8). The set $\mathbb{Z}\text{-}\mathrm{Filt}(L)$ of filtrations of L by $\mathbb{Z}$ is analogous to the set of $\mathbb{Z}$-filtrations of a point x on a good moduli stack $\mathcal{X}$ (or rather, the other way around), and we show that it also can be given the structure of a formal fan $\mathbf{DF}(L)_\bullet$, that we call the *degeneration fan* of L, exactly in the same way that the degeneration fan $\mathbf{DF}(\mathcal{X}, x)_\bullet$ in the case of stacks enhances the set $\mathbb{Z}\text{-}\mathrm{Filt}(\mathcal{X}, x)$ of filtrations (see Section 2.7). We finish the section by showing that the lattice L can be recovered from the degeneration fan $\mathbf{DF}(L)_\bullet$ together with the natural poset structure on $\mathbb{Z}\text{-}\mathrm{Filt}(L)$. This will be an important point in our approach to relate the iterated balanced filtration of a point on a stack with the HKKP filtration of a normed artinian lattice in Chapter 7.

We fix an artinian lattice L for the rest of this section.

6.5.1 GRADINGS

Definition 6.5.1 (Grading of a lattice by a set)**.** Let P be a set. A P*-grading* of L is a family $f = (x_i)_{i \in P}$ of elements of L indexed by P such that

1. $\bigvee_{i \in P} x_i = 1$; and
2. for all $i \in P$, we have $x_i \wedge x_i' = 0$, where $x_i' = \bigvee_{j \in P \setminus \{i\}} x_j$.

We denote $\Gamma^P(L)$ the set of P-gradings of L.

Lemma 6.5.2. *Let $M \subset L$ be a sublattice that is a finite boolean algebra with atoms $a_1, \ldots, a_l$, and let $x \in L \setminus \{0\}$ such that for all $a \in M$ we have $a \wedge x = 0$. Then the sublattice M' of L generated by M and x is a finite boolean algebra with atoms $a_1, \ldots, a_l, x$.*

Proof. If $a, b \in M$, then

$$(a \vee x) \wedge b \leq (a \vee x) \wedge (a \vee b) = (x \wedge (a \vee b)) \vee a = a,$$

so $a \wedge b \leq (a \vee x) \wedge b \leq a \wedge b$ and thus $(a \vee x) \wedge b = a \wedge b$. We also have

$$(a \vee x) \wedge (b \vee x) = ((a \vee x) \wedge b) \vee x = (a \wedge b) \vee x.$$

Let $c \in M$ be the maximal element of M. The formulas above show that the map $M \times \{0, x\} \to M' \colon (a, z) \mapsto a \vee z$ is an isomorphism of lattices with inverse $y \mapsto (y \wedge c, y \wedge x)$. The result follows. □

Proposition 6.5.3. *Let P be a set and let $g = (x_i)_{i \in P}$ be a family of elements of L such that $1 = \bigvee_{i \in P} x_i$. Then the following are equivalent:*

1. *The family g is a P-grading of L.*
2. *The sublattice M of L generated by the x_i is a finite boolean algebra with set of atoms $\{x_i \mid i \in P,\ x_i \neq 0\}$. In particular, this set is finite.*

Proof. It is clear that (2) implies (1). Assume (1), and let $a_1, \dots, a_l$ be different elements in $\{x_i \mid i \in P,\ x_i \neq 0\}$. We prove by induction on l that the sublattice N generated by $a_1, \dots, a_l$ is a finite boolean algebra with atoms $a_1, \dots, a_l$. This is clear for $l = 1$. For $l > 1$, let N' be the sublattice generated by $a_1, \dots, a_{l-1}$. If $a \in N'$, then $a \wedge a_l \leq a_l' \wedge a_l = 0$, where we are using notation from Definition 6.5.1, so $a \wedge a_l = 0$. The result then follows from Lemma 6.5.2. In particular, l is at most the length of L, so the set $\{x_i \mid i \in P,\ x_i \neq 0\}$ is finite and the result is proven by taking $a_1, \dots, a_l$ to be all the elements of this set. □

Proposition 6.5.4. *Let T be a finite boolean algebra, and let A be an abelian group. Then there is a canonical bijection $\Gamma^A(T) \cong \mathrm{K}(T) \otimes_{\mathbb{Z}} A$. In particular, the set $\Gamma^A(T)$ of A-gradings of T is naturally an abelian group. It is an A-module if A is endowed with the structure of a ring.*

Proof. If $a_1, \dots, a_n$ are the atoms of T, then the $v_i = \overline{[0, a_i]}$ are the vertices of T, and they are a canonical basis of $\mathrm{K}(T)$ by Proposition 6.3.4. Let $g = (x_c)_{c \in A}$ be an A-grading of T. For every atom a_i, there is a unique $c_i \in A$ such that $a_i \leq x_{c_i}$. We have $x_c = \bigvee_{c_i = c} a_i$, so the c_i determine g. Therefore the map that sends an A-grading g as before to the element $\sum_{i=1}^n c_i \cdot v_i$ of $\mathrm{K}(T) \otimes_{\mathbb{Z}} A$ gives the desired bijection. □

Remark **6.5.5.** If A is an abelian group and $\lambda = (\lambda_i)_{i \in A}$ and $\mu = (\mu_i)_{i \in A}$ are A-gradings of a finite boolean algebra T, then the formula

$$\lambda + \mu = \left(\bigvee_{a+b=i} \lambda_a \wedge \mu_b \right)_{i \in A}$$

holds for the sum on $\Gamma^A(T)$. If A is an integral domain and $a \in A \setminus \{0\}$, we also have the formula $a\lambda = (\lambda_{i/a})_{i\in A}$ for the scalar product.

6.5.2 FILTRATIONS

DEFINITION 6.5.6 (Filtration associated to a grading)**.** Let L be an artinian lattice, let P be a poset, and let $g = (x_i)_{i\in P}$ be a P-grading of L. The *P-filtration associated to g* is the map $F^g\colon P \to L$ defined by $i \mapsto F^g_{\geq i} := \bigvee_{j\geq i} x_j$.

Note that F^g is a non-increasing, meaning that for $i \leq j$ in P we have $F^g_{\geq j} \leq F^g_{\geq i}$.

DEFINITION 6.5.7 (Admissible grading)**.** Let D be a distributive artinian lattice and let P be a poset. A P-grading $g \in \Gamma^P(T_D)$ is said to be *D-admissible* if the associated filtration $F^g\colon P \to T_D$ factors through the inclusion $D \hookrightarrow T_D$.

DEFINITION 6.5.8 (Filtration of a lattice by a poset)**.** Let L be an artinian lattice and let P be a poset. A *P-filtration of L* is a map $F\colon P \to L\colon i \mapsto F_{\geq i}$ such that there is a total distributive sublattice D of L and a D-admissible P-grading $g \in \Gamma^P(T_D)$ such that $F^g = F$. We denote $P\text{-}\mathrm{Filt}(L)$ the set of P-filtrations of L.

Implicitly, we are asking that the composition $P \xrightarrow{F^g} D \hookrightarrow L$ equals F. If $F\colon P \to L\colon a \mapsto F_{\geq a}$ is a nonincreasing map, we denote $F_{>c} = \bigvee_{a>c} F_{\geq a}$.

Remark **6.5.9** (Conventions on $\mathbb{R}$-filtrations). In [33], an $\mathbb{R}$-filtration of L is defined to be a sequence $0 = a_0 < a_1 < \cdots < a_n = 1$ in L together with an increasing sequence $c_1 < \cdots < c_n$ of real numbers. In this work, however, an $\mathbb{R}$-filtration consists of the chain $0 = a_0 < a_1 < \cdots < a_n = 1$ in L together with a decreasing sequence $d_1 > \cdots > d_n$ of real numbers. The corresponding map $F\colon\mathbb{R} \to L$ as in Definition 6.5.8 is given by $F_{\geq c} = \bigvee_{d_i\geq c} a_i$.

Our convention is better adapted to the point of view of filtrations on stacks. Setting $d_i = -c_i$ gives a bijection between sets of $\mathbb{R}$-filtrations in the two conventions.

LEMMA 6.5.10. *Let P be a poset and let F be a P-filtration of an artinian lattice L. If $D \subset L$ is a total distributive sublattice of L through which $F\colon P \to L$ factors, then there is a unique grading $g \in \Gamma^P(T_D)$ such that $F = F^g$.*

Proof. Let M be a total distributive sublattice of L and let $(x_c)_{c\in P} \in \Gamma^P(T_M)$ be a grading such that $F = F^g$. Let D' be the total distributive sublattice generated by $F(P)$. Recall from Remark 6.3.7 that $T_{D'}$ is naturally a total sublattice of T_M, and thus $\Gamma^P(T_{D'})$ is naturally a subset of $\Gamma^P(T_M)$. From the equality $x_c = F_{\geq c}\setminus F_{>c} \in T_{D'}$, we see that g belongs to $\Gamma^P(T_{D'})$ and that g is uniquely determined by F. On the other hand, D' is a total sublattice of D, so g is naturally a grading of T_D via the inclusion $\Gamma^P(T_{D'}) \subset \Gamma^P(T_D)$. □

PROPOSITION 6.5.11. *Let L be an artinian lattice, let P be a poset and let $F: P \to L: a \mapsto F_{\geq a}$ be a non-increasing map. Suppose further that P is a lattice. Then F is a P-filtration of L if and only if the following conditions hold:*

1. *The sublattice of L generated by $F(P)$ is distributive.*
2. *There exists $a \in P$ such that $F_{\geq a} = 1$.*
3. *For $a, b \in P$, we have*
$$(F_{\geq a}) \wedge (F_{\geq b}) = F_{\geq a \vee b}.$$
4. *There are $a_1, \ldots, a_m \in P$ such that, for all $a \in P$, we have*
$$F_{\geq a} = \bigvee_{a_i \geq a} F_{\geq a_i}.$$

Alternatively, condition (4) can be replaced by the following:

5. *For every $x \in L$ with $x \neq 0$, there is $m_x \in P$ such that $F^{-1}([x,1]) = \{a \in P \mid a \leq m_x\}$.*

Proof. Suppose that F is a P-filtration. Let D be a total distributive sublattice of L and let $g = (x_c)_{c \in P} \in \Gamma^P(T_D)$ be a D-admissible P-grading of T_D such that $F = F^g$. Since $F(P)$ is contained in D, (1) holds. Let $c_1, \ldots, c_n$ be the elements of P such that $x_{c_i} \neq 0$. Then $F^g_{\geq c_1 \wedge \cdots \wedge c_n} = 1$, so (2) holds. Noting that the $x_c \neq 0$ are the atoms of the finite boolean algebra they generate, we have, for all $a, b \in P$,

$$F^g_{\geq a} \wedge F^g_{\geq b} = \left(\bigvee_{c \geq a} x_c\right) \wedge \left(\bigvee_{d \geq b} x_d\right) = \bigvee_{\substack{c \geq a \\ d \geq b}} x_c \wedge x_d = \bigvee_{c \geq a \vee b} x_c = F^g_{\geq a \vee b},$$

so (3) holds. From the definition it follows that $F^g_{\geq c} = \bigvee_{c_i \geq c} F^g_{\geq c_i}$, so (4) holds. For $x \in L$ and $x \neq 0$, let $I_x \subset \{1, \ldots, n\}$ be such that $\bigvee_{i \in I_x} x_{c_i}$ is the smallest element in $D_F \cap [x, 1]$, where D_F is the sublattice of D generated by $F(P)$. Since $x \neq 0$, the set I_x must be nonempty. Let $m_x = \bigwedge_{i \in I_x} c_i$. Then $F^{-1}([x, 1]) = \{c \in P \mid c \leq m_x\}$, so (5) holds.

Conversely, suppose that $F: P \to L$ is a nonincreasing map such that conditions (1) and (2), (3) hold, and that either (4) or (5) holds. Replacing L by a total distributive sublattice containing $F(P)$, which exists by (1) and (2), we may assume that L is distributive. By further replacing L by T_L we may assume that L is a finite boolean algebra. If (5) holds, then (4) holds with the set of a_i's taken to be the set of elements of P of the form m_x, with $x \in L$.

For $c \in P$, let $x_c = F_{\geq c} \setminus F_{>c} \in T_L$. We denote $\operatorname{supp} F := \{c \in P \mid x_c \neq 0\}$ the *support* of F. Let $c, d \in \operatorname{supp} F$ with $F_{\geq c} = F_{\geq d}$. Then $F_{\geq c} = F_{\geq c} \wedge F_{\geq d} = F_{\geq c \vee d}$. Since $c \in \operatorname{supp} F$, we must have $c \vee d = c$, so $d \leq c$. Likewise, $c \leq d$, so $c = d$. Since L is finite, this shows that $\operatorname{supp} F$ is finite.

We now prove that, for all $c \in P$, we have

$$F_{\geq c} = \bigvee_{\substack{d \in \operatorname{supp} F \\ d \geq c}} F_{\geq d}. \tag{6.4}$$

Let $a_1, \ldots, a_m \in P$ be as in (4) with m minimal. For every $i = 1, \ldots, m$, we have $F_{>a_i} = \bigvee_{a_j > a_i} F_{\geq a_j}$, so if we had $F_{\geq a_i} = F_{>a_i}$, then for all $c \in P$ we would have

$$F_{\geq c} = \bigvee_{\substack{a_j \geq c \\ j \neq i}} F_{\geq a_j},$$

contradicting minimality of m. Therefore every a_i is in $\operatorname{supp} F$. Conversely, if $c \in \operatorname{supp} F$, then $\{a_i \mid a_i \geq c\} \neq \{a_i \mid a_i > c\}$, so c must be one of the a_i. Therefore $\operatorname{supp} F = \{a_1, \ldots, a_m\}$ and (6.4) holds for all $c \in P$ by (4).

Consider the family $g = (x_c)_{c \in P}$. By (2) and the previous paragraph, we have $\bigvee_{c \in P} x_c = 1$. For $c, d \in \operatorname{supp} F$ different, we have

$$\begin{aligned} x_c \wedge x_d &= (F_{\geq c} \setminus F_{>c}) \wedge (F_{\geq d} \setminus F_{>d}) = (F_{\geq c} \wedge F_{\geq d}) \setminus (F_{>c} \vee F_{>d}) \\ &= (F_{\geq c \vee d}) \setminus (F_{>c} \vee F_{>d}) = 0, \end{aligned}$$

by (3) and because we have the inequality $F_{\geq c \vee d} \leq F_{>c} \vee F_{>d}$ from the fact that either $c \vee d > c$ or $c \vee d > d$. By distributivity of L, this implies that the x_c are the atoms of the finite boolean algebra they generate. Therefore g is a grading, and we have $F = F^g$ by (6.4), so F is indeed a filtration. □

Example **6.5.12.** Let $L = \{0, 1\}$, and consider $\mathbb{Q}$ with the standard order. Then the map $F\colon \mathbb{Q} \to L$ given by

$$c \mapsto \begin{cases} 0, & c > \sqrt{2}, \\ 1, & c < \sqrt{2}. \end{cases}$$

is *not* a $\mathbb{Q}$-filtration of L, even though it is nonincreasing and it satisfies (1), (2) and (3). For another non-example, consider $\mathbb{R}^2$ with the standard product order. Then the map $G\colon \mathbb{R}^2 \to L$ given by

$$(c_1, c_2) \mapsto \begin{cases} 0, & c_1 > 0, \\ 1, & c_1 \leq 0. \end{cases}$$

is not an $\mathbb{R}^2$-filtration.

Remark **6.5.13.** If P is a poset, F is a P-filtration of an artinian lattice L, and $a \in L$, then we say that a is a *jump* of F if $F_{>a} < F_{\geq a}$. The set of jumps of F is denoted $\operatorname{supp} F$ and called the *support* of F. If $F = F^g$ for $g = (x_c)_{c \in P} \in \Gamma^P(T_D)$, with D a total distributive sublattice of L, then $\operatorname{supp} F = \{c \in P \mid x_c \neq 0\}$. If $a_1, \dots, a_m \in P$ satisfy condition (4) in Proposition 6.5.11 with m minimal, then we have seen in the proof of Proposition 6.5.11 that $\operatorname{supp} F = \{a_1, \dots, a_m\}$.

Definition 6.5.14 (Pushforward of filtrations along a map of posets)**.** Let L be an artinian lattice, let P, P' be posets, let $h\colon P \to P'$ be an order-preserving map and let F be a P-filtration of L. The map $h_* F\colon P' \to L$ defined by

$$(h_* F)_{\geq b} = \bigvee_{h(a) \geq b} F_{\geq a}$$

is a P'-filtration of L, called the *pushforward* of F along h.

Proof that h_*F is a P'-filtration. Let D be a total distributive sublattice of L and let $g = (x_c)_{c \in P} \in \Gamma^P(T_D)$ be a D-admissible P-grading such that $F^g = F$. Let $h_* g = (y_{c'})_{c' \in P'}$ be the P'-grading of T_D given by $y_{c'} = \bigvee_{\substack{c \in P \\ h(c) = c'}} x_c$. We have, for $b \in P'$,

$$(h_* F)_{\geq b} = \bigvee_{\substack{a \in P \\ h(a) \geq b}} F_{\geq a} = \bigvee_{\substack{a \in P \\ h(a) \geq b}} \bigvee_{\substack{c \in P \\ c \geq a}} x_c = \bigvee_{\substack{c' \in P' \\ c' \geq b}} \bigvee_{\substack{c \in P \\ h(c) = c'}} x_c = \bigvee_{\substack{c' \in P' \\ c' \geq b}} y_{c'} = (F^{h_* g})_{\geq b}.$$

Therefore $h_* F = F^{h_* g}$, so $h_* F$ is a P'-filtration. □

6.5.3 THE DEGENERATION FAN OF AN ARTINIAN LATTICE

In this section we show how for an artinian lattice L, the sets $\mathbb{Z}^n\text{-}\mathrm{Filt}(L)$ of $\mathbb{Z}^n$-filtrations for different n can be arranged into a formal fan $\mathbf{DF}(L)_\bullet$, that we call the *degeneration fan* of L in analogy with the degeneration fan of a point in an algebraic stack (see Section 2.7). We will see in Proposition 7.1.3 that if L is the lattice of subrepresentations of a finite dimensional representation E of a quiver Q over a field k, then the degeneration fan of L is canonically isomorphic to the degeneration fan $\mathbf{DF}(\mathcal{R}ep(Q), E)_\bullet$ of the stack $\mathcal{R}ep(Q)$ of representations of Q at the point E.

The degeneration fan of L has similar convexity properties to the degeneration fan of a point in a good moduli stack. We can see $\mathbf{DF}(L)_\bullet$ as a structure similar to that of spherical building, and in fact we will see that $\mathbf{DF}(L)_\bullet$ is determined by the set $\mathbb{Z}\text{-}\mathrm{Filt}(L)$ of filtrations together with the sum of filtrations and the data of the *appartments* of $\mathbb{Z}\text{-}\mathrm{Filt}(L)$.

For the discussion of degeneration fans of lattices, we do not need to restrict ourselves to $\mathbb{Z}^n$-filtrations. We will more generally consider A^n-filtrations, where A

is a subring of $\mathbb{R}$ that we fix for the rest of the section, endowed with the inherited order. We also fix an artinian lattice L.

DEFINITION 6.5.15 (Pullback of A^n-filtrations)**.** Let F be a A^n-filtration of L and let $h\colon A^l \to A^n$ be an A-linear order-preserving map. Let $h^\vee\colon A^n \to A^l$ be the dual map of h. We define the *pullback of F along h* to be the A^l-filtration $h^*F := (h^\vee)_*F$ of L.

Remark **6.5.16.** For a linear map $h\colon A^l \to A^n$, we have that h is order-preserving if and only if $h\left(A^l_{\geq 0}\right) \subset A^n_{\geq 0}$.

Recall from Section 2.7 that the category $\mathbf{Cone_A}$ has objects the A-modules A^n for some $n \in \mathbb{N}$ and morphisms the A-linear order-preserving maps between them. The category of A-linear formal fans (Definition 2.7.2) is the category of presheaves of sets on $\mathbf{Cone_A}$.

DEFINITION 6.5.17 (Degeneration fan of a lattice)**.** The *A-linear degeneration fan* $\mathbf{DF}^A(L)_\bullet$ of L is the A-linear formal fan defined as follows:

- For $n \in \mathbb{N}$, we set $\mathbf{DF}^A(L)_n := A^n\text{-}\mathrm{Filt}(L)$.
- If $h\colon A^l \to A^n$ is a morphism in $\mathbf{Cone_A}$, then the associated map $\mathbf{DF}^A(L)_n \to \mathbf{DF}^A(L)_l$ is given by $F \mapsto h^*F$.

We abbreviate $\mathbf{DF}(L)_\bullet := \mathbf{DF}^{\mathbb{Z}}(L)_\bullet$ and simply call it the *degeneration fan* of L.

The elements of $\mathbf{DF}(L)_1$ are called *integral filtrations* of L, and the elements of $\mathbf{DF}^{\mathbb{Q}}(L)_1$ are called *rational filtrations* of L. The reader can convince themselves that $\mathbf{DF}^{\mathbb{Q}}(L)_\bullet$ is actually the rational formal fan associated to $\mathbf{DF}(L)_\bullet$ as in section Section 2.7.

DEFINITION 6.5.18. Let $F_1, \ldots, F_n \in \mathbf{DF}^A(L)_1$. We say that $F_1, \ldots, F_n$ *commute* if the sublattice of L they generate is distributive. In that case, we define the *box sum* by the formula

$$(F_1 \boxplus \cdots \boxplus F_n)_{\geq d} = (F_1)_{\geq d_1} \wedge \cdots \wedge (F_n)_{\geq d_n},$$

for $d \in A^n$.

LEMMA 6.5.19. *For commuting A-filtrations $F_1, \ldots, F_n \in \mathbf{DF}^A(L)_1$, the box sum $F_1 \boxplus \cdots \boxplus F_n$ is an A^n-filtration*

Proof. Let D be a total distributive sublattice of L through which $F_1, \ldots, F_n$ factor and, for $i \in \{1, \ldots, n\}$, let $g_i = (x^i_c)_{c \in A} \in \Gamma^A(T_D)$ be a grading with $F_i = F^{g_i}$ (Lemma 6.5.10). Let $g_1 \boxplus \cdots \boxplus g_n = (x^1_{c_1} \wedge \cdots \wedge x^n_{c_n})_{(c_1, \ldots, c_n) \in A^n} \in \Gamma^{A^n}(T_D)$, which is readily checked to be an A^n-grading of T_D. We have $F^{g_1 \boxplus \cdots \boxplus g_n} = F_1 \boxplus \cdots \boxplus F_n$, so the box sum is indeed an A^n-filtration. $\square$

Remark **6.5.20.** The sublattice generated by two A-filtrations F_1 and F_2 is always distributive [12, Theorem III.7.9]. In other words, any two A-filtrations commute and, in particular, their box sum can be considered. This can be seen as a convexity property of $\mathbf{DF}(L)_\bullet$, and it is the lattice analogue of the fact that in a reductive algebraic group G the intersection of any two parabolic subgroups contains a maximal torus of G [20, Theorem 10.3.6].

Definition 6.5.21. Let $F_1, F_2 \in \mathbf{DF}^A(L)_1$. We define the *sum* of F_1 and F_2 as $F_1 + F_2 := \binom{1}{1}^* (F_1 \boxplus F_2)$. Here, $\binom{1}{1}$ denotes the linear map $A \to A^2$ with matrix $\binom{1}{1}$.

The following computation readily follows from the definitions.

Proposition 6.5.22. *Let $F, G \in \mathbf{DF}^A(L)_1$. Then the sum $F + G$ is given by the formula*

$$(F+G)_{\geq c} = \bigvee_{a+b \geq c} F_{\geq a} \wedge G_{\geq b} = \bigvee_{a+b=c} F_{\geq a} \wedge G_{\geq b}, \quad c \in A.$$

If $a \in A_{\geq 0}$, multiplication by a defines an order-preserving map $h\colon A^n \to A^n$. We define the scalar multiplication $aF := h^* F$. By definition, if $a \neq 0$ then we have $(aF)_{\geq b} = \bigvee_{ac \geq b} F_{\geq c} = F_{\geq b/a}$, while $0F = 0 := F^0$, the zero filtration, given by

$$0_{\geq c} = \begin{cases} 0, & c > 0, \\ 1, & c \leq 0. \end{cases}$$

Remark **6.5.23.** Our definition of the sum of two $\mathbb{R}$-filtrations agrees with the one introduced in [22, 2.2.3].

Recall (Proposition 6.5.4) that for a distributive sublattice D of L, the set of gradings $\Gamma^A(T_D)$ is bijective to $\mathrm{K}(D) \otimes_{\mathbb{Z}} A$, so it is naturally an A-module.

Proposition 6.5.24. *Let F and G be A-filtrations of L and let $a, b \geq 0$ be elements of A. Let D be a total distributive sublattice of L through which F and G factor and let $f, g \in \Gamma^A(T_D)$ be gradings with $F = F^f$ and $G = F^g$. Then $aF + bG = F^{af+bg}$.*

Proof. Since $Q_{D,0}$ is a basis of the Grothendieck group $\mathrm{K}(D) \otimes_{\mathbb{Z}} A$ (Proposition 6.3.4), we can identify f with a map $f\colon Q_{D,0} \to A$ (Proposition 6.5.4), and similarly with g.

With this notation, $F_{\geq c} = \bigvee_{\substack{v \in Q_{D,0} \\ f(v) \geq c}} v$, and the analogue formula holds for G. Then

$$(F+G)_{\geq c} = \bigvee_{c_1+c_2 \geq c} F_{\geq c_1} \wedge G_{\geq c_2}$$

$$= \bigvee_{c_1+c_2 \geq c} \left(\bigvee_{\substack{v \in Q_{D,0} \\ f(v) \geq c_1}} v \right) \wedge \left(\bigvee_{\substack{v \in Q_{D,0} \\ g(v) \geq c_2}} v \right) = \bigvee_{\substack{v \in Q_{D,0} \\ f(v)+g(v) \geq c}} v = \left(F^{f+g} \right)_{\geq c}$$

for all $c \in A$. If $a \in A \setminus \{0\}$, then

$$(aF)_{\geq c} = F_{\geq c/a} = \bigvee_{\substack{v \in Q_{D,0} \\ f(v) \geq c/a}} v = \bigvee_{\substack{v \in Q_{D,0} \\ af(v) \geq c}} v = \left(F^{af} \right)_{\geq c}$$

for all $c \in A$; while clearly $0F = 0 = F^0 = F^{0f}$. The result follows. □

Remark **6.5.25.** Addition of A-filtrations is in general not associative (see Remark 6.5.33). However, if $F_1, \dots, F_n \in A\text{-}\mathrm{Filt}(L)$ commute, and $a_1, \dots, a_n \in A_{\geq 0}$, then by Proposition 6.5.24 the expression $a_1 F_1 + \cdots + a_n F_n \in A\text{-}\mathrm{Filt}(L)$ is defined unambiguously.

Definition 6.5.26. Let $D_\bullet$ be an A-linear formal fan, and $F \in D_n$. The *components* $v_1 F, \dots, v_n F \in D_1$ of F are the pullbacks of F along the maps $A \to A^n$ given by the standard basis of A^n.

The degeneration fan of L has similar formal properties to that of a point in a good moduli stack. Compare the following result with Propositions 2.7.6 and 2.7.11.

Proposition 6.5.27. *For all $n \in \mathbb{N}$, the assignment $F \mapsto (v_1 F, \dots, v_n F)$ defines a bijection*

$$\mathbf{DF}^A(L)_n \xrightarrow{\sim} \left\{ (F_1, \dots, F_n) \in (A\text{-}\mathrm{Filt}(L))^n \mid F_1, \dots, F_n \text{ commute} \right\}$$

with inverse $(F_1, \dots, F_n) \mapsto F_1 \boxplus \cdots \boxplus F_n$. Under this bijection, the pullback along a nondecreasing linear map $h \colon A^m \to A^n$ with matrix (a_{ij}) is given by the formula

$$h^*(F_1, \dots, F_n) = \left(\sum_i a_{i1} F_i, \dots, \sum_i a_{im} F_i \right).$$

Proof. Injectivity follows from the formula $F = v_1 F \boxplus \cdots \boxplus v_n F$, for $F \in \mathbf{DF}^A(L)_n$. Clearly, the $v_i F$ commute. Moreover, if $F_1, \dots, F_n$ commute, then it follows from the definitions that $v_i\,(F_1 \boxplus \cdots \boxplus F_n) = F_i$ for $i = 1, \dots n$. Also, for h as in the statement,

$$v_j\,(h^*(F_1 \boxplus \ldots \boxplus F_n))_{\geq c} = \bigvee_{\substack{(c_1,\dots,c_m) \in A^m \\ c_j \geq c}} \ \bigvee_{\substack{(d_1,\dots,d_n) \in A^n \\ h^\vee(d_1,\dots,d_n) \geq (c_1,\dots,c_m)}} (F_1)_{\geq d_1} \wedge \cdots \wedge (F_n)_{\geq d_n}$$

$$= \bigvee_{\substack{(d_1,\dots,d_n) \in A^n \\ \sum_i a_{ij} d_i \geq c}} (F_1)_{\geq d_1} \wedge \cdots \wedge (F_n)_{\geq d_n} = \left(\sum_i a_{ij} F_i \right)_{\geq c}.$$

The last equality above can be seen inductively:

$$\bigvee_{\substack{(d_1,\ldots,d_n)\in A^n\\ \sum_i a_{ij}d_i\geq c}} (F_1)_{\geq d_1}\wedge\cdots\wedge(F_n)_{\geq d_n}$$

$$= \bigvee_{c_1+c_2\geq c}\left(\bigvee_{\substack{(d_1,\ldots,d_{n-1})\in A^{n-1}\\ \sum_{i=1}^{n-1} a_{ij}d_i\geq c_1}} (F_1)_{\geq d_1}\wedge\cdots\wedge(F_{n-1})_{\geq d_{n-1}}\right)\wedge\left(\bigvee_{\substack{d_n\in A\\ a_{nj}d_n\geq c_2}}(F_n)_{\geq d_n}\right) =$$

$$\bigvee_{c_1+c_2\geq c}\left(a_{1j}F_1+\cdots a_{(n-1)j}F_{n-1}\right)_{\geq c_1}\wedge(a_{nj}F_n)_{\geq c_2} = \left(\sum_i a_{ij}F_i\right)_{\geq c}.$$

□

From Proposition 6.5.27 we see that the degeneration fan $\mathbf{DF}^A(L)_\bullet$ determines and is completely determined by:

1. the set $\mathbf{DF}^A(L)_1$ of A-filtrations of L;
2. the sum of A-filtrations and the multiplication by scalars in $A_{\geq 0}$; and
3. the data of which finite sequences $F_1,\ldots,F_n$ of A-filtrations of L commute.

The last piece of data leads to the concept of appartment.

Definition 6.5.28 (Appartment)**.** An *appartment* of $\mathbf{DF}^A(L)_\bullet$ is a subset $S\subset\mathbf{DF}^A(L)_1$ such that any finite sequence $F_1,\ldots,F_n$ of A-filtrations in S commutes and such that S is maximal with this property.

Proposition 6.5.29. *For every maximal distributive sublattice $D\subset L$, the set of A-filtrations of D is an appartment $\mathbf{DF}^A(D)_1\subset\mathbf{DF}^A(L)$. Moreover every appartment of $\mathbf{DF}^A(L)_\bullet$ is of this form.*

Proof. In a distributive lattice, any finite sequence of filtrations commutes, so the first statement is clear. If $S\subset\mathbf{DF}^A(L)_1$ is an appartment, then the lattice D generated by the elements $F_{\geq c}$ for $F\in S$ and $c\in A$ is distributive, and $S\subset\mathbf{DF}^A(D)_1$. By maximality of S, we have $S=\mathbf{DF}^A(D)_1$, and again by maximality we must have that D is a maximal distributive sublattice of L. □

Remark **6.5.30.** In [22, 2.2.6], the set of $\mathbb{R}$-filtrations $\mathbb{R}\text{-}\mathrm{Filt}(L)$ of an artinian lattice L is studied and it is given a spherical building like structure where the appartments are defined to be subsets of the form $\mathbb{R}\text{-}\mathrm{Filt}(D)$ with $D\subset L$ a maximal distributive sublattice. Cornut also defines a notion of chamber [22, 2.2.2], but we note that the degeneration fan does not contain enough information to recover the chambers.

The degeneration fan of L contains information about L. For example, $\mathbf{DF}(L)_\bullet$ can detect whether L is complemented (Proposition 6.5.34). To prove this, we introduce a couple of definitions.

DEFINITION 6.5.31 (Connected formal fan)**.** We say that a formal fan $D_\bullet$ is *connected* if D_0 has a single element, that we denote 0. If $f\colon [n] \to [0]$ is the unique map between $[n]$ and $[0]$, we also denote $0 = f^*0 \in D_n$.

Note that $\mathbf{DF}^A(L)_\bullet$ is connected. The zero n-filtration is given by, for $c \in A^n$,

$$0_{\geq c} = \begin{cases} 1, & c \leq 0^n \\ 0, & \text{else.} \end{cases}$$

DEFINITION 6.5.32. If $F \in \mathbf{DF}^A(L)_1$ is a filtration, an *opposite* of F is an element $G \in \mathbf{DF}^A(L)_1$ such that $F + G = 0$.

Remark **6.5.33.** A filtration may have several opposites, as the following example shows. Suppose that L is the lattice of vector subspaces of $\mathbb{C}^2$. Let F be the filtration given by $\mathbb{C}\langle(1,0)\rangle \leq \mathbb{C}^2$ and weights $1, 0$. Let $v \in \mathbb{C}^2 \setminus \mathbb{C}\langle(1,0)\rangle$ and let G_v be the filtration given by $\mathbb{C}\langle v\rangle \leq \mathbb{C}^2$ and weights $0, -1$. Then $F + G_v = 0$. In fact every opposite of F is of the form G_v for some $v \in \mathbb{C}^2 \setminus \mathbb{C}\langle(1,0)\rangle$. This also shows that the sum filtrations is in general not associative, since associativity would imply uniqueness of the opposite filtration.

We can now state the following proposition, which is the lattice analogue of Proposition 2.7.14.

PROPOSITION 6.5.34. *The artinian lattice L is complemented if and only if every element of $\mathbf{DF}^A(L)_1$ has an opposite.*

Proof. If L is complemented and F is an A-filtration of L, we can find a maximal distributive sublattice $D \subset L$ such that $F \in \mathbf{DF}^A(D)_1$. Note that D is complemented by Proposition 6.4.6 and thus a boolean algebra. If $x_1, \dots, x_n$ are the atoms of D, an A-grading g on D is a map $g\colon \{x_1, \dots, x_n\} \to A$, so the set of A-gradings of D is A^n. The map $A^n \to \mathbf{DF}^A(D)_1\colon g \mapsto F^g$ is a bijection by Lemma 6.5.10, and we have $F^{g_1+g_2} = F^{g_1} + F^{g_2}$ by Proposition 6.5.24. Thus every element of $\mathbf{DF}^A(D)_1$ has a (unique) opposite in $\mathbf{DF}^A(D)_1$, and in particular F has an opposite in $\mathbf{DF}^A(L)_1$.

Conversely, suppose that every filtration of L has an opposite, and let $x \in L$.

Consider the filtration F of L given by

$$F_{\geq c} = \begin{cases} 0, & c > 1 \\ x, & 0 < c \leq 1 \\ 1, & c \leq 0. \end{cases}$$

Let $G \in \mathbf{DF}^A(L)_1$ be an opposite of F. It follows from Proposition 6.5.22 that $x \wedge G_{\geq 0} = 0$ and $x \vee G_{\geq 0} = 1$. Thus x has a complement in L. □

6.5.4 RECOVERING A LATTICE FROM ITS FILTRATIONS

We now show that an artinian lattice L is canonically determined by its degeneration fan $\mathbf{DF}(L)_\bullet$ and the natural poset structure on the set of its $\mathbb{Z}$-filtrations, even though the degeneration fan $\mathbf{DF}(L)_\bullet$ alone does not have enough information to recover L. This observation will be important in our approach to compare the iterated balanced filtration for stacks and the HKKP filtration for lattices in Chapter 7.

Definition 6.5.35. Let $F, G \in \mathbf{DF}(L)_1$. We say $F \leq G$ if for all $c \in \mathbb{Z}$ we have $F_{\geq c} \leq G_{\geq c}$.

This defines a poset structure on $\mathbf{DF}(L)_1$.

Definition 6.5.36. A *formal fan with relation* is a pair $(D_\bullet, \leq)$ where $D_\bullet$ is a connected formal fan and $\leq$ is a relation on D_1.

If $(D_\bullet, \leq)$ is a formal fan with relation, we define the set

$$L_D = \{F \in D_1 \mid F \geq 0 \text{ and } \forall G \in D_1,\ 0 \leq 2G \leq F \implies G = 0\},$$

endowed with the relation $\leq$ inherited from D_1.

Definition 6.5.35 gives a formal fan with relation $(\mathbf{DF}(L)_\bullet, \leq)$.

Proposition 6.5.37. *Let $(D_\bullet, \leq)$ be a formal fan with relation. Suppose that there is an artinian lattice M and an isomorphism $\varphi\colon (D_\bullet, \leq) \to (\mathbf{DF}(M)_\bullet, \leq)$ of formal fans with relation. Then $(L_D, \leq)$ is an artinian lattice isomorphic to M. Moreover, there is canonical isomorphism $(\mathbf{DF}(L_D)_\bullet, \leq) \to (D_\bullet, \leq)$ of formal fans with relation, independent of M and φ.*

Proof. For every $a \in M$, we denote $f(a) \in \mathbf{DF}(M)_1$ the $\mathbb{Z}$-filtration of M given by

$$f(a)_{\geq c} = \begin{cases} 0, & c > 1, \\ a, & 0 < c \leq 1, \\ 1, & c \leq 0. \end{cases}$$

If $a \leq b$, then $f(a) \leq f(b)$, so the map $f\colon M \to \mathbf{DF}(M)_1\colon a \mapsto f(a)$ is an injection of posets.

Claim **6.5.38.** The image of f is $L_{\mathbf{DF}(M)}$.

Indeed, let $f \in L_{\mathbf{DF}(M)}$ with jumps $c_0 < \ldots < c_n$ (in $\mathbb{Z}$). Since $F \geq 0$, we have $c_i \geq 0$ for all i. Let $a = F_{\geq c_n} > 0$. Then $rf(a) \leq F$ for all $r = 1, \ldots, c_n$, so c_n is 0 or 1. If $c_n = 0$, then $n = 0$ and $F = 0 = f(0)$. If $c_n = 1$, then $F = f(a)$. Thus $L_{\mathbf{DF}(M)} \subset f(M)$.

Let $a \in M$ and suppose there is $G \in \mathbf{DF}(M)_1$ with $0 \leq G$ and $2G \leq f(a)$ but $G \neq 0$. If $c_0 < \cdots < c_n$ are the jumps of G, then $c_n > 0$, since $G \neq 0$. Thus $2c_n$ is a jump of $2G$, but

$$0 \leq (2G)_{\geq 2c_n} \leq F_{\geq 2c_n} = 0,$$

since $2c_n > 1$. Therefore $(2G)_{\geq 2c_n} = 0$, contradicting that $2c_n$ is a jump of $2G$. It follows that $f(a) \in L_{\mathbf{DF}(M)}$, and thus the claim is established.

Therefore f gives an isomorphism $M \cong L_{\mathbf{DF}(M)}$. Since φ induces an isomorphism $L_{\mathbf{DF}(M)} \cong L_D$, we have an isomorphism $M \cong L_D$ of sets with relation, and thus L_D is an artinian lattice.

Let $F \in \mathbf{DF}(M)_1$ be given by jumps $c_0 < \cdots < c_n$ in $\mathbb{Z}$ and a chain $1 > a_1 > \cdots > a_n > 0$ in M. Then $F = c_0 f(1) + (c_1 - c_0) f(a_1) + \cdots + (c_n - c_{n-1}) f(a_n)$ and the sum is associative because $f(1), f(a_1), \ldots, f(a_n)$ live inside a common distributive sublattice of M. We can use this observation to define an isomorphism $\alpha\colon \mathbf{DF}(L_D)_\bullet \to D_\bullet$ independently of φ.

Note that the map $v^{(n)}\colon D_n \to (D_1)^n$ that maps an element $F \in D_n$ to its components $(v_1 F, \ldots, v_n F) \in (D_1)^n$ is injective for all $n \in \mathbb{N}$ and bijective for $n = 2$ by Proposition 6.5.27 and Remark 6.5.20. Therefore we can define sums and box sums of elements of D_1 as follows. We say that $F_1, \ldots, F_n \in D_1$ *commute* if $(F_1, \ldots, F_n)$ is in the image of $v^{(n)}$. In that case we denote $F_1 \boxplus \cdots \boxplus F_n$ the unique preimage. We define the *sum* of $F, G \in D_1$ as $F + G := \binom{1}{1}^*(F \boxplus G)$.

If $F \in \mathbf{DF}(L_D)_1$ is given by jumps $c_0 < \cdots < c_n$ in $\mathbb{Z}$ and a chain $1 > b_1 > \cdots > b_n > 0$ in L_D, we define $\alpha_1(F) = c_0 1 + (c_1 - c_0) b_1 + \cdots + (c_n - c_{n-1}) b_n$. The sum is associative because it is so in the case $D_\bullet = \mathbf{DF}(M)_\bullet$, and the map $\alpha_1\colon \mathbf{DF}(L_D)_1 \to D_1$ is an isomorphism since it is so in the case $D_\bullet = \mathbf{DF}(M)_\bullet$. We define $\alpha_n(F_1 \boxplus \cdots \boxplus F_n) = \alpha_1(F_1) \boxplus \cdots \boxplus \alpha_1(F_n)$. This is well defined and a bijection between $\mathbf{DF}(L_D)_n$ and D_n since it is so in the case $D_\bullet = \mathbf{DF}(M)_\bullet$. Thus $\alpha\colon (\mathbf{DF}(L_D)_\bullet, \leq) \to (D_\bullet, \leq)$ is an isomorphism of formal fans with relation and it is defined solely in terms of $(D_\bullet, \leq)$. □

Remark **6.5.39.** It is clear from the definitions and from Proposition 6.5.37 that the poset $\mathbb{Z}$ - Filt(L) of an artinian lattice L, endowed with the multiplication of scalars in $\mathbb{Z}_{\geq 0}$, is enough to recover L, and that the full structure of formal fan of $\mathbf{DF}(L)_{\bullet}$ is not needed for this. However, for our applications to Chapter 7, the formulation using formal fans is more convenient.

6.6 NORMED LATTICES AND THE HKKP FILTRATION

In this section, we prove the main result of this chapter, which is a new characterisation of the *weight filtration* of Haiden–Katzarkov–Kontsevich–Pandit (called the *HKKP filtration* here) [33] as the unique minimiser of a certain norm function (Definition 6.6.13) on the set of so-called paracomplemented filtrations (Theorems 6.6.15 and 6.6.26). We fix an artinian lattice L and a subring A of $\mathbb{R}$, endowed with the induced order.

DEFINITION 6.6.1 (Complementedness)**.** Let $F \in \mathbf{DF}^A(L)_1$ be an A-filtration of L. We define the *complementedness* $\langle F, \wr \rangle$ of F to be

$$\langle F, \wr \rangle := \sup \{ b \in A_{\geq 0} \mid \forall c \in A,\ [F_{\geq c+b}, F_{\geq c}] \text{ is complemented} \} \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

The filtration F is said to be *paracomplemented* if $\langle F, \wr \rangle \geq 1$. We denote $\mathcal{B}_A(L)$ the set of all paracomplemented A-filtrations of L

The notion of complementedness for lattices is an analogue of that for states (Definition 5.1.13).

Remark **6.6.2.** The notion of paracomplementedness for $\mathbb{R}$-filtrations coincides with the original definition [33, Section 4.3].

LEMMA 6.6.3. *Let D be a maximal distributive sublattice of L and let $F \in \mathbf{DF}^A(L)_1$ be a filtration that factors through D. If $\langle F, \wr_D \rangle$ denotes the complementedness of F seen as a filtration of D, then $\langle F, \wr_D \rangle = \langle F, \wr \rangle$.*

Proof. This follows directly from Corollary 6.4.8. □

As the lemma suggests, in order to understand complementedness for L it will be key to study first the distributive case.

DEFINITION 6.6.4. Let M be a finite boolean algebra. We define the module $\Gamma_A(M)$ of A-*linear characters* of M to be the dual of the A-module $\Gamma^A(M)$ of A-gradings of M.

Let D be a distributive artinian lattice. The family $(v)_{v\in Q_{D,0}}$ of vertices of D is a canonical basis of $\Gamma^A(T_D)$. We denote $(v^\vee)_{v\in Q_{D,0}}$ the basis of $\Gamma_A(T_D)$ dual to $(v)_{v\in Q_{D,0}}$.

Definition 6.6.5. Let D be a distributive artinian lattice. The *state* of D is the finite subset

$$\Xi_D = \left\{ v_2^\vee - v_1^\vee \;\middle|\; v_1, v_2 \in Q_{D,0} \text{ and there is an arrow } v_1 \to v_2 \text{ in } Q_{D,1} \right\}$$

of $\Gamma_{\mathbb{Z}}(T_D)$.

Remark **6.6.6.** There is a canonical injection $\Gamma_{\mathbb{Z}}(T_D) \subset \Gamma_A(T_D)$, and thus we can see the state Ξ_D as a subset of $\Gamma_A(T_D)$ as well.

Every filtration of D can be seen as a filtration of T_D. Thus there is an injection $\mathbf{DF}^A(D)_1 \to \mathbf{DF}^A(T_D)_1 = \Gamma^A(T_D)$ through which we see $\mathbf{DF}^A(D)_1$ as a subset of $\Gamma^A(T_D)$.

Proposition 6.6.7. *Let D be a distributive artinian lattice. We have the equality*

$$\mathbf{DF}^A(D)_1 = \{\lambda \in \Gamma^A(T_D) \mid \; \forall w \in \Xi_D,\ \langle \lambda, w\rangle \geq 0\}$$

via the canonical injection $\mathbf{DF}^A(D)_1 \to \Gamma^A(T_D)$.

Proof. We see D as the lattice of closed subgraphs of Q_D via the injection $D \hookrightarrow T_D$ (Theorem 6.3.5). If $\lambda \in \Gamma^A(T_D)$ is a grading and $c \in A$ we have

$$F^\lambda_{\geq c} = \bigvee_{\substack{v\in Q_{D,0}\\ \langle \lambda, v^\vee\rangle \geq c}} v.$$

This is an element of D precisely if for each arrow $v_1 \to v_2$ in Q_D we have

$$\langle \lambda, v_1^\vee\rangle \geq c \implies \langle \lambda, v_2^\vee\rangle \geq c$$

This condition holds for all $c \in A$ if and only if $\langle \lambda, v_1^\vee\rangle \leq \langle \lambda, v_2^\vee\rangle$. Thus F^λ is a filtration of D precisely when $\langle \lambda, v_2^\vee - v_1^\vee\rangle \geq 0$ for all arrows $v_1 \to v_2$. □

Remark **6.6.8.** Proposition 6.6.7 precisely says that the filtrations of D coincide with the filtrations of the trivially polarised state $(\Gamma_A(T_D), \Xi_D, 0)$ in the sense of Definition 5.1.3, while Proposition 6.6.9 states that the complementedness of a filtration of D in the sense of lattices agrees with the complementedness in the sense of states (Definition 5.1.13).

Proposition 6.6.9. *Let D be a distributive lattice and $F^\lambda \in \mathbf{DF}^A(D)_1$ a filtration of D, with $\lambda \in \Gamma^A(T_D)$. We have the following formula for the complementedness of F^λ:*

$$\langle F^\lambda, \mathfrak{l}\rangle = \inf\{\langle \lambda, w\rangle \mid w \in \Xi_D\}.$$

In particular, F^λ is paracomplemented if and only if for all $w \in \Xi_D$ we have $\langle \lambda, w\rangle \geq 1$.

Proof. Again we see D as the lattice of closed subgraphs of Q_D. For $b, c \in A$ with $b \geq 0$, the interval $[F_{\geq c+b}, F_{\geq c}]$ is isomorphic to the full subgraph of Q with set of vertices $\{v \in Q_{D,0} \mid c \leq \langle \lambda, v^\vee\rangle < c+b\}$. Thus $[F_{\geq c+b}, F_{\geq c}]$ is complemented if and only if there are no arrows $v_1 \to v_2$ with $c \leq \langle \lambda, v_1^\vee\rangle, \langle \lambda, v_2^\vee\rangle < c+b$. It follows that $[F_{\geq c+b}, F_{\geq c}]$ is complemented for all $c \in A$ if and only if $\langle \lambda, v_2^\vee - v_1^\vee\rangle \geq b$ for all arrows $v_1 \to v_2$. This is seen by setting $c = \langle \lambda, v_1^\vee\rangle$. We thus have

$$b \leq \langle F^\lambda, \mathfrak{l}\rangle \iff b \leq \inf\{\langle \lambda, w\rangle \mid w \in \Xi_D\}.$$

Since this holds for all $b \geq 0$ in A, we have the desired equality. □

Corollary 6.6.10. *The set $\mathcal{B}_A(L)$ of paracomplemented A-filtrations is convex, in the sense that for all $F, G \in \mathcal{B}_A(L)$ and $t \in [0,1] \cap A$, we have $(1-t)F + tG \in \mathcal{B}_A(L)$.*

Proof. Take a maximal distributive sublattice D of L containing F and G. By Proposition 6.6.9, the complementedness $\langle -, \mathfrak{l}\rangle$ is a convex function on $\mathbf{DF}^A(D)_1$. The result follows. □

We recall the definition of norm on a lattice used in [33].

Definition 6.6.11 (Norm on a lattice)**.** An A-valued *norm* X on L is the data of a nonnegative number $X([a,b]) \in A_{\geq 0}$ for every interval $[a,b]$ of L such that

1. $X([a,b]) = 0$ if and only if $a = b$;
2. $X([a,b]) = X([a,c]) + X([c,b])$ if $a \leq c \leq b$; and
3. $X([a,b]) = X([a',b'])$ if $[a,b] \sim [a',b']$.

An *A-normed artinian lattice* is a pair (L, X) where L is an artinian lattice and X is an A-valued norm on L.

From now on, we fix an A-valued norm X on L.

Remark **6.6.12.** The conditions in Definition 6.6.11 amount to X being defined by a homomorphism $X\colon \mathrm{K}(L) \to A$ such that $X\left(\overline{[a,b]}\right) > 0$ for $a < b$.

Definition 6.6.13 (Norm of a filtration)**.** For every filtration $F \in \mathbf{DF}^A(L)_1$, we define its *norm squared* $\|F\|^2$ to be

$$\|F\|^2 = \sum_{c \in A} c^2 X([F_{>c}, F_{\geq c}]).$$

Remark **6.6.14.** For any distributive sublattice, X induces a norm on D by restriction and on T_D because $\mathrm{K}(D) = \mathrm{K}(T_D)$. Since $\mathrm{K}(D)$ has for basis the set of vertices of D, the restriction of X to D is determined by the values $X(v)$ for $v \in Q_{D,0}$. The norm on T_D gives an inner product $(-,-)$ on $\Gamma^A(T_D)$ that, in the basis of vertices $Q_{D,0}$, has diagonal matrix $(\delta_{vw}X(v))_{v,w\in Q_{D,0}}$. If $\lambda \in \Gamma^A(T_D)$ is a D-admissible grading, then we have $\|\lambda\|^2 = \|F^\lambda\|^2$.

Theorem 6.6.15. *Suppose A is a field. Then there is a unique paracomplemented filtration $F \in \mathcal{B}_A(L)$ minimising $\|-\|^2$ on the set $\mathcal{B}_A(L)$ of paracomplemented A-filtrations of L.*

Proof. Let D be a maximal distributive sublattice of L. We first study the problem of minimising $\|-\|^2$ in $\mathbf{DF}^A(D)_1$. By Proposition 6.6.9 and Remark 6.6.14, we are looking to minimise $\|\lambda\|^2$ for $\lambda \in R_A$, where $R_A = \{\lambda \in \Gamma^A(T_D) \mid \inf\langle\lambda, \Xi_D\rangle \geq 1\}$ is the set of those D-admissible A-gradings $\lambda \in \Gamma^A(T_D)$ such that F^λ is paracomplemented. A unique minimiser of $\|-\|^2$ on R_A exists by the case of states (Proposition 5.1.14).

From Proposition 6.3.2 we see that there are only finitely many isomorphism types of normed distributive lattices that appear as maximal distributive sublattices of L. Indeed, once we fix a maximal chain $0 = a_0 < \cdots < a_n = 1$ of L, distributive sublattices D of L containing the a_i are determined by the set of arrows of the corresponding directed acyclic graph with vertices $\{1,\ldots,n\}$. The norm on D is determined by attaching the number $c_i = X([a_{i-1},a_i]) \in A_{>0}$ to each vertex i. If we choose a different maximal chain $0 = b_0 < \cdots < b_n = 1$, by the theorem of Jordan–Hölder–Dedekind, we have $X([a_{i-1},a_i]) = X([b_{\sigma(i)-1},b_{\sigma(i)}])$ for a permutation σ of $\{1,\ldots,n\}$. Therefore all maximal distributive sublattices D of L are isomorphic, as normed lattices, to the lattice of closed subgraphs of some directed acyclic graph with set of vertices $\{1,\ldots,n\}$ and norm given by the numbers $c_1,\ldots,c_n$. We conclude that the minimum of $\|-\|^2$ on $\mathcal{B}_A(L)$ is attained at some $F \in \mathcal{B}_A(L)$.

Suppose that there are two paracomplemented filtrations $F, G \in \mathcal{B}_A(L)$ minimising $\|-\|^2$. By [12, Theorem III.7.9], the lattice generated by the $F_{\geq c}$ and the $G_{\geq c}$ is distributive. Therefore there is a maximal distributive sublattice D of L such that both F and G factor through D. We must have $F = G$ from uniqueness of the minimiser in the distributive case. □

We now study the notion of linear form on a lattice.

Definition 6.6.16 (Linear form on a lattice)**.** An *A-valued linear form* ℓ on L is the data of a number $\ell([a,b]) \in A$ for each interval $[a,b]$ of L such that:

1. $\ell([a,b]) = \ell([a,c]) + \ell([c,b])$ if $a \leq c \leq b$; and

2. $\ell([a,b]) = \ell([a',b'])$ if $[a,b] \sim [a',b']$.

If ℓ is an A-valued linear form on ℓ and $F \in \mathbf{DF}^A(L)_1$ is a filtration, we define the pairing

$$\langle F, \ell\rangle := \sum_{c\in A} c\ \ell([F_{>c}, F_{\geq c}]).$$

Remark **6.6.17.** Equivalently, an A-valued linear form on L is an A-module homomorphism $\ell\colon \mathrm{K}(L) \to A$.

The word *linear* in the definition is justified by the following property.

Lemma 6.6.18. *Let ℓ be an A-valued linear form on L. For $a, b \in A_{\geq 0}$ and $F, G \in \mathbf{DF}^A(L)_1$ we have*

$$\langle aF + bG, \ell\rangle = a\langle F, \ell\rangle + b\langle G, \ell\rangle.$$

Proof. By taking a distributive sublattice D of L through which both F and G factor, and subsequently taking the associated boolean algebra T_D, we may assume that $L = M$ is a finite boolean algebra. Then ℓ is given by an element $\alpha \in \Gamma_A(M)$, and for $\lambda \in \Gamma^A(M)$ we have the equality $\langle F^\lambda, \ell\rangle = \langle \lambda, \alpha\rangle$. In this context, linearity is clear. □

Definition 6.6.19 (Semistable lattice)**.** Let ℓ be a linear form on L. We say that L is *semistable with respect to ℓ* if for all $F \in \mathbf{DF}^A(L)_1$ we have $\langle F, \ell\rangle \leq 0$.

The following proposition relates the notion of semistability above with what in [33] is called *semistable of phase 0*.

Proposition 6.6.20. *Let ℓ be a linear form on L. Then L is semistable with respect to ℓ if and only if $\ell(L) = 0$ and for all $x \in L$ we have $\ell([0,x]) \leq 0$.*

Proof. Suppose first that L is semistable with respect to L. Let $a \in A$ and consider the filtration F of L characterised by the fact that $F_{>a} = 0$, $F_{\geq a} = 1$. We have $\langle F, \ell\rangle = a\ell(L)$. Thus $a\ell(L) \leq 0$ for all $a \in A$, which forces $\ell(L) = 0$.

Now let $x \in L$ and consider the filtration F with jumps 0 and 1 such that $F_{\geq 1} = x$ and $F_{\geq 0} = 1$. We have $\ell([0,x]) = \langle F, \ell\rangle \leq 0$, as desired.

Conversely, suppose that $\ell(L) = 0$ and for all $x \in L$ we have $\ell([0,x]) \leq 0$. Let $F \in \mathbf{DF}^A(L)_1$ be any filtration with jumps $c_1 < \ldots < c_n$. We have

$$\langle F, \ell\rangle = \sum_{i=1}^{n} c_i \ell([F_{>c_i}, F_{\geq c_i}]) = \sum_{i=1}^{n-1} c_i \left(\ell([0, F_{\geq c_i}]) - \ell([0, F_{\geq c_{i+1}}])\right) + c_n \ell([0, F_{\geq c_n}]) =$$

$$= \sum_{i=2}^{n} (c_i - c_{i-1})\ell([0, F_{\geq c_i}]) + c_1 \ell([0, F_{\geq c_1}]) \leq 0,$$

since all $c_i - c_{i-1} > 0$ and $\ell([0, F_{\geq c_1}]) = \ell(L) = 0$. □

If $L' \subset L$ is a sublattice and ℓ is a linear form on L, there is an induced linear form $\ell|_{L'}$ on L' given by $\ell|_{L'}([a, b]) = \ell([a, b])$ for $a \leq b$ in L'.

Remark **6.6.21.** If $D \subset L$ is a distributive sublattice, since $\Gamma^A(T_D) = \mathrm{K}(D)$, we identify $\ell|_D$ with an element of $\Gamma_A(T_D)$. Therefore we have a polarised state $(\Gamma_A(T_D), \Xi_D, -\ell|_D)$ over A, which is semistable precisely when D is semistable with respect to $\ell|_D$. If L is endowed with an A-valued norm X, then $(\Gamma_A(T_D), \Xi_D, -\ell|_D)$ is a normed polarised state.

Definition 6.6.22 (Lattice of semistable elements)**.** Let ℓ be a linear form on L and suppose that L is semistable with respect to L. We denote $L^{\mathrm{ss}(\ell)}$ (or just L^{ss} if ℓ is clear from the context) the sublattice of L consisting of those elements $x \in L$ such that $[0, x]$ is semistable with respect to $\ell|_{[0,x]}$.

Since in the definition L itself is assumed to be semistable, $x \in L$ is in L^{ss} if and only if $\ell([0, x]) = 0$. If $x, y \in L^{\mathrm{ss}}$, then

$$\ell([0, x \vee y]) = \ell([x, x \vee y]) = \ell([x \wedge y, y]) = -\ell([0, x \wedge y]),$$

therefore $\ell([0, x \vee y]) = \ell([0, x \wedge y]) = 0$, since both are ≤ 0. Thus L^{ss} is indeed a sublattice of L, and it contains 0 and 1.

Definition 6.6.23 (Associated graded lattice)**.** Let $F \in \mathbf{DF}^A(L)_1$ be an A-filtration of L. We define the *associated graded lattice* $\mathrm{Grad}_F(L)$ to be the product lattice

$$\mathrm{Grad}_F(L) := \prod_{c \in A} [F_{>c}, F_{\geq c}].$$

The associated graded lattice is normed by setting

$$X\left([(x_c)_{c \in A}, (y_c)_{c \in A}]\right) = \sum_{c \in A} X([x_c, y_c]).$$

We still denote by X the norm on $\mathrm{Grad}_F(L)$. The lattice $\mathrm{Grad}_F(L)$ carries naturally a linear form $F^\vee$, defined by

$$F^\vee([(x_c)_{c \in A}, (y_c)_{c \in A}]) = \sum_{c \in A} c X([x_c, y_c]).$$

Suppose that D is a maximal distributive sublattice of L containing F. Then each $[F_{>c}, F_{\geq c}] \cap D$ is a maximal distributive sublattice of $[F_{>c}, F_{\geq c}]$ by Proposition 6.4.5, and thus $D_{\mathrm{gr}} = \prod_{c \in A} [F_{>c}, F_{\geq c}] \cap D$ is a maximal distributive sublattice of $\mathrm{Grad}_F(L)$. Note that D and D_{gr} have the same associated boolean algebra T_D.

Suppose that G is an A-filtration of $\mathrm{Grad}_F(L)$ contained in D_{gr}. Then we have the formula

$$\langle G, F^\vee \rangle = (G, F), \tag{6.5}$$

where in the right hand side, G and F are seen as elements in $\Gamma^A(T_D)$ and $(-,-)$ is the inner product on $\Gamma^A(T_D)$ given by X.

In [33], the local structure of $\mathscr{B}_A(L)$ around a paracomplemented filtration F is described in terms of another lattice $\Lambda(F)$, whose definition we recall now.

Definition 6.6.24. Suppose that $F \in \mathscr{B}_A(L)$ is a paracomplemented A-filtration. We define the sublattice $\Lambda(F)$ of $\mathrm{Grad}_F(L)$ to be the set of those $(x_c)_{c\in A} \in \mathrm{Grad}_F(L)$ such that for each $c \in A$ the interval $[x_{c+1}, x_c]$ of L is complemented.

It is not obvious that $\Lambda(F)$ is actually a sublattice of $\mathrm{Grad}_F(L)$. This fact is [33, Proposition 4.5], although it can also be deduced from the proof of the following lemma by reducing to the distributive case.

Lemma 6.6.25. *Let $F \in \mathscr{B}_A(L)$ be a paracomplemented A-filtration of L, and let D be a maximal distributive sublattice of L through which F factors. Then $D_{\Lambda(F)} := D_{\mathrm{gr}} \cap \Lambda(F)$ is a maximal distributive sublattice of $\Lambda(F)$ with associated boolean algebra canonically isomorphic to T_D. Moreover, its state is*

$$\Xi_{D_{\Lambda(F)}} = \{w \in \Xi_D \mid \langle \lambda, w \rangle = 1\} \subset \Gamma_A(T_D),$$

where $\lambda \in \Gamma^A(T_D)$ is such that $F^\lambda = F$

Proof. Let D' be a maximal distributive sublattice of $\Lambda(F)$ containing $D_{\Lambda(F)}$. The projections $p_c\colon D' \to [F_{>c}, F_{\geq c}]$ are maps of lattices whose image contains $D \cap [F_{>c}, F_{\geq c}]$. Indeed, we have a section $s_c\colon D \cap [F_{>c}, F_{\geq c}] \to D_{\Lambda(F)} \subset D'$ defined by

$$s_c(x)_d = \begin{cases} F_{\geq d}, & d > c, \\ x, & d = c, \\ F_{>d}, & d < c. \end{cases}$$

Since $D \cap [F_{>c}, F_{\geq c}]$ is a maximal distributive sublattice of $[F_{>c}, F_{\geq c}]$ and $p_c(D')$ is distributive, we must have $p_c(D') = D \cap [F_{>c}, F_{\geq c}]$. Therefore $D' = D_{\Lambda(F)}$.

Denote F_D the filtration F seen as a filtration of D. From Corollary 6.4.8 it follows that $D_{\Lambda(F)} = \Lambda(F_D)$, that is, $D_{\Lambda(F)}$ consists of the elements $(y_c)_{c\in A} \in D_{\mathrm{gr}}$ such that $[y_{c+1}, y_c] \cap D$ is complemented for all $c \in A$. We can regard D canonically as the lattice of closed subgraphs of the directed acyclic graph Q_D. The filtration F_D is of the form F^λ for an element $\lambda \in \Gamma^A(T_D)$. Given $(y_c)_{c\in A} \in D_{\mathrm{gr}}$ and $c \in A$, the interval $[y_{c+1}, y_c] \cap D$ is complemented if and only if there are no arrows

$\alpha: u \to v$ with $u, v \in y_c \setminus y_{c+1}$. Note that, since F_D is paracomplemented, for any such arrow we have $\langle \lambda, u \rangle = c$ and $\langle \lambda, v \rangle = c + 1$, and thus we must have $u \in y_c \setminus F_{>c}$ and $v \in F_{\geq c+1} \setminus y_{c+1}$. Therefore $(y_c)_{c \in A}$ is in $D_{\Lambda(F)}$ if and only if for every arrow $\alpha: u \to v$ with $\langle \lambda, v^\vee - u^\vee \rangle = 1$ and $u \in y_c$, for some $c \in A$, we have $v \in y_{c+1}$. Let Q' be the subgraph of Q_D with vertices $Q_{D,0}$ and arrows $\alpha: u \to v$ such that $\langle \lambda, v^\vee - u^\vee \rangle = 1$. The condition above is precisely that the subset $\bigcup_{c \in A} y_c \setminus F_{>c}$ of $Q_{D,0}$ (which corresponds to the element $(y_c)_{c \in A} \in D_{\mathrm{gr}}$ itself via the identification $T_D = T_{D_{\mathrm{gr}}}$) is a closed subgraph of Q'. Therefore $D_{\Lambda(F)}$ is identified with the lattice of closed subgraphs of Q'. On the other hand, the state of Q' is by definition $\{w \in \Xi_D \mid \langle \lambda, w \rangle = 1\}$. □

The linear form $F^\vee$ on $\mathrm{Grad}_F(L)$ induces a linear form on $\Lambda(L)$ that we still denote by $F^\vee$.

Theorem 6.6.26 (Existence and characterisation of the HKKP filtration)**.** *Let A be a field, and let L be an A-normed artinian lattice. Let $F \in \mathcal{B}_A(L)$ be a paracomplemented A-filtration of L. Then the following are equivalent:*

1. *The filtration F minimises the function $\|-\|^2$ on $\mathcal{B}_A(L)$.*
2. *The lattice $\Lambda(F)$ is semistable with respect to the linear form $-F^\vee$.*

Moreover, there is a unique $F \in \mathcal{B}_A(L)$ satisfying these conditions.

Definition 6.6.27 (HKKP filtration of a normed lattice)**.** In the context of Theorem 6.6.26, the unique $F \in \mathcal{B}_A(L)$ satisfying the two equivalent conditions is called the *HKKP filtration* of the A-normed lattice (L, X).

Remark **6.6.28.** The proof of Theorem 6.2.2 in [33] uses completeness of $\mathbb{R}$ in an essential way. Our methods however allow us to prove that the weight filtration is defined over the field A if the norm takes values in A. The characterisation of the HKKP filtration as the unique minimiser of $\|-\|^2$ on $\mathcal{B}_A(L)$ is new and will be important in our approach to relate the iterated HKKP filtration with the iterated balanced filtration in Chapter 7.

Proof. The existence and uniqueness statement follows from Theorem 6.6.15. We now prove the equivalence of the two conditions. The first one is equivalent to, for every $G \in \mathcal{B}_A(L)$, the function

$$f(t) = \frac{1}{2}\|(1-t)F + tG\|^2, \quad t \in [0,1] \cap A,$$

having a local minimum at $t = 0$. This follows by convexity of $\mathcal{B}_A(L)$ and strict convexity of $\|-\|^2$, which can be seen by taking a maximal distributive sublattice D of L such that both F and G factor through D.

Claim **6.6.29.** There is $H \in \mathbf{DF}^A(\Lambda(L))_1$ such that $f'(0) = \langle H, F^\vee \rangle$.

Indeed, take $\lambda, \mu \in \Gamma^A(T_D)$ such that $F = F^\lambda$ and $G = G^\mu$. Then $f(t) = \frac{1}{2}\|(1-t)\lambda + t\mu\|^2$, where now the norm is associated to the inner product $(-,-)$ in $\Gamma^A(T_D)$. Therefore $f'(0) = (\mu - \lambda, \lambda) = \langle H, F^\vee \rangle$, where H is the filtration of $T_D = D_{\mathrm{gr}} \subset \mathrm{Grad}_F(L)$ associated to $\mu - \lambda$. Using the description of $\Xi_{D_{\Lambda(F)}}$ from Lemma 6.6.25, we see that for all $w \in \Xi_{D_{\Lambda(F)}}$ we have $\langle \mu - \lambda, w \rangle = \langle \mu, w \rangle - 1 \geq 0$, since μ is a filtration of D. Hence H is a filtration of $\Lambda(F)$ by Proposition 6.6.7. This proves the claim.

If $\Lambda(F)$ is semistable with respect to $-F^\vee$, we have that $f'(0) \geq 0$, so 0 is a local minimum of f. Since this holds for all $G \in \mathcal{B}_A(L)$, we have that F minimises $\|-\|^2$ on $\mathcal{B}_A(L)$.

Conversely, suppose that F maximises $\|-\|^2$ on $\mathcal{B}_A(L)$, and let H be an A-filtration of $\Lambda(F)$. Take a maximal distributive sublattice D of L containing F such that H factors through $D_{\Lambda(F)}$, and let $\mu, \lambda \in \Gamma^A(T_D)$ correspond to H and F. Since $\langle \mu, w \rangle \geq 0$ for all $w \in \Xi_D$ such that $\langle \lambda, w \rangle = 1$, we have, for small enough $t \in A \cap (0, 1]$ that $\langle \lambda + t\mu, \Xi_D \rangle \geq 1$. Thus $\lambda + t\mu$ corresponds to a filtration G of $D \subset L$. Defining f as above, we see that $f'(0) = \langle tH, F^\vee \rangle$. Since f has a local minimum at 0, it must be $f'(0) \geq 0$ and thus $\langle H, F^\vee \rangle \geq 0$. This proves that $\Lambda(F)$ is semistable with respect to $-F^\vee$. □

We can get more intuition about why Theorem 6.6.26 is true by interpreting $\Lambda(F)$ as describing $\mathcal{B}_A(L)$ locally around F, as we now explain.

Keep assuming A is a field. For $F \in \mathcal{B}_A(L)$ and $\varepsilon > 0$ a real number, we define the set $B(F, \varepsilon)$ as

$$B(F, \varepsilon) = \{H \in \mathcal{B}_A(L) \mid \forall c \in A,\ d(\operatorname{supp} F, c) \geq \varepsilon \implies H_{\geq c} = F_{\geq c}\}.$$

Recall that $\operatorname{supp} F$ denotes the support of F, which is defined as the set of its jumps.

Suppose now that

$$\varepsilon < \frac{1}{2} \inf\left\{|c - c'| \mid c, c' \in \operatorname{supp} F,\ c \neq c'\right\}.$$

If G is an A-filtration of $\Lambda(F)$ with support inside $(-\varepsilon, \varepsilon)$ we define a filtration $F + G$ of L as follows. Let $c_1 < \ldots < c_n$ be the jumps of F. For $t \in A$ with $|t| < \varepsilon$ and $i \in \{1, \ldots, n\}$ we set

$$(F + G)_{\geq c_i + t} = (G_{\geq t})_{c_i},$$

and for $c \in A$ with $d(\operatorname{supp} F, c) \geq \varepsilon$ we set $(F + G)_{\geq c} = F_{\geq c}$. This yields a well-defined element $F + G \in \mathbf{DF}^A(L)_1$.

The following result is [33, Proposition 4.8] for the case $A = \mathbb{R}$, but the same proof is valid for all subfields A of $\mathbb{R}$.

Proposition 6.6.30. *Suppose A is a field and let $F \in \mathcal{B}_A(L)$ be a paracomplemented A-filtration. For small enough $\varepsilon > 0$, the map $G \mapsto F + G$ establishes a bijection between the set of A-filtrations of $\Lambda(F)$ with support contained in $(-\varepsilon, \varepsilon)$ and $B(F, \varepsilon)$.*

Intuitively the proof of Theorem 6.6.26 goes as follows. By convexity, F minimises $\|-\|^2$ on $\mathcal{B}_A(L)$ if and only if F is a local minimum of $\|-\|^2$ on $\mathcal{B}_A(L)$. Since locally around F elements of $\mathcal{B}_A(L)$ are of the form $F + H$, with $H \in \mathbf{DF}^A(\Lambda(F))_1$, this is equivalent to the function

$$f(t) = \frac{1}{2}\|F + tH\|^2, \quad t \in [0,1] \cap A$$

having a local minimum at 0, that is $f'(0) \geq 0$. Computing the derivative we get $f'(0) = \langle H, F^\vee \rangle$, from which the result follows. A concrete way to establish the computation of $f'(0)$ is by working with maximal distributive sublattices, which is what we do in the proof of Theorem 6.6.26.

6.7 THE HKKP CHAIN

In analogy with the notion of chain for stacks (Definition 4.1.1), we define the concept of chain of lattices. The operation of taking the HKKP filtration of a normed artinian lattice L can be iterated, and the result is best expressed as a chain of lattices. As a shadow of this procedure, we obtain a $\mathbb{Q}^\infty$-filtration of the original lattice L.

Definition 6.7.1. A *chain of lattices* is data $(L_n, F_n, c_n)_{n \in \mathbb{N}}$ where

1. for every $n \in \mathbb{N}$, L_n is an artinian lattice endowed with a rational norm;
2. $F_n \in \mathbf{DF}^{\mathbb{Q}}(L_n)_1$ is a $\mathbb{Q}$-filtration of L_n;
3. $c_n \colon L_{n+1} \hookrightarrow \mathrm{Grad}_{F_n}(L_n)$ is a sublattice of $\mathrm{Grad}_{F_n}(L_n)$ in such a way that the norm on L_{n+1} is the restriction along c_n of the norm on $\mathrm{Grad}_{F_n}(L_n)$, which is itself inherited from L_n.

Definition 6.7.2. Let L be an artinian lattice endowed with a rational norm. The *HKKP chain of L* is the chain $(L_n, F_n, c_n)_{n \in \mathbb{N}}$ defined inductively as follows:

1. $L_0 = L$ as normed lattices;
2. if $n \in \mathbb{N}$ and L_n is defined, we let $F_n \in \mathbf{DF}^{\mathbb{Q}}(L_n)_1$ be its HKKP filtration (Definition 6.6.27). We let $L_{n+1} = \Lambda(F_n)^{\mathrm{ss}(-F_n^\vee)}$ (see Definitions 6.6.22 and 6.6.24) and c_n to be the inclusion

$$c_n \colon L_{n+1} \hookrightarrow \Lambda(F_n) \hookrightarrow \mathrm{Grad}_{F_n}(L_n).$$

We endow L_{n+1} with the norm induced from $\mathrm{Grad}_{F_n}(L_n)$.

For every $n \in \mathbb{N}$, there is an inclusion

$$\iota_n\colon L_{n+1} \hookrightarrow \mathrm{Grad}^{n+1}_{F_n,\dots,F_0}(L) := \mathrm{Grad}_{F_n}(\mathrm{Grad}_{F_{n-1}}(\cdots(\mathrm{Grad}_{F_0}(L))\cdots)),$$

which allows us to see F_{n+1} as a $\mathbb{Q}$-filtration of $\mathrm{Grad}^{n+1}_{F_n,\dots,F_0}$. The ι_n are constructed inductively by letting $\iota_0 = c_0$ and ι_{n+1} to be the composition

$$L_{n+2} \xrightarrow{c_{n+1}} \mathrm{Grad}_{F_{n+1}}(L_{n+1}) \xrightarrow{\mathrm{Grad}_{F_{n+1}}(\iota_{n+1})} \mathrm{Grad}_{F_{n+1}}\left(\mathrm{Grad}^{n+1}_{F_n,\dots,F_0}(L)\right),$$

noting that $\mathrm{Grad}_{F_{n+1}}\left(\mathrm{Grad}^{n+1}_{F_n,\dots,F_0}(L)\right) = \mathrm{Grad}^{n+2}_{F_{n+1},\dots,F_0}(L)$.

The data of the F_n, seen as $\mathbb{Q}$-filtrations of $\mathrm{Grad}^{n+1}_{F_n,\dots,F_0}(L)$, is the same as the data of a $\mathbb{Q}^\infty$-filtration of L in the sense of Definition 6.5.8. Here $\mathbb{Q}^\infty = \mathbb{Q}^{\oplus\mathbb{N}}$ with lexicographic order. Since $\mathbb{Q}^\infty$ is a totally ordered set, a $\mathbb{Q}^\infty$-filtration of L can be described in more concrete terms as a chain

$$0 < a_1 < a_2 < \cdots < a_n = 1$$

of elements of L together with a chain

$$c_1 > c_2 > \cdots > c_n$$

of elements of $\mathbb{Q}^\infty$.

Definition 6.7.3. The *iterated HKKP filtration* of L is the $\mathbb{Q}^\infty$-filtration of L determined by the F_n as above.

Remark **6.7.4** (Refined Harder–Narasimhan filtrations for lattices). For artinian lattice L, a rational norm X and a rational linear form ℓ give rise to a *polarisation* $Z\colon \mathrm{K}(L) \to \mathbb{C}$ in the sense of [33] by setting $Z([a,b]) = -\ell([a,b]) + iX([a,b])$. The *Harder–Narasimhan filtration* F of (L,Z) is then defined, and it is characterised by the fact that $\mathrm{Grad}_F(L)$ is semistable with respect to the linear form $\ell - F^\vee$. We can define the *refined Harder–Narasimhan chain* $(L_n, F_n, c_n)_{n\in\mathbb{N}}$ by letting $L_0 = L$, $L_1 = \mathrm{Grad}_F(L)^{\mathrm{ss}(\ell-F^\vee)}$, $F_0 = F$, $c_0\colon L_1 \to \mathrm{Grad}_F(L) = \mathrm{Grad}_{F_0}(L_0)$ the natural inclusion and $(L_{n+1}, F_{n+1}, c_{n+1})_{n\in\mathbb{N}}$ to be the HKKP chain of L_1. The associated $\mathbb{Q}^\infty$-filtration of this chain is by definition the *refined Harder–Narasimhan filtration* of (L,Z).

We note however that for many moduli problems of objects in abelian categories, like that of vector bundles on a smooth projective curve, the Harder–Narasimhan filtration does not arise in this way because the relevant lattice of subobjects is not artinian. Artinian lattices are still suitable for defining *refinements* of Harder–Narasimhan filtrations in these cases though.

CHAPTER 7

COMPARISON WITH THE ITERATED HKKP FILTRATION

This chapter is devoted to establishing a correspondence between the iterated balanced filtration for stacks and the iterated HKKP filtration for lattices. We introduce a class of stacks $\mathcal{X}$ endowed with some extra structure, called *linearly lit good moduli stacks* (Definition 7.3.7), for which every point x has an associated artinian lattice L_x. Morally, linearly lit stacks parametrise objects in some linear category. We give many examples of linearly lit stacks and provide a general method to establish linear litness in Section 7.4. We prove that the sets of sequential filtrations $\mathbb{Q}^\infty\text{-}\operatorname{Filt}(L_x) = \mathbb{Q}^\infty\text{-}\operatorname{Filt}(\mathcal{X}, x)$ are canonically isomorphic and that, in the presence of a norm, the iterated HKKP filtration of L_x agrees with the iterated balanced filtration of $(\mathcal{X}, x)$ (Theorem 7.5.9 and Corollary 7.5.11).

7.1 AGREEMENT OF DEGENERATION FANS

In this section, we will exhibit how the degeneration fan of the lattice of subrepresentations of a quiver representation can be read off from the moduli stack of representations. The simplest case is that of subspaces of a given vector space.

Proposition 7.1.1. *Let k be a field, let $d \in \mathbb{N}$, and let $\mathrm{pt}\colon \operatorname{Spec} k \to B\mathrm{GL}(d)_k$ be the standard point, corresponding to the vector space k^d. Let L be the lattice of vector subspaces of k^d. Then there is a canonical isomorphism*

$$\mathbf{DF}(B\mathrm{GL}(d)_k, \mathrm{pt})_\bullet \cong \mathbf{DF}(L)_\bullet$$

of formal fans.

Proof. We write $\mathbf{DF}(\mathrm{GL}(d)_k)_\bullet = \mathbf{DF}(B\mathrm{GL}(d)_k, \mathrm{pt})_\bullet$. The formal fan $\mathbf{DF}(\mathrm{GL}(d)_k)_\bullet$ can be explicitly described as

$$\mathbf{DF}(\mathrm{GL}(d)_k)_n = \mathrm{Hom}(\mathbb{G}_{m,k}^n, \mathrm{GL}(d)_k)/\sim$$

where for $\lambda, \gamma \in \mathrm{Hom}(\mathbb{G}_{m,k}^n, \mathrm{GL}(d)_k)$, we have $\lambda \sim \gamma$ if there is $g \in P(\lambda)(k)$ such that $\gamma = \lambda^g$. This follows from Example 2.2.13.

A homomorphism $\lambda: \mathbb{G}_{m,k}^n \to \mathrm{GL}(d)_k$ is the same data as grading $V := k^d = \bigoplus_{\chi \in \Gamma_{\mathbb{Z}}(\mathbb{G}_{m,k}^n)} V_\chi^\lambda$ of V by $\Gamma_{\mathbb{Z}}(\mathbb{G}_{m,k}^n)$, where each $V_\chi^\lambda \leq V$. We identify $\Gamma_{\mathbb{Z}}(\mathbb{G}_{m,k}^n) = \mathbb{Z}^n$ canonically, and see it as a poset with product order. Thus there is a canonical bijection between the set of homomorphisms $\lambda: \mathbb{G}_{m,k}^n \to \mathrm{GL}(d)_k$ and the set of $\mathbb{Z}^n$-gradings of L. For such a λ, we have the associated filtration given by $F_{\geq c}^\lambda = \bigoplus_{\chi \in \mathbb{Z}^n} V_\chi^\lambda$, for $c \in \mathbb{Z}^n$. It is a $\mathbb{Z}^n$-filtration by Definition 6.5.6.

Claim **7.1.2.** For all $g \in \mathrm{GL}(d)(k)$, we have that $g \in P(\lambda)(k)$ if and only if for all $c \in \mathbb{Z}^n$, $g(F_{\geq c}^\lambda) \subset F_{\geq c}^\lambda$.

Indeed, $g \in P(\lambda)(k)$ if and only if $\lim_{t \to 0} \lambda(t) g \lambda(t)^{-1}$ exists. Write $g = (g_{\chi,\chi'})_{\chi,\chi' \in \mathbb{Z}^n}$, where $g_{\chi,\chi'}: V_{\chi'}^\lambda \to V_\chi^\lambda$. Then $(\lambda(t) g \lambda(t)^{-1})_{\chi,\chi'} = (\chi - \chi')(t) g_{\chi,\chi'}$. Thus $g \in P(\lambda)(k)$ if and only if for all $\chi, \chi' \in \mathbb{Z}^n$, if $g_{\chi,\chi'} \neq 0$ then $\chi \geq \chi'$. This condition is equivalent to

$$\forall \chi \in \mathbb{Z}^n,\ g V_\chi^\lambda \subset F_{\geq \chi}^\lambda,$$

which is in turn equivalent to

$$\forall \chi \in \mathbb{Z}^n,\ g F_{\leq \chi}^\lambda \subset F_{\geq \chi}^\lambda.$$

This proves the claim. Note the equalities $V_\chi^{\lambda^g} = g V_\chi^\lambda$ and $F_{\geq \chi}^{\lambda^g} = g F_{\geq \chi}^\lambda$ for any $g \in \mathrm{GL}(d)(k)$. If $g \in P(\lambda)(k)$, then $F^{\lambda^g} = F^\lambda$. Thus the assignment $\lambda \mapsto F^\lambda$ gives a well-defined map $f_n: \mathbf{DF}(\mathrm{GL}(d)_k)_n \to \mathbf{DF}(L)_n$.

Suppose $\lambda_1, \lambda_2: \mathbb{G}_{m,k}^n \to \mathrm{GL}(d)_k$ are such that $F := F^{\lambda_1} = F^{\lambda_2}$. Let $\mathcal{C}$ be the set of jumps of F, and let $\chi_1, \ldots, \chi_l$ be a labelling of the elements of $\mathcal{C}$ in such a way that $\mathcal{X}_i \leq \mathcal{X}_j$ implies $i \leq j$. Note that $\mathcal{C} = \{\chi \in \mathbb{Z}^n \mid V_\chi^{\lambda_i} \neq 0\}$ for $i = 1, 2$. Let $U_i = \sum_{j \geq i} F_{\geq \chi_j}$. We have $U_l \subsetneq U_{l-1} \subsetneq \cdots \subsetneq U_1 = V$.

We define inductively an automorphism $g_i: U_i \to U_i$ such that $g_i(V_{\chi_i}^{\lambda_1}) = V_{\chi_i}^{\lambda_2}$ and $g_i|_{U_{i+1}} = g_{i+1}$. We set $g_l = \mathrm{id}_{U_1}$. Suppose g_i is defined, $i \geq 2$. We have two direct sum decompositions, $U_i = U_{i-1} \oplus V_{\chi_i}^{\lambda_1} = U_{i-1} \oplus V_{\chi_i}^{\lambda_2}$, so there is an isomorphism $o: V_{\chi_i}^{\lambda_1} \to V_{\chi_i}^{\lambda_2}$. We set $g_{i-1} = g_i \oplus o$. Now let $g = g_1$. We have $g(F_{\geq c}) = g\left(\sum_{\chi \geq c} V_\chi^{\lambda_1}\right) = \sum_{\chi \geq c} g\left(V_\chi^{\lambda_1}\right) = \sum_{\chi \geq c} V_\chi^{\lambda_2} = F_{\geq c}$. Thus $g \in P(\lambda_1)$ and $\lambda_2 = \lambda_1^g$, since $V_\chi^{\lambda_2} = g V_\chi^{\lambda_1}$ for all $\chi \in \mathbb{Z}^n$. This proves $f_n: \mathbf{DF}(\mathrm{GL}(d)_k)_n \to \mathbf{DF}(L)_n$ is injective.

Let $F \in \mathbf{DF}(L)_n$, and let M be a maximal distributive sublattice of L containing each $F_{\geq c}$ for $c \in \mathbb{Z}^n$. By Proposition 6.4.6, M is complemented. By Lemma 6.5.10, there is a grading λ of M, which is also a grading of L, such that $F = F^\lambda$. The grading λ can be seen as a map $\lambda\colon \mathbb{G}_{m,k}^n \to \mathrm{GL}(d)_k$, and thus $F = f_n(\lambda)$. The map f_n is therefore surjective.

It is only left to check naturality of the maps f_n. Let $h\colon \mathbb{Z}^l \to \mathbb{Z}^n$ be an order-preserving linear map and $\lambda\colon \mathbb{G}_{m,k}^n \to \mathrm{GL}(d)_k$. We denote $h^*(\lambda) := \lambda \circ D(h^\vee)$, where D denotes the Cartier dual map. This descends to give the pullback map $h^*\colon \mathbf{DF}(\mathrm{GL}(d)_k)_n \to \mathbf{DF}(\mathrm{GL}(d)_k)_l$. By Cartier duality, for $\chi \in \mathbb{Z}^n$, the torus $\mathbb{G}_{m,k}^l$ acts on V_χ^λ as $h^\vee(\chi)$, so $V_\alpha^{h^*\lambda} = \bigoplus_{h^\vee(\chi)=\alpha} V_\chi^\lambda$ for all $\alpha \in \mathbb{Z}^l$. Therefore for all $c \in \mathbb{Z}^l$ we have

$$F_{\geq c}^{h^*\lambda} = \bigoplus_{\substack{\chi \in \mathbb{Z}^n \\ h^\vee(\chi) \geq c}} V_\chi^\lambda = \sum_{\substack{\chi \in \mathbb{Z}^n \\ h^\vee(\chi) \geq c}} F_{\geq \chi}^\lambda = (h^* F^\lambda)_{\geq c},$$

as desired. □

With the aim of generalising Proposition 7.1.1 to the quiver setting, we recall certain basic definitions and set notations. We fix a field k.

A *quiver* Q is a finite directed graph. It consists of a finite set Q_0 of *vertices*, a finite set Q_1 of *arrows* and *source* and *target* maps $s, t\colon Q_1 \to Q_0$. If $\alpha \in Q_1$, we write $\alpha\colon i \to j$ if $i = s(\alpha)$ and $j = t(\alpha)$. A *representation* E of Q over the field k consists of a vector space E_i for every vertex $i \in Q_0$ and a linear map $x_\alpha\colon E_i \to E_j$ for every arrow $\alpha\colon i \to j$ of Q. We will sometimes denote $E = ((E_i)_{i \in Q_0}, (x_\alpha)_{\alpha \in Q_1})$. We say that E is *finite dimensional* if every E_i is. A morphism of representations $E \to E'$, where $E' = ((E_i')_i, (x_\alpha')_\alpha)$, is a family $(f_i)_{i \in Q_0}$ of maps $f_i\colon E_i \to E_i'$ such that $x_\alpha' \circ f_i = f_j \circ x_\alpha$ for every arrow $\alpha\colon i \to j$.

A *dimension vector* d for Q is the data of a number $d_i \in \mathbb{N}$ for every vertex i of Q. The *representation space* for the dimension vector d is

$$\mathrm{Rep}(Q, d) = \bigoplus_{\alpha: i \to j} \mathrm{Hom}(k^{d_i}, k^{d_j}).$$

The group $G(d) = \prod_{i \in Q_0} \mathrm{GL}_{d_i,k}$ acts on $\mathrm{Rep}(Q, d)$ by

$$(g_i) \cdot (x_\alpha)_\alpha = (g_j x_\alpha g_i^{-1})_{\alpha: i \to j}.$$

The quotient stack $\mathcal{R}ep(Q, d) := \mathrm{Rep}(Q, d)/G(d)$ is referred to as the *moduli stack of representations of Q of dimension vector d*.

We now fix a quiver Q, a dimension vector d and we denote $G = G(d)$, $V = \mathrm{Rep}(Q, d)$ and $\mathcal{X} = \mathcal{R}ep(Q, d)$. We fix a k-point x of V, and, by slight abuse of notation, we denote the image of x under $V \to \mathcal{X}$ also by x. The point x gives a representation $E = ((k^{d_i})_i, (x_\alpha)_\alpha)$, and we will denote $L = L_E$ the lattice of subrepresentations of E, which is an artinian lattice (Definition 6.1.1).

Proposition 7.1.3. *There is a canonical isomorphism*

$$\mathbf{DF}(\mathcal{X}, x)_\bullet \cong \mathbf{DF}(L)_\bullet$$

of formal fans.

Proof. By Proposition 2.7.5, there is a injection $\mathbf{DF}(\mathcal{X}, x)_\bullet \to \mathbf{DF}(BG, \mathrm{pt})_\bullet$ that identifies $\mathbf{DF}(\mathcal{X}, x)_\bullet$ with a subfan of $\mathbf{DF}(BG, \mathrm{pt})_\bullet$ by

$$\mathbf{DF}(\mathcal{X}, x)_n = \{\gamma \in \mathbf{DF}(BG, \mathrm{pt})_n \mid \lim \gamma x \text{ exists}\},$$

and where

$$\mathbf{DF}(BG, \mathrm{pt})_n = \mathrm{Hom}(\mathbb{G}_{m,k}^n, G)/\sim,$$

with $\gamma \sim \gamma^g$ if $g \in P(\gamma)(k)$.

Let Q° be the quiver with $Q_0^\circ = Q_0$ and $Q_1^\circ = \varnothing$. Then E defines a representation E° of Q° which is just the vector space $\bigoplus_{i \in Q_0} E_i$ endowed with its grading by the set Q_0. The lattice L_{E° of subrepresentations of E° is the product

$$L_{E^\circ} = \prod_{i \in Q_0} L_{E_i}$$

of the lattices of vector subspaces of each E_i. Proposition 7.1.1 gives a canonical isomorphism

$$\mathbf{DF}(BG, \mathrm{pt})_\bullet = \prod_{i \in Q_0} \mathbf{DF}(B\mathrm{GL}_{d_i,k}, \mathrm{pt})_\bullet \cong \prod_{i \in Q_0} \mathbf{DF}(L_{E_i})_\bullet \cong \mathbf{DF}(L^\circ)_\bullet \colon \gamma \mapsto F^\gamma.$$

The injection of lattices $L \to L^\circ$ gives an injection of formal fans $\mathbf{DF}(L)_\bullet \to \mathbf{DF}(L^\circ)_\bullet$. Now if $\gamma \in \mathbf{DF}(BG, \mathrm{pt})_n$ is represented by a group homomorphism $\mathbb{G}_{m,k}^n \to G$ (that we also denote γ), then the associated filtration $F^\gamma \in \mathbf{DF}(L^\circ)_n$ is given by

$$(F^\gamma_{\geq c})_i = \bigoplus_{\substack{\chi \in \mathbb{Z}^n \\ \chi \geq c}} (E_i)^\gamma_\chi, \quad c \in \mathbb{Z}^n,$$

where $(E_i)^\gamma_\chi$ is the eigenspace of E_i where $\mathbb{G}_{m,k}^n$ acts, via γ, through the character $\chi \in \Gamma_{\mathbb{Z}}(\mathbb{G}_{m,k}^n) \cong \mathbb{Z}^n$. The direct sum decomposition of the E_i allows us to write x in coordinates $x = (x_{\alpha,\chi,\chi'})_{\alpha \in Q_1, \chi, \chi' \in \mathbb{Z}^n}$ where, if $\alpha \colon i \to j$, then $x_{\alpha,\chi,\chi'} \colon (E_i)^\gamma_{\chi'} \to (E_j)^\gamma_\chi$ is the corresponding component of x_α. For F^γ to be a filtration of E we need that

$$x_\alpha\left((E_i)^\gamma_{\chi'}\right) \subset \bigoplus_{\substack{\chi \in \mathbb{Z}^n \\ \chi \geq \chi'}} (E_j)^\gamma_\chi$$

for all $\alpha: i \to j$ and all $\chi' \in \mathbb{Z}^n$. This is equivalent to $x_{\alpha,\chi,\chi'}$ being 0 whenever $\chi < \chi'$, for all $\alpha \in Q_1$. On the other hand,

$$\gamma(t)x = ((\chi - \chi')(t)x_{\alpha,\chi,\chi'})_{\alpha,\chi,\chi'},$$

so $\lim \gamma x$ exists if and only if $x_{\alpha,\chi,\chi'} = 0$ whenever $\chi < \chi'$. Therefore $\mathbf{DF}(\mathcal{X}, x)_\bullet$ and $\mathbf{DF}(L)_\bullet$ are the same subfan under the isomorphism $\mathbf{DF}(BG, \mathrm{pt})_\bullet \cong \mathbf{DF}(L^\circ)_\bullet$, so we get an isomorphism between $\mathbf{DF}(\mathcal{X}, x)_\bullet$ and $\mathbf{DF}(L)_\bullet$, as desired. □

We denote $\mathbf{DF}(\mathcal{X}, x)_n \to \mathbf{DF}(L)_n : \gamma \mapsto F^\gamma$ the isomorphism above.

7.2 THE CASE OF NILPOTENT QUIVER REPRESENTATIONS

We will now relate, in a particular quiver setting, the complementedness of a filtration in the sense of lattices (Definition 6.6.1) with the Kempf number of a filtration in the sense of stacks (Definition 3.1.1). We fix a base algebraically closed field k, a quiver Q, and a dimension vector d such that the stack $\mathcal{Rep}(Q, d)$ of representations (over k) has a good moduli space. If k is of characteristic 0, this last condition is automatic, while if k has positive characteristic then we are imposing that d is a vector of 0s and 1s, so that $\mathrm{GL}(d)$ is a torus.

Definition 7.2.1 (Null representation)**.** We say that a representation $M = ((M_i)_{i \in Q_0}, (y_\alpha)_{\alpha \in Q_1})$ of Q is *null* if $y_\alpha = 0$ for all $\alpha \in Q_1$.

Proposition and Definition 7.2.2. Let $M = ((M_i)_{i \in Q_0}, (y_\alpha)_{\alpha \in Q_1})$ be a finite dimensional representation of Q of dimension vector d, corresponding to the k-point y of $\mathcal{Rep}(Q, d)$. The following are equivalent:

1. There is a $\mathbb{Z}$-filtration of M whose associated graded is null.
2. We have $0 \in \overline{\{y\}}$ inside $\mathcal{Rep}(Q, d)$.
3. There exists $n \in \mathbb{N}$ such that for every sequence of composable arrows $\alpha_1, \ldots, \alpha_n$ in Q we have $y_{\alpha_n} \circ \cdots \circ y_{\alpha_1} = 0$.

If these conditions hold we say that M is a *nilpotent* representation of Q.

Proof. Since $\mathcal{Rep}(Q, d)$ has a good moduli space, $0 \in \overline{\{y\}}$ if and only if there is $\lambda \in \mathbf{DF}(\mathcal{Rep}(Q, d), y)_1$ with $\mathrm{ev}_0\, \lambda = 0$ by [8, Lemma 3.24]. This gives a filtration F^λ of M whose associated graded representation is null, and conversely any such filtration gives a suitable λ. This shows that the first two conditions are equivalent.

For the third one, let M' be the subrepresentation of M consisting of the vectors $v \in M_i$ such that $y_\alpha(v) = 0$ for all arrows α in Q with $s(\alpha) = i$. By definition, M' is null. Let $M_1 = M'$ and let $M_n \subset M$ be defined by $M_n / M_{(n-1)} = (M/M_{(n-1)})'$. Then

M satisfies the third condition if and only if $M_n = M$ for some $n \in \mathbb{N}$. In that case, $0 \subset M_1 \subset \cdots \subset M_n = M$ is a filtration with null associated graded. On the other hand, if $0 \subset N_1 \subset N_2 \subset \cdots \subset N_m = M$ is a filtration with null associated graded, then $N_i \subset M_i$ for all i, so $M_m = M$ and the third condition holds. □

Proposition 7.2.3. *Let M be a nilpotent representation of Q of dimension vector d. Then:*

1. *Any subquotient of M is also nilpotent.*
2. *The representation M is semisimple if and only if M is null.*

Proof. A Jordan–Hölder filtration of a subquotient N of M can be extended to a Jordan–Hölder filtration of M, whose associated graded must be null by hypothesis. Thus the associated graded of a Jordan–Hölder filtration of N is also null.

The representation M is semisimple if it is isomorphic to the associated graded representation of a Jordan–Hölder filtration, which is the null representation by hypothesis. □

We now denote $\mathcal{X} := \mathcal{Rep}(Q, d)$ and let x be a k-point of $\mathcal{X}$. We denote $E = ((k^{d_i})_i, (x_\alpha)_\alpha)$ the representation of Q corresponding to x and $L = L_E$ the lattice of subrepresentations of E.

Proposition 7.2.4. *Suppose that E is nilpotent and let $\lambda \in \mathbf{DF}^{\mathbb{Q}}(\mathcal{X}, x)_1$. Then*

$$\langle \lambda, \mathcal{X}^{\max} \rangle = \langle F^\lambda, \mathfrak{l} \rangle,$$

where $\langle \lambda, \mathcal{X}^{\max} \rangle$ denotes Kempf's intersection number (Definition 3.1.1), F^λ is the $\mathbb{Q}$-filtration of L corresponding to λ under Proposition 7.1.3, and $\langle F^\lambda, \mathfrak{l} \rangle$ is the complementedness of F^λ as a filtration of the lattice L (Definition 6.6.1).

Proof. It is enough to assume that $\lambda \in \mathbf{DF}(\mathcal{X}, x)_1$. We choose a cocharacter $\mathbb{G}_{m,k} \to G$ representing λ that we still denote by λ. Then λ induces a direct sum decomposition $\mathrm{Rep}(Q, d) = V = \bigoplus_{l \in \mathbb{Z}} V_l$. Let $p_l : V \to V_l$ be the induced projections and let $\Xi_{x,\lambda} = \{l \in \mathbb{Z} \mid p_l(x) \neq 0\}$. Note that $\Xi_{x,\lambda} \subset \mathbb{N}$ since $\lim \lambda x$ exists.

Claim **7.2.5.** $\langle \lambda, \mathcal{X}^{\max} \rangle = \inf \Xi_{x,\lambda}$.

Note that $\mathcal{X}^{\max} = V^G / G$. Since G is linearly reductive, there is a unique splitting $V = V^G \oplus V'$, with V' a G-subrepresentation of V. Choose a basis $v_1, \ldots, v_m$ of V of eigenvectors for λ and $0 \leq r \leq r' \leq m$ such that $v_1, \ldots, v_r$ is a basis of V^G, $v_1, \ldots, v_{r'}$ is a basis of V_0 and the v_i with $i > r$ are in V'. Let $n_i \in \mathbb{Z}$ such that $v_i \in V_{n_i}$. Taking

Spec of the cartesian square

$$\begin{array}{ccc} R & \longrightarrow & V^G \\ \big\downarrow & \ulcorner & \big\downarrow \\ \mathbb{A}^1_k & \xrightarrow{t \mapsto \lambda(t)x} & V \end{array}$$

we get

$$\begin{array}{ccc} k[t]/I & \longleftarrow & k[v_1^\vee, \dots, v_r^\vee] \\ \big\uparrow & \ulcorner & \big\uparrow \\ k[t] & \xleftarrow[x_i t^{n_i} \leftarrow\!\shortmid\, v_i^\vee]{} & k[v_1^\vee, \dots, v_m^\vee], \end{array}$$

where I is the ideal generated by the t^{n_i} with $x_i \neq 0$ and $i > r$. Since $0 \in \overline{Gx}$, we have $x_i = 0$ for $i \leq r$. Therefore $I = (t^a \mid a \in \Xi_{x,\lambda}) = \left(t^{\inf \Xi_{x,\lambda}}\right)$ and the claim follows.

Claim **7.2.6.** For any $a \in \mathbb{N}$, we have that $F^\lambda_{\geq c}/F^\lambda_{\geq c+a}$ is semisimple for all $c \in \mathbb{Z}$ if and only if $\inf \Xi_{x,\lambda} \geq a$.

The cocharacter $\lambda\colon \mathbb{G}_{m,k} \to \mathrm{GL}(d)$ induces a direct sum decomposition $E_i = \bigoplus_{l \in \mathbb{Z}} (E_i)_l$ of every E_i, $i \in Q_0$. The subquotient $F^\lambda_{\geq c}/F^\lambda_{\geq c+a}$ can be written in coordinates as $(x_{\alpha,nm})_{\alpha, c \leq n, m < c+a}$ where, if $\alpha\colon i \to j$, then $x_{\alpha,nm}\colon (E_i)_m \to (E_j)_n$ is the corresponding component of $x_\alpha\colon E_i \to E_j$. Since it is nilpotent, the representation $F^\lambda_{\geq c}/F^\lambda_{\geq c+a}$ is semisimple if and only if $x_{\alpha,nm} = 0$ for all $\alpha \in Q_1$ and $c \leq n, m < c+a$. This holds for all $c \in \mathbb{Z}$ if and only if $x_{\alpha,nm} = 0$ whenever $|n-m| < a$. Note that $\lambda(t)x = (t^{n-m} x_{\alpha,nm})_{\alpha \in Q_1, n,m \in \mathbb{Z}}$. Therefore $\Xi_{x,\lambda} = \{n - m \mid \exists \alpha \in Q_1,\ x_{\alpha,nm} \neq 0\}$. The claim follows.

The complementedness of F^λ is the supremum of the set of those $a \in \mathbb{N}$ such that $F_{\geq c}/F_{\geq c+a}$ is semisimple for all $c \in \mathbb{Z}$. Thus $\inf \Xi_{x,\lambda} = \langle F^\lambda, \mathfrak{l} \rangle$ by the second claim, and the result follows from the first claim. □

7.3 LAMPS AND LINEARLY LIT GOOD MODULI STACKS

In this section we define a kind of algebraic stacks $\mathcal{X}$, that we call *linearly lit good moduli stacks*, for which there is an artinian lattice L_x associated to every geometric point x, and a canonical isomorphism between the set of $\mathbb{Q}$-filtrations of L_x and the set $\mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, x)$ of $\mathbb{Q}$-filtrations of x, the latter defined in the stacky sense. For the lattice L_x to be canonical, we need to introduce a piece of extra structure on stacks, what we call a lamp. Morally, a lamp contains the information of what graded points have nonnegative weights only. In Section 7.4 we will see that stacks parameterising objects in an abelian category and admitting a good moduli space are naturally linearly lit.

Definition 7.3.1 (Lamp)**.** Let $\mathcal{X}$ be an algebraic stack satisfying Assumption 2.2.3. A *lamp* on $\mathcal{X}$ is a closed and open substack $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})_{\geq 0}$ of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$. A *lit stack* $\mathcal{X}$ over an algebraic space B is a stack $\mathcal{X}$ over B satisfying Assumption 2.2.3 endowed with a lamp.

If $f\colon \mathcal{X} \to \mathcal{Y}$ is a morphism between stacks satisfying Assumption 2.2.3 and $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{Y})_{\geq 0}$ is a lamp on $\mathcal{Y}$, then $f^*(\mathrm{Grad}_{\mathbb{Q}}(\mathcal{Y})_{\geq 0}) := \mathrm{Grad}(f)^{-1}(\mathrm{Grad}_{\mathbb{Q}}(\mathcal{Y})_{\geq 0}) \subset \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})_{\geq 0}$ is a lamp on $\mathcal{X}$, called the *pullback lamp*. If $\mathcal{X}$ is lit with lamp $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})_{\geq 0}$, then we say that the morphism f is *lit* if $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})_{\geq 0} = f^*(\mathrm{Grad}_{\mathbb{Q}}(\mathcal{Y})_{\geq 0})$.

Example **7.3.2.** Let G be a linear algebraic group over a field k. One source of lamps on BG are partial compactifications $\overline{G}$ of G. By this we mean an affine algebraic monoid $(\overline{G}, e)$ over k such that the unit group $\overline{G}^{\times}$, which is an open subscheme of $\overline{G}$, is dense in $\overline{G}$, together with an isomorphism $G \cong \overline{G}^{\times}$ of algebraic groups (compare with [48, Definition 2.1]).

Given such a partial compactification, we can define the associated lamp to be the closed and open substack $\mathrm{Grad}_{\mathbb{Q}}(BG)_{\geq 0}$ of $\mathrm{Grad}_{\mathbb{Q}}(BG)$ whose components form the image of the map

$$\mathbb{Q}\text{-}\mathrm{Filt}\left(\overline{G}/G, e\right) \to \pi_0\left(\mathrm{Grad}_{\mathbb{Q}}(\overline{G}/G)\right) \to \pi_0(\mathrm{Grad}_{\mathbb{Q}}(BG))$$

induced by taking associated graded points.

Example **7.3.3.** Let k be a field. A particularly important example of the above is the partial compactification $\mathrm{GL}_{n,k} \subset \mathrm{Mat}_{n\times n,k}$ of $\mathrm{GL}_{n,k}$ given by the algebraic monoid of $n \times n$ matrices. More generally, we consider the partial compactification $\mathrm{GL}(d) = \prod_{i\in Q_0} \mathrm{GL}_{d_i,k} \subset \mathrm{Mat}(d) := \prod_{i\in Q_0} \mathrm{Mat}_{d_i\times d_i,k}$ of $\mathrm{GL}(d)$, where d is a dimension vector of some quiver Q. This gives a lamp $\mathrm{Grad}_{\mathbb{Q}}(B\mathrm{GL}(d))_{\geq 0}$ that we refer to as the *standard lamp* on $B\mathrm{GL}(d)$.

A point p of $\mathrm{Grad}_{\mathbb{Q}}(B\mathrm{GL}_{n,k})$ represented by a rational cocharacter

$$\lambda(t) = \mathrm{diag}(t^{a_1}, \ldots, t^{a_n})$$

of $\mathrm{GL}_{n,k}$ is in $\mathrm{Grad}_{\mathbb{Q}}(B\mathrm{GL}_{n,k})_{\geq 0}$ precisely when $a_i \geq 0$ for all i.

Example **7.3.4.** We can use the example above to define a canonical lamp on stacks of quiver representations over a field k. Let Q be a quiver and let d be a dimension vector for Q. We have a canonical map

$$\mathcal{R}ep(Q, d) \to B\mathrm{GL}(d)$$

and we define the *standard lamp* on $\mathcal{R}ep(Q,d)$ to be the pullback of the standard lamp on $B\mathrm{GL}(d)$ along this map.

Remark **7.3.5.** In all the examples above and all other examples that we will consider, the lamp on the lit stack $\mathcal{X}$ satisfies the following convexity property: if k is a field and $f \in \mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X})(k)$ is a k-point such that for the maps $e_i \colon \mathbb{Q} \to \mathbb{Q}^n$ corresponding to the standard basis of $\mathbb{Q}^n$ we have that $e_i^* f$ is in $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})_{\geq 0}$, then for every order-preserving map $h \colon \mathbb{Q} \to \mathbb{Q}^n$ we have that $h^* f$ is in $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})_{\geq 0}$.

We will consider the following assumption on algebraic stacks $\mathcal{X}$ defined over an algebraically closed field k.

Assumption **7.3.6.** The stack $\mathcal{X}$ is of finite presentation over k, it has affine diagonal and a good moduli space $\pi \colon \mathcal{X} \to X$.

Definition 7.3.7. Let $\mathcal{X}$ be an algebraic stack over an algebraically closed field k satisfying Assumption 7.3.6, endowed with a lamp $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})_{\geq 0}$. Let us denote $\pi \colon \mathcal{X} \to X$ the good moduli space of $\mathcal{X}$. We say that $\mathcal{X}$ is *linearly lit* if for every closed k-point y of $\mathcal{X}$, there is a quiver Q, a dimension vector d of Q and a lit pointed closed immersion

$$i \colon (\mathcal{F}, y) \to (\mathcal{R}ep(Q,d), 0),$$

where $\mathcal{F} = \pi^{-1}\pi(y)$ is the fibre of π containing y.

By *linearly lit good moduli stack* over k we will mean an algebraic stack $\mathcal{X}$ over k satisfying Assumption 7.3.6 and endowed with a lamp that makes it linearly lit.

Remark **7.3.8.** Since every point x of $\mathcal{F}$ above specialises to y, $i(x)$ specialises to 0, that is, it corresponds to a nilpotent representation of Q.

Definition 7.3.9. Let $\mathcal{X}$ be a linearly lit good moduli stack over k, and let $x \in \mathcal{X}(k)$. We define a relation $\leq$ on $\mathbf{DF}(\mathcal{X},x)_1$ as follows. For $\lambda, \mu \in \mathbf{DF}(\mathcal{X},x)_1$, let $\gamma \in \mathbf{DF}(\mathcal{X},x)_2$ be the unique element such that $v_1\gamma = \lambda$ and $v_2\gamma = \mu$. Consider the graded point g given by the composition

$$B\mathbb{G}_{m,k} \xrightarrow{B\binom{-1}{1}} B\mathbb{G}^2_{m,k} \xrightarrow{0} \Theta^2_k \xrightarrow{\gamma} \mathcal{X}.$$

Then we say $\lambda \leq \mu$ if g lies in $\mathrm{Grad}_{\geq 0}(\mathcal{X})$. Here we denote $\binom{-1}{1} \colon \mathbb{G}_{m,k} \to \mathbb{G}^2_{m,k} \colon t \mapsto (t^{-1}, t)$.

This gives $(\mathbf{DF}(\mathcal{X},x)_\bullet, \leq)$ the structure of a formal fan with relation (Definition 6.5.36). We denote $L_x = L_{\mathbf{DF}(\mathcal{X},x)}$, endowed with the relation $\leq$.

Unravelling the definitions, we observe the following fact.

Lemma 7.3.10. *Let Q be a quiver and d a dimension vector for Q. Let x be a point of $\mathcal{R}ep(Q,d)$ corresponding to a representation E with lattice of subrepresentations L. Then the isomorphism*

$$\mathbf{DF}(\mathcal{R}ep(Q,d),x)_\bullet \xrightarrow{\sim} \mathbf{DF}(L)_\bullet$$

of Proposition 7.1.3 is indeed an isomorphism of formal fans with relation, where $\mathcal{R}ep(Q,d)$ is equipped with the standard lamp. □

Proposition 7.3.11. *Let $\mathcal{X}$ be a linearly lit good moduli stack over k, and let $x \in \mathcal{X}(k)$. Then L_x is an artinian lattice and there is a canonical isomorphism $(\mathbf{DF}(L_x)_\bullet, \leq) \to (\mathbf{DF}(\mathcal{X},x)_\bullet, \leq)$ of formal fans with relation.*

Proof. By Definition 7.3.7, there is a quiver Q, a dimension vector d and a point $z \in \mathcal{R}ep(Q,d)(k)$ corresponding to a nilpotent representation E of Q such that $(\mathbf{DF}(\mathcal{X},x)_\bullet, \leq)$ is isomorphic to $(\mathbf{DF}(\mathcal{R}ep(Q,d),z)_\bullet, \leq)$ as formal fans with relation. By Proposition 7.1.3 and Lemma 7.3.10, $(\mathbf{DF}(\mathcal{R}ep(Q,d),z)_\bullet, \leq)$ is isomorphic to $(\mathbf{DF}(M)_\bullet, \leq)$, where M is the lattice of subrepresentations of E, which is an artinian lattice. Therefore, by Proposition 6.5.37, $(L_x, \leq)$ is isomorphic to $(M, \leq)$ and there is a canonical isomorphism $(\mathbf{DF}(L_x)_\bullet, \leq) \cong (\mathbf{DF}(\mathcal{X},x)_\bullet, \leq)$ of formal fans with relation, independent of the choice of Q, d, E or of the isomorphism $(\mathbf{DF}(\mathcal{X},x)_\bullet, \leq) \cong (\mathbf{DF}(\mathcal{R}ep(Q,d),z)_\bullet, \leq)$. □

If $\lambda \in \mathbf{DF}^{\mathbb{Q}}(\mathcal{X},x)_1$, we will denote F^λ the $\mathbb{Q}$-filtration of L_x associated to λ under the isomorphism above.

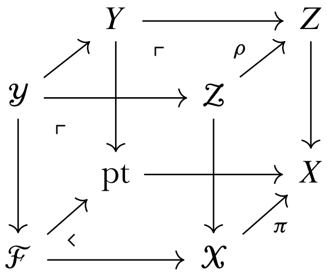
If $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})_{\geq 0} \subset \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ is a lamp on the algebraic stack $\mathcal{X}$, we endow $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ with the pullback lamp along the canonical map $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}) \to \mathcal{X}$.

Proposition 7.3.12. *Let $\mathcal{X}$ be a linearly lit good moduli stack over k. Then every quasi-compact component of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ is linearly lit. Moreover, if $x \in \mathcal{X}(k)$ and $\lambda \in \mathbf{DF}^{\mathbb{Q}}(\mathcal{X},x)_1$, then there is a canonical isomorphism (defined in the proof) of lattices $L_{\mathrm{gr}\,\lambda} \cong \mathrm{Grad}_{F^\lambda}(L_x)$.*

Remark **7.3.13.** If $\mathcal{X}$ has a norm on graded points, which we assume for the main constructions in this work, then every component of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ is quasi-compact by [36, Proposition 3.8.2].

Proof. A quasi-compact component $\mathcal{Z}$ of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ satisfies Assumption 7.3.6, by Lemma 2.6.11. Let $\pi\colon \mathcal{X} \to X$ be the good moduli space of $\mathcal{X}$, and let $\rho\colon \mathcal{Z} \to Z$ be that of $\mathcal{Z}$. A point $g \in \mathcal{Z}(k)$ lies over a k-point $x = u(g)$ of $\mathcal{X}$. Let $\mathcal{F} = \pi^{-1}\pi(x)$. Then $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{F}) \to \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ is a closed immersion (for example by the same argument as in Proposition 2.2.16). Let $\mathcal{Y}$ be the component of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{F})$ containing

g. The good moduli space of $\mathcal{F}$ is $\operatorname{Spec}(k) = \mathrm{pt}$. Let $Y = \mathrm{pt} \times_{\pi(x), X} Z$ and form a cube



where the marked faces are cartesian. Thus the face $Y, Z, \mathcal{Y}, \mathcal{Z}$ is also cartesian and therefore $\mathcal{Y} \to Y$ is a good moduli space. It is thus enough to prove the statement for the linearly lit stack $\mathcal{F}$. By embedding $\mathcal{F} \hookrightarrow \mathcal{R}ep(Q, d)$, we may assume $\mathcal{X} = \mathcal{R}ep(Q, d)$, which is linearly lit by Theorem 7.4.12.

The connected component of $\operatorname{Grad}_{\mathbb{Q}}(\mathcal{R}ep(Q, d))$ containing g is isomorphic to $\operatorname{Rep}(Q, d)^{\lambda, 0}/L(\lambda)$ for some rational one-parameter subgroup λ of $G(d)$ [36, Theorem 1.4.8]. Now λ induces a direct sum decomposition $k^{d_i} = \bigoplus_{c \in \mathbb{Q}} k_c^{d_i}$ for each vertex $i \in Q_0$. Define a quiver Q' with set of vertices

$$Q'_0 = \{(i, c) \in Q_0 \times \mathbb{Q} \mid k_c^{d_i} \neq 0\}.$$

An arrow $\alpha\colon (i, c) \to (j, c)$ in Q' is just an arrow $\alpha\colon i \to j$ in Q, and there are no arrows $(i, c) \to (j, c')$ for $c \neq c'$.

We have a dimension vector d' on Q' given by $d'_{i,c} = \dim k_c^{d_i}$. The fixed points of λ in $\operatorname{Hom}(k^{d_i}, k^{d_j})$ are $\bigoplus_{c \in \mathbb{Q}} \operatorname{Hom}(k_c^{d_i}, k_c^{d_j})$, and the centraliser $L(\lambda)$ of λ in $G(d)$ is identified with $G(d')$. Therefore we have an isomorphism $\operatorname{Rep}(Q, d)^{\lambda, 0}/L(\lambda) \cong \mathcal{R}ep(Q', d')$.

If $\beta\colon \mathbb{G}_{m,k} \to L(\lambda)$ is a cocharacter, β defines a grading $k_c^{d_i} = \bigoplus_{c' \in \mathbb{Z}} (k_c^{d_i})_{c'}^{\beta}$. If $\beta'\colon \mathbb{G}_{m,k} \to L(\lambda) \to G(d)$ denotes the composition of β and the inclusion $L(\lambda) \to G(d)$, then β' defines a grading $k^{d_i} = \bigoplus_{c' \in \mathbb{Z}} (k^{d_i})_{c'}^{\beta'}$ and we have $(k^{d_i})_{c'}^{\beta'} = \bigoplus_{c \in \mathbb{Q}} (k_c^{d_i})_{c'}^{\beta}$. We have $\beta \geq 0$ if and only if $(k_c^{d_i})_{c'}^{\beta} = 0$ whenever $c' < 0$, while $\beta' \geq 0$ if and only if $(k^{d_i})_{c'}^{\beta'} = 0$ whenever $c' < 0$. Thus $\beta \geq 0$ if and only if $\beta' \geq 0$ and hence the isomorphism $\operatorname{Rep}(Q, d)^{\lambda, 0}/L(\lambda) \cong \mathcal{R}ep(Q', d')$ is compatible with lamps. This proves the first part of the proposition.

Now let $\lambda \in \mathbf{DF}^{\mathbb{Q}}(\mathcal{X}, x)_1$. We define a map

$$o\colon \mathbf{DF}^{\mathbb{Q}}(\mathcal{X}, x)_1 \to \mathbf{DF}^{\mathbb{Q}}(\operatorname{Grad}_{\mathbb{Q}}(\mathcal{X}), \operatorname{gr} \lambda)_1$$

as follows. If $\mu \in \mathbf{DF}^{\mathbb{Q}}(\mathcal{X}, x)_1$, let $\gamma \in \mathbf{DF}^{\mathbb{Q}}(\mathcal{X}, x)_2$ be the unique element such that $v_1 \gamma = \mu$ and $v_2 \gamma = \lambda$ (Proposition 2.7.6). Noting that γ can be seen as an element of $\mathbf{DF}^{\mathbb{Q}}(\operatorname{Filt}_{\mathbb{Q}}(\mathcal{X}), \lambda)_1$, we define $o(\mu) \in \mathbf{DF}^{\mathbb{Q}}(\operatorname{Grad}_{\mathbb{Q}}(\mathcal{X}), \operatorname{gr} \lambda)_1$ to be the image of γ

under the associated graded map $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) \to \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$. The definition of o does not depend on any choices.

If $\mathcal{X} = \mathcal{R}ep(Q, d)$, then the point x corresponds to a representation E of Q and $L_x \cong L_E$. One then sees easily by hand that there is an identification $v\colon L_{\mathrm{gr}\,\lambda} \xrightarrow{\sim} \mathrm{Grad}_{F^\lambda}(L_x)$ which a priori depends on the choices made. The map o is then identified with

$$o'\colon L_x \to \mathrm{Grad}_{F^\lambda}(L_x)\colon a \mapsto ((a \wedge F^\lambda_{\geq c}) \vee F^\lambda_{>c})_{c\in\mathbb{Q}}.$$

This proves that o is injective, that $o(L_x) \subset L_{\mathrm{gr}\,\lambda}$ and that the induced map $o\colon L_x \to L_{\mathrm{gr}\,\lambda}$ is a map of lattices.

In general, a choice of embedding $\mathcal{F} \hookrightarrow \mathcal{R}ep(Q, d)$ gives a commuting triangle

$$\begin{array}{ccc} & \overset{o}{\nearrow} & L_{\mathrm{gr}\,\lambda} \\ L_x & & \downarrow{\scriptstyle v}\,\wr \\ & \underset{o'}{\searrow} & \mathrm{Grad}_{F^\lambda}(L_x). \end{array}$$

For each $c \in \mathbb{Q}$, $o(F^\lambda_{>c})$ has a unique complement $o(F^\lambda_{>c})^\circ$, since this is true for o'. We define a map

$$\begin{aligned} u\colon \mathrm{Grad}_{F^\lambda}(L_x) &\to L_{\mathrm{gr}\,\lambda} \\ (a_c)_{c\in\mathbb{Q}} &\mapsto \bigvee_{c\in\mathbb{Q}} o(a_c) \wedge o(F^\lambda_{>c})^\circ. \end{aligned}$$

This map is an isomorphism, since the analogous map for o' is. Now u does not depend on choices, and it is the desired canonical isomorphism. In fact, $u = v^{-1}$. □

Using Proposition 7.3.12 iteratively, we get a canonical isomorphism between sets of sequential filtrations.

Corollary 7.3.14. *Let $\mathcal{X}$ be a linearly lit good moduli stack over k endowed with a norm on graded points, and let $x \in \mathcal{X}(k)$. Then there is a canonical isomorphism $\mathbb{Q}^\infty\text{-}\mathrm{Filt}(\mathcal{X}, x) \cong \mathbb{Q}^\infty\text{-}\mathrm{Filt}(L_x)$ between the set of sequential filtrations of x and that of sequential filtrations of the lattice L_x.* □

Note that we make the assumption that $\mathcal{X}$ has a norm on graded points in Corollary 7.3.14 to guarantee that the connected components of $\mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X})$ are quasi-compact.

7.4 EXAMPLES OF LINEARLY LIT STACKS

We show here that many algebraic stacks parametrising objects in an abelian category and admitting a good moduli space are naturally linearly lit. Our main tool for that purpose is Theorem 7.4.4, that we hope can be applied to essentially all good moduli stacks of linear origin. We apply the theorem to moduli stacks of objects in an abelian category in the sense of Artin–Zhang and to moduli stacks of quiver representations.

7.4.1 A CRITERION FOR LINEAR LITNESS

Let k be an algebraically closed field and let $\mathcal{X}$ be a quasi-separated algebraic stack locally of finite type over k. If $x \in \mathcal{X}(k)$ is a closed point, then there is a unique reduced closed substack $\mathcal{Z}$ of $\mathcal{X}$ whose topological space is just $\{x\}$. In fact, $\mathcal{Z} \cong BG_x$ is the residual gerbe at x. Let $\mathcal{I}$ be the ideal sheaf defining $\mathcal{Z}$ and let $\iota\colon BG_x \to \mathcal{X}$ denote the closed immersion. Then $\iota^*\mathcal{I} = \mathcal{I}/\mathcal{I}^2$ is a coherent sheaf on BG_x, that is, a finite dimension representation of G_x. Its dual $N_x = (\iota^*\mathcal{I})^\vee$ is called the *normal space* of $\mathcal{X}$ at x. If G_x is smooth, then N_x is also the tangent space of $\mathcal{X}$ at x. We may also consider the quotient stack $\mathcal{N}_x := N_x/G_x$, which equals the relative spectrum $\mathcal{N}_x = \operatorname{Spec}_{BG_x} \operatorname{Sym} \iota^*\mathcal{I}$, and call it the *normal stack* of $\mathcal{X}$ at x. We have representable morphisms $\mathcal{N}_x \to BG_x$ and $BG_x \to \mathcal{X}$, so if $\mathcal{X}$ is lit, then BG_x and $\mathcal{N}_x$ inherit lamps from $\mathcal{X}$ by pullback along these maps.

PROPOSITION 7.4.1. *Let k be an algebraically closed field, and let $\mathcal{X}$ be a quasi-separated algebraic stack locally of finite type over k. Suppose $\mathcal{X}$ has a good moduli space $\pi\colon \mathcal{X} \to X$ with affine diagonal. Let $x \in \mathcal{X}(k)$ be a closed point and denote $\mathcal{F} = \pi^{-1}\pi(x)$ the fibre of π containing x. Let $\mathcal{N}_x$ be the normal stack of $\mathcal{X}$ at x.*

Then there is a pointed closed immersion $\iota\colon (\mathcal{F}, x) \to (\mathcal{N}_x, 0)$ and a commuting triangle

$$\begin{array}{ccc} & BG_x & \\ \swarrow & & \searrow \\ \mathcal{F} & \xrightarrow{\ \iota\ } & \mathcal{N}_x \end{array} \tag{7.1}$$

where the arrows $BG_x \to \mathcal{F}$ and $BG_x \to \mathcal{N}_x$ are the natural identifications of BG_x with the residual gerbes of $\mathcal{F}$ and $\mathcal{N}_x$ at x and 0.

Moreover, if $\mathcal{X}$ is lit, then ι is a lit closed immersion, where $\mathcal{F}$ and $\mathcal{N}_x$ are endowed with the induced lamps.

Proof. Since x is closed and $\mathcal{X}$ has a good moduli space, the stabiliser G_x is linearly reductive and thus $\mathcal{N}_x$ has a good moduli space $\rho\colon \mathcal{N}_x \to W$. Call $\mathcal{R} = \rho^{-1}\rho(0)$ the fibre. By [6, Theorem 4.16, (3)], there are cartesian diagrams

$$\begin{array}{ccc} \widehat{\mathcal{X}}_x & \longrightarrow & \mathcal{X} \\ {\scriptstyle \pi'}\downarrow & \ulcorner & \downarrow{\scriptstyle \pi} \\ \operatorname{Spec} \widehat{\mathcal{O}}_{X,\pi(x)} & \longrightarrow & X \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \widehat{\mathcal{N}}_x & \longrightarrow & \mathcal{N}_x \\ {\scriptstyle \rho'}\downarrow & \ulcorner & \downarrow{\scriptstyle \rho} \\ \operatorname{Spec} \widehat{\mathcal{O}}_{W,\rho(0)} & \longrightarrow & W \end{array}$$

where $\widehat{\mathcal{X}}_x$ is the completion of $\mathcal{X}$ at x (that is, the completion of the adic sequence of thickenings of x, see [7, Definition 1.9]) and $\widehat{\mathcal{N}}_x$ is the completion of $\mathcal{N}_x$ at 0. Thus $\mathcal{F}$ and $\mathcal{R}$ are the special fibres of π' and ρ'.

In [6, Proof of Theorem 1.1, p.688-689] it is shown that there is a closed immersion $\widehat{\mathcal{X}}_x \to \widehat{\mathcal{N}}_x$. We now recall the argument, with some modifications, for the convenience of the reader.

For $n \in \mathbb{N}$, let us denote $\mathcal{X}^{[n]}$ the nth infinitesimal thickening of $\mathcal{X}$ at x, that is, if $\mathcal{I}$ is the ideal sheaf of the closed substack BG_x of $\mathcal{X}$, then $\mathcal{X}^{[n]} = \operatorname{Spec}_{\mathcal{X}} \mathcal{O}_{\mathcal{X}}/\mathcal{I}^{n+1}$. Similarly, let $\mathcal{N}^{[n]}$ be the nth infinitesimal thickening of $\mathcal{N}_x$ at 0. We have natural identifications $\mathcal{X}^{[0]} \cong BG_x$ and $\mathcal{N}^{[0]} \cong BG_x$. We denote $g_{ij} \colon \mathcal{X}^{[j]} \to \mathcal{X}^{[i]}$ the relevant closed immersions for $j \leq i$. We have a diagram of solid arrows

$$\begin{array}{ccccc} & & \mathcal{X}^{[0]} & & \\ & {\scriptstyle g_{10}}\swarrow & & \searrow{\scriptstyle \sim} & \\ \mathcal{X}^{[1]} & & \overset{f}{\dashrightarrow} & & BG_x \end{array}$$

Since g_{10} is a square zero extension of ideal sheaf $\mathcal{I}/\mathcal{I}^2$, by [67, Theorem 8.5] the obstruction to the existence of the dotted arrow f lives in $\operatorname{Ext}^1_{BG_x}(\mathbb{L}_{BG_x/k}, \mathcal{I}/\mathcal{I}^2)$, which is 0 because BG_x is smooth, and hence the cotangent complex $\mathbb{L}_{BG_x/k}$ is concentrated in degrees $[0,1]$, and G_x is linearly reductive. Since $\mathcal{X}^{[0]} \to BG_x$ is affine and $\mathcal{X}^{[1]}$ is a square zero extension of $\mathcal{X}^{[0]}$, by dévissage and Serre's criterion for algebraic spaces, we have that f is affine too. We have a short exact sequence

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{O}_{\mathcal{X}^{[1]}} \underset{\dashleftarrow}{\overset{}{\longrightarrow}} \mathcal{O}_{\mathcal{X}^{[0]}} \longrightarrow 0$$

in $\operatorname{QCoh}(BG_x)$, of which f provides the dotted splitting. Thus $\mathcal{O}_{\mathcal{X}^{[1]}} \cong \mathcal{O}_{BG_x} \oplus \varepsilon(\mathcal{I}/\mathcal{I}^2)$, where $\varepsilon^2 = 0$, and hence $\mathcal{X}^{[1]} \cong \mathcal{N}^{[1]}$ under BG_x.

We now produce compatible morphisms $g_n \colon \mathcal{X}^{[n]} \to \mathcal{N}_x$. For $i = 0, 1$, we let g_i be the composition $\mathcal{X}^{[i]} \cong \mathcal{N}^{[i]} \to \mathcal{N}_x$. If g_n has been constructed, the obstruction for g_n to lift to a map $\mathcal{X}^{[n+1]} \to \mathcal{N}_x$ is an element of $\operatorname{Ext}^1_{\mathcal{X}^{[n]}}(g_n^*\mathbb{L}_{\mathcal{N}_x/k}, (g_{n0})_*\mathcal{I}^{n+1}/\mathcal{I}^{n+2})$. By adjunction, we have

$$\operatorname{Ext}^1_{\mathcal{X}^{[n]}}(g_n^*\mathbb{L}_{\mathcal{N}_x/k}, (g_{n0})_*\mathcal{I}^{n+1}/\mathcal{I}^{n+2}) = \operatorname{Ext}^1_{BG_x}(g_0^*\mathbb{L}_{\mathcal{N}_x/k}, \mathcal{I}^{n+1}/\mathcal{I}^{n+2}).$$

This group is 0 because $\mathcal{N}_x$ is smooth, and so $\mathbb{L}_{\mathcal{N}_x/k}$ has Tor-amplitude $[0,1]$, and BG_x is linearly reductive. Hence the g_n have been constructed for all n. By [6, Proposition A.8, (1)] applied to each g_n, we see that g_n is a closed immersion for all n. Each of the g_n factors through the closed immersion $\mathcal{N}^{[n]} \to \mathcal{N}_x$, and after passing to the completion the closed immersions $\mathcal{X}^{[n]} \to \mathcal{N}^{[n]}$ induce, by Tannaka duality [6, Theorem 2.7], a closed immersion $\widehat{\mathcal{X}}_x \to \widehat{\mathcal{N}}_x$.

The map $\widehat{\mathcal{X}}_x \to \widehat{\mathcal{N}}_x$ yields a closed immersion $\mathcal{F} \to \mathcal{R}$, and thus the desired result, since $\mathcal{R}$ is a closed substack of $\mathcal{N}_x$. By construction, there is a triangle

$$\begin{array}{ccc} & BG_x & \\ \swarrow & & \searrow \\ \widehat{\mathcal{X}}_x & \longrightarrow & \widehat{\mathcal{N}}_x \end{array}$$

identifying BG_x with the 0th infinitesimal thickenings $\mathcal{X}^{[0]}$ and $\mathcal{N}^{[0]}$. After taking special fibres of the good moduli spaces, we get the triangle (7.1).

The map $BG_x \to \mathcal{F}$ induces a bijection $\pi_0(\mathrm{Grad}_{\mathbb{Q}}(BG_x)) \to \pi_0(\mathrm{Grad}_{\mathbb{Q}}(\mathcal{F}))$ by Proposition 3.1.7, and the same holds for the maps $BG_x \to \mathcal{N}_x$ and $\mathcal{N}_x \to BG_x$. This implies, together with the commutativity of (7.1), that $\iota\colon \mathcal{F} \to \mathcal{N}_x$ is lit. □

Now let R be a finite-dimensional associative k-algebra. We can give the groups of units $R^\times$ a structure of algebraic group $\underline{R}^\times$ over k by setting $\underline{R}^\times(A) = (R \otimes_k A)^\times$ for a commutative k-algebra A. Actually, we can enhance R to an algebraic monoid $\underline{R}$ by setting $\underline{R}(A) = R \otimes_k A$ with the multiplication. Note that $\underline{R}$ is isomorphic to $\mathbb{A}_k^{\dim R}$. There is a map $l\colon \underline{R} \to \underline{\mathrm{End}}_k(R)$, sending an element a of $R \otimes_k A$ to the endomorphism of $R \otimes_k A$ given by left multiplication by a. Then $\underline{R}^\times = l^{-1}(\mathrm{GL}(R))$. Therefore $\underline{R}^\times$ is an affine open subscheme of $\underline{R}$, and it is therefore an affine smooth connected algebraic group over k.

Lemma 7.4.2. *Let R be a finite dimensional associative algebra over k. Then R is semisimple if and only if $\underline{R}^\times$ is reductive. Moreover, in that case $\underline{R}^\times \cong \prod_{i=1}^n \mathrm{GL}_{n_i,k}$, for some $n_i \in \mathbb{Z}_{>0}$.*

Proof. By definition, R is semisimple if its Jacobson radical $\mathrm{Jac}(R)$ is trivial. The Jacobson radical is a nilpotent two-sided ideal, so $1 + \underline{\mathrm{Jac}(R)}$ is a smooth connected unipotent normal subgroup of $\underline{R}^\times$. Therefore if $\underline{R}^\times$ is reductive, $\mathrm{Jac}(R)$ has to be trivial and hence R is semisimple.

Conversely, if R is semisimple, by the Artin–Wedderburn Theorem [47, Structure theorem for semi-primitive artinian rings, §4.4], together with the fact that there are no nontrivial finite-dimensional division algebras over the algebraically closed field k, there is a k-algebra isomorphism $R \cong \prod_i \mathrm{Mat}_{n_i \times n_i, k}$, and thus $\underline{R}^\times \cong \prod_i \mathrm{GL}_{n_i,k}$, which is reductive. □

Lemma 7.4.3. *Let R be a semisimple finite dimensional associative algebra over k and let M be a finite-dimensional R-R-bimodule. Then the quotient stack $\underline{M}/\underline{R}^\times$, where $\underline{R}^\times$ acts on $\underline{M}$ by $g \cdot m = gmg^{-1}$, is isomorphic as a lit stack to $\mathcal{R}ep(Q, d)$ for some quiver Q and some dimension vector d.*

Note that above, since $\underline{R}^\times$ is the group of units of an algebraic monoid $\underline{R}$, there is a natural lamp on $B\underline{R}^\times$ from Example 7.3.2 and thus a lamp on $\underline{M}/\underline{R}^\times$ by pulling back along $\underline{M}/\underline{R}^\times \to B\underline{R}^\times$.

Proof. Again, by the Artin–Wedderburn theorem there are finite-dimensional vector spaces $V_1, \dots, V_n$ and a k-algebra isomorphism $R \cong \mathrm{End}(V_1) \times \cdots \times \mathrm{End}(V_n)$. Let us call $e_i = (0, \cdots, \mathrm{id}_{V_i}, \cdots, 0)$. The e_i are orthogonal central idempotents and $1 = \sum_{i=1}^n e_i$. Therefore each $e_i M e_j$ is an R-R-bimodule and there is a splitting $M = \bigoplus_{i,j} e_i M e_j$ as bimodules. The R-R-bimodule structure on $e_i M e_j$ is induced by restriction of scalars from the obvious $\mathrm{End}(V_i)$-$\mathrm{End}(V_j)$-bimodule structure, which is the same as a left $\mathrm{End}(V_i \otimes V_j^\vee)$-module structure, since $\mathrm{End}(V_i) \otimes_k \mathrm{End}(V_j)^{\mathrm{op}} = \mathrm{End}(V_i) \otimes_k \mathrm{End}(V_j^\vee) = \mathrm{End}(V_i \otimes V_j^\vee)$. The $\mathrm{End}(V_i \otimes V_j^\vee)$-$k$-bimodule $V_i \otimes V_j^\vee$ gives a Morita equivalence between k and $\mathrm{End}(V_i \otimes V_j^\vee)$, so there is $n_{ij} \in \mathbb{N}$ and an isomorphism $e_i M e_j \cong (V_i \otimes V_j^\vee)^{\oplus n_{ij}}$ of left $\mathrm{End}(V_i \otimes V_j^\vee)$-modules, or equivalently of $\mathrm{End}(V_i)$-$\mathrm{End}(V_j)$-bimodules. Thus M is isomorphic to $\bigoplus_{i,j} \mathrm{Hom}(V_j, V_i)^{n_{ij}}$ as an R-R-bimodule, and thus also as $\underline{R}^\times = \prod_i \mathrm{GL}(V_i)$-representations, where the action is as in the statement of the lemma. Let Q be the quiver with set of vertices $\{1, \dots, n\}$ and with n_{ij} arrows from j to i, and take the dimension vector d with $d_i = \dim V_i$. Then

$$M/\underline{R}^\times \cong \bigoplus_{i,j} \mathrm{Hom}(V_j, V_i)^{\oplus n_{ij}} / \prod_i \mathrm{GL}(V_i) \cong \mathcal{R}ep(Q, d),$$

as desired. The isomorphism is lit because the standard lamp on $B \prod_i \mathrm{GL}(V_i)$ comes from the partial compactification $\prod_i \mathrm{End}(V_i)$ of $\prod_i \mathrm{GL}(V_i)$, exactly as the lamp in $B\underline{R}^\times$. □

The following theorem is our main tool for proving that a lit stack is linearly lit.

Theorem 7.4.4. *Let $\mathcal{X}$ be an algebraic stack over k satisfying Assumption 7.3.6, endowed with a lamp and having a good moduli space $\pi \colon \mathcal{X} \to X$. Suppose that for every closed k-point x of $\mathcal{X}$ there exists a finite-dimensional associative k-algebra R and an isomorphism $G_x \cong \underline{R}^\times$ such that the lamp on BG_x inherited from $\mathcal{X}$ agrees with the one coming from the partial compactification $\underline{R}$ of G_x as in Example 7.3.2. Suppose further that the G_x-action on the normal space N_x comes from an R-R-bimodule structure on N_x. Then $\mathcal{X}$ is linearly lit.*

Proof. This follows readily from Proposition 7.4.1 and Lemmas 7.4.2 and 7.4.3. □

Remark **7.4.5.** In positive characteristic, for the condition in Theorem 7.4.4 to hold, the stabiliser G_x of $\mathcal{X}$ at x should be linearly reductive, and hence a torus.

7.4.2 THE MODULI STACK OF ARTIN AND ZHANG

We fix an algebraically closed field k and consider a k-linear locally noetherian Grothendieck abelian category $\mathcal{A}$. This means that $\mathcal{A}$ is a k-linear abelian category that is cocomplete, where filtered colimits are exact, and that has a set of noetherian generators (we recall the notion of noetherian object below). An object E in $\mathcal{A}$ is said to be

1. of *finite type*, if for every filtered diagram $(F_i)_{i\in I}$ in $\mathcal{A}$ where all maps $F_i \to F_j$ in the diagram are monomorphisms, we have a canonical isomorphism
$$\operatorname{Hom}(E, \operatorname{colim}_{i\in I} F_i) \cong \operatorname{colim}_{i\in I} \operatorname{Hom}(E, F_i);$$
2. of *finite presentation*, if for every filtered diagram $(F_i)_{i\in I}$ in $\mathcal{A}$, we have a canonical isomorphism
$$\operatorname{Hom}(E, \operatorname{colim}_{i\in I} F_i) \cong \operatorname{colim}_{i\in I} \operatorname{Hom}(E, F_i);$$
3. *noetherian*, if every subobject of E is of finite type.

For every commutative k-algebra R, there is a notion of base change category $\mathcal{A}_R$ which is an R-linear abelian category. An object of $\mathcal{A}_R$ is a pair (E, ρ) where E is an object of $\mathcal{A}$ and $\rho\colon R \to \operatorname{End}(E)$ is a k-algebra homomorphism. A map from (E, ρ) to (E', ρ') is a map $f\colon E \to E'$ in $\mathcal{A}$ such that for all $r \in R$ we have $f \circ \rho(r) = \rho'(r) \circ f$. It turns out that $\mathcal{A}_R$ is an R-linear Grothendieck abelian category [9, Proposition B2.2] and that it is noetherian if R is essentially of finite type [9, Corollary B6.3].

If R is a commutative k-algebra, N is an R-module, and M is an object of $\mathcal{A}_R$, there is a notion of tensor product $N \otimes_R M$. One can describe it by taking a free presentation
$$R^{\oplus J} \longrightarrow R^{\oplus I} \longrightarrow N \longrightarrow 0$$
of N. Then $N \otimes_R M$ can be defined by exactness of the sequence
$$M^{\oplus J} \longrightarrow M^{\oplus I} \longrightarrow N \otimes_R M \longrightarrow 0.$$
In fact, $(-) \otimes_R M\colon R\text{-Mod} \to \mathcal{A}_R$ is characterised by being the left adjoint to $\operatorname{Hom}(M, -)\colon \mathcal{A}_R \to R\text{-Mod}$.

The tensor product can be used to give an alternative description of the base change categories $\mathcal{A}_R$. An object of $\mathcal{A}_R$ is given by an object E of $\mathcal{A}$ and a map $R \otimes_k E \to E$ such that the usual diagrams commute.

If R' is a commutative R-algebra, we have the base-change functor $R' \otimes_R (-)\colon \mathcal{A}_R \to \mathcal{A}_{R'}$, left adjoint to the restriction of scalars functor $\mathcal{A}_{R'} \to \mathcal{A}_R$.

Definition 7.4.6. An object $M \in \mathcal{A}_R$ is said to be *flat* (over R), if the functor $(-) \otimes_R M : R\text{-}\mathrm{Mod} \to \mathcal{A}_R$ is exact.

The properties of being flat or finitely presented are preserved under base-change. The following is [8, Definition 7.8].

Definition 7.4.7. The *moduli stack $\mathcal{M}_\mathcal{A}$ of objects of $\mathcal{A}$* is the prestack over k defined by setting, for a commutative k-algebra R, the groupoid $\mathcal{M}_\mathcal{A}(R)$ to be that of finitely presented flat objects in $\mathcal{A}_R$.

The prestack $\mathcal{M}_\mathcal{A}$ is actually a stack for the fppf topology [9, Theorem C8.6].

Suppose now that $\mathcal{M}_\mathcal{A}$ is algebraic and locally of finite type over k. By [8, Lemma 7.20], $\mathcal{M}_\mathcal{A}$ has affine diagonal, and thus it satisfies Assumption 2.2.3. We can define a lamp $\mathrm{Grad}_\mathbb{Q}(\mathcal{M}_\mathcal{A})_{\geq 0} \subset \mathrm{Grad}(\mathcal{M}_\mathcal{A})$ on $\mathcal{M}_\mathcal{A}$ as follows. A graded point $\mathrm{Spec}\, R \to \mathrm{Grad}(\mathcal{M}_\mathcal{A})$ is a $\mathbb{Z}$-graded object $\bigoplus_{n\in\mathbb{Z}} E_n$ in $\mathcal{A}_R$ with $E_n = 0$ for all but finitely many n and such that each E_n is flat and finitely presented [8, Proposition 7.12]. Suppose that R is a finite type k-algebra. Then the condition that $E_n = 0$ for $n < 0$ is open on $\mathrm{Spec}\, R$ by Nakayama's lemma [9, Theorem C4.3] and thus it defines an open substack $\mathrm{Grad}(\mathcal{M}_\mathcal{A})_{\geq 0}$ of $\mathrm{Grad}(\mathcal{M}_\mathcal{A})$.

Denoting $E = \bigoplus_{n\in\mathbb{Z}} E_n$ the underlying object of the grading, we may define $\underline{\mathrm{End}}(E)$ as a functor from R-algebras to sets by $\underline{\mathrm{End}}(E)(R') = \mathrm{End}_{R'}(R' \otimes_R E)$. It is proven in [8, Proof of Lemma 7.19] that $\underline{\mathrm{End}}(E)$ is represented by an affine scheme of finite type over R. In fact, $\underline{\mathrm{End}}(E)$ is an algebraic monoid with unit group $\mathrm{Aut}(E)$ The grading $E = \bigoplus_{n\in\mathbb{Z}} E_n$ corresponds to a one-parameter subgroup $\lambda : \mathbb{G}_{m,R} \to \mathrm{Aut}(E)$. The multiplicative group acts on E diagonally with respect to the direct sum decomposition, and it acts in E_n with weight n. Therefore, the condition that $E_n = 0$ for $n < 0$ is equivalent to the existence of $\lim_{t\to 0} \lambda(t)$ in $\underline{\mathrm{End}}(E)$. This is in turn equivalent to the identity section $\mathrm{id}_E \in \underline{\mathrm{End}}(E)(R)$ factoring through the attractor $\underline{\mathrm{End}}(E)^+$ (where the $\mathbb{G}_m$-action is given by λ). Since $\underline{\mathrm{End}}(E)$ is affine, $\underline{\mathrm{End}}(E)^+$ is a closed subscheme of $\underline{\mathrm{End}}(E)$. Therefore the open substack $\mathrm{Grad}(\mathcal{M}_\mathcal{A})_{\geq 0}$ is also closed. Since $\mathrm{Grad}(\mathcal{M}_\mathcal{A})_{\geq 0}$ is equivariant for the $\mathbb{N}_{>0}$-action on $\mathrm{Grad}(\mathcal{X})$, it defines a closed and open substack $\mathrm{Grad}_\mathbb{Q}(\mathcal{M}_\mathcal{A})_{\geq 0}$ of $\mathrm{Grad}_\mathbb{Q}(\mathcal{M}_\mathcal{A})$.

Definition 7.4.8. Suppose $\mathcal{M}_\mathcal{A}$ is an algebraic stack locally of finite type over k. The *standard lamp* on $\mathcal{M}_\mathcal{A}$ is the lamp $\mathrm{Grad}_\mathbb{Q}(\mathcal{M}_\mathcal{A})_{\geq 0}$ defined above.

Remark **7.4.9.** By the description in terms of $\underline{\mathrm{End}}(E)$, we can regard the lamp on $\mathcal{M}$ as a shadow of the enhancement of $\mathcal{M}_\mathcal{A}$ to a *stack in categories*, where we remember all morphisms between objects and not just isomorphisms. See [16, Section 4] for a precise definition of this notion.

THEOREM 7.4.10. *Suppose that $\mathcal{M}_{\mathcal{A}}$ is an algebraic stack, locally of finite type over k, and let $\mathcal{X}$ be a quasi-compact open substack of $\mathcal{M}_{\mathcal{A}}$, endowed with the lamp inherited from the standard lamp on $\mathcal{M}_{\mathcal{A}}$. Suppose that $\mathcal{X}$ has a good moduli space $\pi\colon \mathcal{X} \to X$. Then $\mathcal{X}$ has affine diagonal and is linearly lit.*

Proof. By [8, Lemma 7.20], the algebraic stack $\mathcal{M}_{\mathcal{A}}$ has affine diagonal over k, and thus the same holds for $\mathcal{X}$. Therefore $\mathcal{X}$ satisfies Assumption 7.3.6.

Let $x \in \mathcal{X}(k)$ be a point with linearly reductive stabiliser, corresponding to an object $E \in \mathcal{A}$. By [9, Proposition B3.21], we have $\underline{\mathrm{End}}(E)(R) = \mathrm{End}(E) \otimes_k R = \underline{\mathrm{End}(E)}(R)$ for any k-algebra R, and thus the stabiliser group G_x is the group of units of the associative k-algebra $\mathrm{End}(E)$, which has to be finite-dimensional because $\underline{\mathrm{End}}(E)$ is an affine scheme of finite type over k. In particular, G_x is smooth, and thus the normal space N_x coincides with the tangent space T_x of $\mathcal{M}_{\mathcal{A}}$ at x. By [9, Proposition E1.1], the tangent space T_x is canonically identified with $\mathrm{Ext}^1(E, E)$, which is thus finite dimensional, since $\mathcal{M}_{\mathcal{A}}$ is locally of finite type over the field k.

An element of T_x is a pair (E', α) where $E' \in \mathcal{M}_{\mathcal{A}}(k[\varepsilon])$, with $\varepsilon^2 = 0$, and $\alpha\colon k \otimes_{k[\varepsilon]} E' \to E$ is an isomorphism. An element $g \in G_x(k)$ acts on T_x by $g(E', \alpha) = (E', g\alpha)$. One gets an element u of $\mathrm{Ext}^1(E, E)$ by tensoring the short exact sequence

$$0 \longrightarrow k \longrightarrow k[\varepsilon] \longrightarrow k \longrightarrow 0$$

of $k[\varepsilon]$-modules with E', obtaining a self-extension

$$0 \longrightarrow E \longrightarrow E' \longrightarrow E \longrightarrow 0.$$

The $\mathrm{End}(E)$ - $\mathrm{End}(E)$-bimodule structure on $\mathrm{Ext}^1(E, E)$ can be described as follows. If $a \in \mathrm{End}(E)$, then au corresponds to the pulled back short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & E'' & \longrightarrow & E & \longrightarrow & 0 \\ & & \downarrow & \ulcorner & \downarrow & \ulcorner & \downarrow{\scriptstyle a} & & \\ 0 & \longrightarrow & E & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & 0, \end{array}$$

while ua corresponds to the pushed forward sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & 0 \\ & & {\scriptstyle a}\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow & & \\ 0 & \longrightarrow & E & \longrightarrow & E'' & \longrightarrow & E & \longrightarrow & 0. \end{array}$$

From these descriptions we see that the G_x-action on T_x is the one coming from the $\mathrm{End}(E)$ - $\mathrm{End}(E)$-bimodule structure on $\mathrm{Ext}^1(E, E)$. Note that it is indeed enough to check the previous statement on k-points of G_x, since it is smooth.

We conclude by Theorem 7.4.4. □

Remark **7.4.11.** There are other constructions of moduli stacks of objects in linear categories, notably Lieblich's stack of universally gluable perfect complexes [62] and Toën and Vaquié's stack of objects in a DG-category [75]. We expect that Theorem 7.4.4 can be applied in these cases without difficulties to show that good moduli stacks arising as (classical truncations) of open substacks of these moduli stacks are naturally linearly lit when we work over a base algebraically closed field. We have not pursued proofs of these statements here.

7.4.3 MODULI OF SEMISTABLE QUIVER REPRESENTATION

We fix an algebraically closed field k. Let Q be a quiver and let d be a dimension vector for Q. The stack of finite dimensional representations of Q does not coincide with Artin–Zhang's stack of objects in the abelian category of representations of Q unless Q is acyclic. Nevertheless, $\mathcal{Rep}(Q,d)$ is still linearly lit.

THEOREM 7.4.12. *Let $\mathcal{X}$ be an open substack of $\mathcal{Rep}(Q,d)$ that admits a good moduli space. Then $\mathcal{X}$ is linearly lit with the lamp inherited from $\mathcal{Rep}(Q,d)$.*

Proof. It is clear that $\mathcal{X}$ satisfy Assumption 7.3.6 because $\mathcal{Rep}(Q,d)$ does. Let $x \in \mathcal{X}(k)$ be a point with linearly reductive stabiliser. The point x corresponds to a representation E of Q. The automorphism group G_x is identified with the group of units of $\mathrm{End}(E)$, so it is in particular smooth and thus the normal space at x coincides with the tangent space. The normal stack is then $\mathcal{N}_x = \mathrm{Ext}^1(E,E)/G_x$ by deformation theory, where the action on G_x on $\mathrm{Ext}^1(E,E)$ comes from the natural $\mathrm{End}(E)$-$\mathrm{End}(E)$-bimodule structure of $\mathrm{Ext}^1(E,E)$ by the same argument as in the proof of Theorem 7.4.10. We conclude by Theorem 7.4.4. □

Remark **7.4.13.** The prototypical example of open substack $\mathcal{X}$ as in the Theorem is the semistable locus for a line bundle on $\mathcal{Rep}(Q,d)$.

7.5 MAIN COMPARISON RESULT

While living in seemingly very different worlds, the iterated balanced filtration for stacks and the iterated HKKP filtration for lattices agree when both make sense. This section is devoted to the proof of such comparison result (Corollary 7.5.11).

7.5.1 LINEAR NORMS

We now study a class of norms on linearly lit stacks that interact well with the lamp.

We fix an algebraically closed field k, and we let Q be a quiver, d a dimension vector for Q, and we fix a family $(m_i)_{i\in Q_0}$ of positive rational numbers $m_i \in \mathbb{Q}_{>0}$. We assume as usual that $G(d)$ is a torus if k is of positive characteristic. Let x be a k-point of $\mathcal{Rep}(Q,d)$, corresponding to a quiver representation E, and let λ be a $\mathbb{Q}$-grading of x (i.e. a rational one-parameter subgroup of $\mathrm{Aut}(x)$), corresponding to direct sum decompositions

$$E_i = \bigoplus_{c\in\mathbb{Q}} E_{i,c}$$

for each $i \in Q_0$. We set

$$\|\lambda\|^2 = \sum_{i\in Q_0}\sum_{c\in\mathbb{Q}} m_i c^2 \dim(E_{i,c}).$$

It is not hard to see that this formula defines a norm on graded points of $\mathcal{Rep}(Q,d)$.

Definition 7.5.1. A norm on graded points of $\mathcal{Rep}(Q,d)$ is said to be *standard* if it is induced by a family $(m_i)_{i\in Q_0} \in (\mathbb{Q}_{>0})^{Q_0}$ as above.

Now let $\mathcal{X}$ be a linearly lit good moduli stack over k endowed with a norm on graded points q. For any k-point of $\mathcal{X}$, recall that we have a canonical isomorphism $\mathbf{DF}(\mathcal{X},x)_\bullet \cong \mathbf{DF}(L_x)_\bullet$ of formal fans and an inclusion $L_x \hookrightarrow \mathbf{DF}(\mathcal{X},x)_1$, so we see elements a of the lattice L_x as filtrations of x and in particular we can consider their norm $\|a\|^2 = q(a)$.

Definition 7.5.2 (Linear norm)**.** We say that the norm q on $\mathcal{X}$ is *linear* if the following holds: for every k-point x of $\mathcal{X}$ and for every $\mathbb{Q}$-grading λ of x, corresponding to a $\mathbb{Q}$-grading $(a_c)_{c\in\mathbb{Q}}$ of L_x (Definition 6.5.1) we have

$$\|\lambda\|^2 = \sum_{c\in\mathbb{Q}} c^2\|a_c\|^2. \tag{7.2}$$

Proposition 7.5.3. *A norm q on graded points of $\mathcal{Rep}(Q,d)$ is linear if and only if it is standard.*

Proof. If q is standard, then it clearly is linear. To prove the converse, we can assume that Q has no arrows, and thus that we are working with $BG(d)$. Indeed, since $\mathrm{Grad}^n(\mathcal{Rep}(Q,d))$ and $\mathrm{Grad}^n(BG(d))$ have the same connected components for all n, norms on graded points of $\mathcal{Rep}(Q,r)$ are the same as norms on $BG(d)$. We can also assume that $d_i \geq 1$ for all $i \in Q_0$.

Fix a k-point x corresponding to a representation E. Let $V(i)$ be the skyscraper representation at vertex i. For any embedding $\alpha\colon V(i) \hookrightarrow E$, consider the corresponding filtration with weights $1,0$ and set $m_i = \|F_i\|^2$. This number does not depend on the choice of α, since any two different α's give conjugate F_i's.

Let E_1, E_2, E_3 be subrepresentations of E with $E = E_1 \oplus E_2 \oplus E_3$. Recall we are regarding subrepresentations of E as filtrations, and thus we can consider their norm. Let F_1 (resp. F_2) be the grading E_1, E_2, E_3 with weights $1, 0, 0$ (resp. $0, 1, 0$). We have $q(F_1) = q(E_1)$, $q(F_2) = q(E_2)$, $q(F_1 + F_2) = q(E_1 \oplus E_2)$ and $q(F_1 - F_2) = q(E_1) + q(E_2)$ by linearity of q. From the equality $q(F_1 + F_2) + q(F_1 - F_2) = 2q(F_1) + 2q(F_2)$, it follows that $q(E_1 \oplus E_2) = q(E_1) + q(E_2)$. By applying this formula repeatedly, we get that

$$q(E') = \sum_{i \in Q_0} m_i \dim(E'_i)$$

for any subrepresentation E' of E. Combining this with (7.2), it follows that q is standard. □

PROPOSITION 7.5.4. *Let $\pi\colon \mathcal{X} \to X$ be the good moduli space of $\mathcal{X}$. The norm q on $\mathcal{X}$ is linear if and only for all closed k-points y of $\mathcal{X}$ the lit embedding*

$$\iota\colon (\pi^{-1}\pi(y), y) \hookrightarrow (\mathcal{R}ep(Q, d), 0)$$

in Definition 7.3.7 can be chosen to preserve norms for some (uniquely determined) standard norm on $\mathcal{R}ep(Q, d)$. Equivalently, all such embeddings have this property.

Proof. Call $\mathcal{F} = \pi^{-1}\pi(x)$. Given a lit embedding ι as above, the stacks $\mathrm{Grad}^n(\mathcal{F})$ and $\mathrm{Grad}^n(\mathcal{R}ep(Q, d))$ have the same components for all n. Thus giving a norm on $\mathcal{F}$ is equivalent to giving a norm on $\mathcal{R}ep(Q, d)$. If the norm on $\mathcal{X}$ is linear, then so will be the induced norm on $\mathcal{R}ep(Q, d)$, and thus it will be standard by Proposition 7.5.3. □

DEFINITION 7.5.5. A *linearly lit normed good moduli stack* is a linearly lit good moduli stack endowed with a linear norm on graded points.

DEFINITION 7.5.6 (Induced norm on lattice)**.** Let $\mathcal{X}$ be a linearly lit normed good moduli stack and let x be a k-point of $\mathcal{X}$. For $a \leq b$ in L_x, we let $X_x([a, b]) = \|b\|^2 - \|a\|^2$. Then X_x is a norm on the lattice L_x, and we will regard L_x as a normed lattice endowed with X_x.

That X_x defines a norm on L_x is immediate in the case $\mathcal{X} = \mathcal{R}ep(Q, d)$, and the general case follows from this by the embedding in Definition 7.3.7.

LEMMA 7.5.7. *Let $(\mathcal{X}, x)$ be a pointed linearly lit normed good moduli stack with norm on graded points q, and let $\lambda \in \mathbf{DF}^{\mathbb{Q}}(\mathcal{X}, x)_1$. Let ℓ_q be the canonical linear form on graded points of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ (Definition 2.4.7), and let $(F^\lambda)^\vee$ be the linear form on $\mathrm{Grad}_{F^\lambda}(L_x)$ induced by the norm on L_x (see discussion after Definition 6.6.23). Then for all $\mu \in \mathbf{DF}(\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}), \mathrm{gr}\,\lambda)_1$, we have the equality*

$$\langle \mu, \ell_q \rangle = \langle F^\mu, (F^\lambda)^\vee \rangle.$$

Proof. We may assume $\mathscr{X} = \mathscr{Rep}(Q, d)$ with standard norm given by $(m_i)_{i \in Q_0}$. The point x corresponds to a representation $E = ((E_i)_{i \in Q_0}, (f_\alpha)_{\alpha \in Q_1})$, and the filtration λ of x is represented by a rational one-parameter subgroup β of $G(d)$ that induces a grading $E_i = \bigoplus_{c \in \mathbb{Q}} E_{i,c}$ for each $i \in Q_0$. With respect to this decomposition, each f_α is written in coordinates as $(f_{\alpha,c,c'})_{c,c' \in \mathbb{Q}}$.

The component of $\mathrm{Grad}_{\mathbb{Q}}(\mathscr{X})$ containing $\mathrm{gr}\,\lambda$ is of the form $\mathscr{Rep}(Q', d')$, with Q', d' as in the proof of Proposition 7.3.12. The point $\mathrm{gr}\,\lambda$ has coordinates $(f_{\alpha,c,c})_{c \in \mathbb{Q}}$. If a rational cocharacter μ of $L(\beta)$ defines a filtration of $\mathrm{gr}\,\lambda$, then $\langle \mu, \ell_q \rangle = \langle \mu, \beta^\vee \rangle = (\mu, \beta)$. A grading $E_{i,c} = \bigoplus_{c \in \mathbb{Q}} E_{i,c,c'}$ for each $E_{i,c}$ is determined by μ. From the definitions, we have the formula

$$(\mu, \beta) = \sum_{i,c,c'} cc' \dim(E_{i,c,c'}) m_i.$$

Also from the definitions, we have

$$\begin{aligned} \langle F^\mu, (F^\lambda)^\vee \rangle &= \sum_{c' \in \mathbb{Q}} c' (F^\lambda)^\vee([F^\mu_{>c'}, F^\mu_{\geq c'}]) = \sum_{c' \in \mathbb{Q}} c' \sum_{i \in Q_0} (F^\lambda)^\vee(E_{i,\bullet,c'}) \\ &= \sum_{c' \in \mathbb{Q}} c' \sum_{i \in Q_0} \sum_{c \in \mathbb{Q}} c m_i \dim(E_{i,c,c'}), \end{aligned}$$

as desired. □

7.5.2 A LEMMA ABOUT SEMISTABILITY

We briefly turn our attention to a lemma in Geometric Invariant Theory that we will need later.

Let k be an algebraically closed field. Let G be a linearly reductive algebraic group over k, endowed with a norm q on cocharacters, and let V be a finite dimensional representation of G. Let $\lambda \in \Gamma^{\mathbb{Q}}(G)$ be a rational cocharacter of G, inducing a direct sum decomposition $V = \bigoplus_{s \in \mathbb{Q}} V_s$. The commutator subgroup $L(\lambda)$ acts on V_1. We endow $V_1/L(\lambda)$ with the linear form $\ell_1 = -\lambda^\vee$ on graded points, where $\langle \gamma, \lambda^\vee \rangle = (\gamma, \lambda)$ for any $\gamma \in \Gamma^{\mathbb{Q}}(L(\lambda))$, and $(-,-)$ is the inner product that q induces in some split subtorus of $L(\lambda)$ containing γ and λ. Let ℓ be the linear form on $\mathbb{P}(V_1)/L(\lambda)$ determined by $\mathscr{O}(1)$, and we consider semistability on $\mathbb{P}(V_1)/L(\lambda)$ with respect to the linear form $\ell_2 = \ell - \lambda^\vee/\|\lambda\|^2$.

Lemma 7.5.8. *In the situation above, we have a cartesian square*

$$\begin{array}{ccc} (V_1/L(\lambda))^{ss(\ell_1)} & \longrightarrow & (V_1 \setminus \{0\})/L(\lambda) \\ \downarrow & \ulcorner & \downarrow \\ (\mathbb{P}(V_1)/L(\lambda))^{ss(\ell_2)} & \longrightarrow & \mathbb{P}(V_1)/L(\lambda). \end{array}$$

Proof. Let $V_1 \setminus \{0\} \to \mathbb{P}(V_1)\colon x \mapsto [x]$ be the projection and let $v\colon \operatorname{Spec}(k) \to V_1 \setminus \{0\}$ be a geometric point. We wish to show that v is semistable for ℓ_1 if and only if $[v]$ is semistable for ℓ_2.

If T is a maximal torus of $L(\lambda)$, then v (resp. $[v]$) is semistable for $L(\lambda)$ if and only if gv (resp. $g[v]$) is semistable for T for all $g \in L(\lambda)(k)$. It is thus enough to assume that $L(\lambda) = T$ is a torus.

We write $\Xi \subset \Gamma_{\mathbb{Q}}(T)$ for the *state* of v. That is, if $V_1 = \bigoplus_{\chi\in\Gamma_{\mathbb{Q}}(T)}(V_1)_\chi$ is the grading induced by the T-action on V_1 and $p_\chi\colon V_1 \to (V_1)_\chi$ are the projections, then $\Xi = \{\chi \in \Gamma_{\mathbb{Q}}(T)\colon p_\chi(v) \neq 0\}$. Now we have

1. $v \in V_1^{\mathrm{ss}(\ell_1)}(k)$ if and only if $\lambda^\vee \in \operatorname{cone}(\Xi)$, and
2. $[v] \in \mathbb{P}(V_1)^{\mathrm{ss}(\ell_2)}(k)$ if and only if $\lambda^\vee/\|\lambda\|^2 \in \operatorname{conv}(\Xi)$ or, equivalently, if $0 \in \operatorname{conv}\left(\Xi - \lambda^\vee/\|\lambda\|^2\right)$.

Here, $\operatorname{cone}(\Xi)$ is the convex cone in $\Gamma_{\mathbb{Q}}(T)$ generated by Ξ, while $\operatorname{conv}(\Xi)$ is the convex hull of Ξ. The result follows from Lemma 5.2.17. □

7.5.3 THE TORSOR CHAIN AND THE HKKP CHAIN

We get to the main result of this chapter.

THEOREM 7.5.9. *Let $(\mathcal{X}, x)$ be a pointed linearly lit normed good moduli stack over k, and let $(\mathcal{Y}_n, y_n, \eta_n, v_n)$ be its torsor chain (Definition 4.3.3). Then:*

1. *Each $\mathcal{Y}_n$ is, with the lamp inherited from $\mathcal{X}$, a linearly lit normed good moduli stack.*
2. *Under the canonical identification $\mathbf{DF}^{\mathbb{Q}}(\mathcal{Y}_n, y_n)_1 \cong \mathbf{DF}^{\mathbb{Q}}(L_{y_n})_1\colon \lambda \mapsto F^\lambda$, the balanced filtration η_n of $(\mathcal{Y}_n, y_n)$ (Definition 4.2.2) coincides with the HKKP filtration of the normed lattice L_{y_n} (Definition 6.6.27).*
3. *Each map $v_n\colon (\mathcal{Y}_{n+1}, y_{n+1}) \to (\operatorname{Grad}_{\mathbb{Q}}(\mathcal{Y}_n), \operatorname{gr}\eta_n)$ induces a norm-preserving injection of lattices*
$$c_n\colon L_{y_{n+1}} \hookrightarrow \operatorname{Grad}_{F^{\eta_n}}(L_{y_n}),$$
thus giving a chain of lattices $(L_{y_n}, F^{\eta_n}, c_n)_{n\in\mathbb{N}}$.
4. *The chain $(L_{y_n}, F^{\eta_n}, c_n)_{n\in\mathbb{N}}$ is the HKKP chain of the normed artinian lattice L_x.*

Proof. We may assume that the good moduli space of $\mathcal{X}$ is a point after replacing $\mathcal{X}$ with the fibre of its good moduli space containing x, since this does not change the $(\mathcal{Y}_n, y_n)$ for $n \geq 1$. By embedding $\mathcal{X}$ in quiver moduli, we can assume that $\mathcal{X} = \mathcal{R}ep(Q, d)$ for some quiver Q and dimension vector d, endowed with the standard lamp and a standard norm on graded points given by positive rational numbers $(m_i)_{i\in Q_0}$, and that x corresponds to a nilpotent representation E of Q.

If x is a closed point, then 1 holds trivially. In fact, x is a closed point if and only if L_x is complemented, by Propositions 2.7.14, 6.5.34 and 7.3.11. In this case both the balanced filtration and the HKKP filtration are zero, so 2 holds, and 3 and 4 hold trivially. We now assume that x is not closed.

We now prove 2 for $n = 0$. We identify $L_x \cong L_E$, the lattice of subrepresentations of E. We have $\|\lambda\|^2 = \|F^\lambda\|^2$, where $\|\lambda\|^2$ is the norm-squared of λ with respect to the norm on graded points on $\mathcal{X}$, and $\|F^\lambda\|^2$ is the norm-squared of F^λ with respect to the induced norm on the lattice L_x.

On the other hand, Kempf's intersection number of λ and $\mathcal{X}^{\max}$ agrees with the complementedness of F^λ, $\langle \lambda, \mathcal{X}^{\max} \rangle = \langle F^\lambda, \mathfrak{l} \rangle$, by Proposition 7.2.4. The balanced filtration of $(\mathcal{X}, x)$ is the element λ of $\mathbf{DF}^{\mathbb{Q}}(\mathcal{X}, x)_1$ with $\langle \lambda, \mathcal{X}^{\max} \rangle \geq 1$ and smallest norm. The HKKP filtration of L_x is the element F of $\mathbf{DF}^{\mathbb{Q}}(L_x)_1$ with $\langle F, \mathfrak{l} \rangle \geq 1$ and smallest norm. Thus λ is the balanced filtration of $(\mathcal{X}, x)$ if and only if F^λ is the HKKP filtration of L_x. This proves 2 for $n = 0$.

We write $\operatorname{Rep}(Q, d) = V$, $G = G(d)$ and $V = W \oplus V^G$, where W is the Reynolds operator of G applied to V, that is, the sum of all simple nontrivial subrepresentations of V.

Let $\pi\colon \mathcal{X} = V/G \to (W /\!\!/ G) \times V^G$ be the good moduli space. Since E is nilpotent, x lies in W, and the fibre $\mathcal{F} = \pi^{-1}\pi(x) = \pi^{-1}(0)$ is also the fibre of $W/G \to W /\!\!/ G$ at 0, and so a closed substack of W/G containing 0. We may replace $\mathcal{F}$ by W/G since this will only change $\mathcal{Y}_1$ by a stack of which it is a closed substack. To construct $\mathcal{Y}_1$, we blow-up W/G at $(W/G)^{\max} = \{0\}/G$ and consider the centre $\mathcal{Z}$ of the weak Θ-stratum of $\operatorname{Bl}_0 W/G$ containing x. By Lemma 4.3.2, $\mathcal{Z}$ is an open substack of $\operatorname{Grad}_{\mathbb{Q}}(\mathbb{P}(W)/G)$, since $\mathbb{P}(W)/G$ is the exceptional divisor. In fact, $\mathcal{Z}$ is the centre of the weak Θ-stratum of $\mathbb{P}(W)/G$ containing $u(\operatorname{gr} \eta_0)$. Here, the linear form ℓ on graded points considered is given by the line bundle $\mathcal{O}_{\mathbb{P}(W)/G}(1)$. The balanced filtration η_0 is given by a rational cocharacter β of G which induces a direct sum decomposition $W = \bigoplus_{c \in \mathbb{Q}} W_c$. Since $\langle \eta_0, \ell \rangle = 1$, the component of $\operatorname{Grad}_{\mathbb{Q}}(\mathbb{P}(W)/G)$ containing $u(\operatorname{gr} \eta_0)$ is $\mathbb{P}(W_1)/L(\beta)$. We also denote ℓ the linear form on $\mathbb{P}(W_1)/L(\beta)$ given by $\mathcal{O}_{\mathbb{P}(W_1)/L(\beta)}(1)$, and we consider also the linear form $\beta^\vee$ given by $\langle \lambda, \beta^\vee \rangle = (\lambda, \beta)$, where $(-,-)$ is the inner product induced by the norm on graded points.

By [55, Remark 12.21], $\mathcal{Z}$ is the open substack $\mathcal{Z} = \mathbb{P}(W_1)^{\mathrm{ss}(\ell - \beta^\vee/\|\beta\|^2)}/L(\beta)$, and $\mathcal{Y}_1$ is the $\mathbb{G}_{m,k}$-torsor over $\mathcal{Z}$ determined by $\mathcal{O}(1)$. Thus $\mathcal{Y}_1 = W_1^{\mathrm{ss}(-\beta^\vee)}/L(\beta)$ by Lemma 7.5.8.

Claim **7.5.10.** There is a quiver Q', a dimension vector d' and an isomorphism $W_1/L(\beta)$

$\cong \mathcal{R}ep(Q', d')$ as lit stacks such that the induced norm on $\mathcal{R}ep(Q', d')$ is linear.

Proof of Claim. Let $V = \bigoplus_{c \in \mathbb{Q}} V_c$ be the grading that β induces on V. Note that $W_1 = V_1$.

The rational cocharacter β also induces a direct sum decomposition $E_i = \bigoplus_{c \in \mathbb{Q}} E_{i,c}$ for each $i \in Q_0$. We define a quiver Q' as follows. Its set of vertices is

$$Q'_0 = \{(i, c) \in Q_0 \times \mathbb{Q} \mid E_{i,c} \neq 0\}.$$

The set of arrows from (i, c) to (j, c') is empty if $c' \neq c + 1$ and otherwise it is the set of arrows from i to j in Q. We set a dimension vector d' of Q' by $d'_{(i,c)} = \dim E_{i,c}$.

In this way we have identifications $V_1 = \mathrm{Rep}(Q', d')$ and $L(\beta) = G(d')$, so we get an isomorphism $V_1/L(\beta) \cong \mathcal{R}ep(Q', d')$. The induced lamp on $V_1/L(\beta)$ is the standard one on $\mathcal{R}ep(Q', d')$: if $\lambda: \mathbb{G}_{m,k} \to L(\beta)$ is a cocharacter, then the induced grading on $E_{i,c}$ has only nonnegative weights for all i and c if and only if the grading on E_i has only nonnegative weights for all i. The induced norm is standard with $m_{(i,c)} = m_i$. □

By the claim, $\mathcal{Y}_1$ is an open substack of $\mathcal{R}ep(Q', d')$ admitting a good moduli space, so it is linearly lit by Theorem 7.4.12. This proves 1 for $n = 1$ and, by induction, for all n. Since we knew 2 for $n = 0$, now we know it for all n.

It is left to show that the map $(\mathcal{Y}_1, y_1) \to (\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}), \mathrm{gr}\, \eta_0)$ induces a map $L_{y_1} \hookrightarrow \mathrm{Grad}_{F^{\eta_0}}(L_x)$ of lattices that identifies L_{y_1} with $\Lambda(F^{\eta_0})^{\mathrm{ss}(-(F^{\eta_0})^\vee)}$ (Definitions 6.6.22 and 6.6.24). By induction, this is enough to prove 3 and 4 for all n.

Recall that $V_1 = W_1$, so $\mathcal{Y}_1 = V_1^{\mathrm{ss}}/L(\beta)$, where semistability is with respect to $\beta^\vee$. Let $\mathcal{U} = V_1/L(\beta) = \mathcal{R}ep(Q', d')$, and let $u \in \mathcal{U}(k)$ be the point y_1 seen as a point of $\mathcal{U}$. The map $(\mathcal{Y}_1, y_1) \to (\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}), \mathrm{gr}\, \eta_0)$ factors as

$$(\mathcal{Y}_1, y_1) \to (\mathcal{U}, u) \to (BL(\beta), 0) \to (V_0/L(\beta), 0) \to (\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}), \mathrm{gr}\, \eta_0).$$

By Proposition 7.3.12, the lattice $L_{(V_0/L(\beta),0)} = L_{(BL(\beta),0)}$ is canonically identified with $\mathrm{Grad}_{F^{\eta_0}}(L_x)$. First we show that the map $(\mathcal{U}, u) \to (BL(\beta), 0)$ induces an injection of lattices $L_u \hookrightarrow \mathrm{Grad}_{F^{\eta_0}}(L_x)$ that identifies L_u with $\Lambda(F^{\eta_0})$.

The representation E of Q, corresponding to the point x, can be written in coordinates as $E = ((E_i)_{i \in Q_0}, (f_\alpha)_{\alpha \in Q_1})$, where $f_\alpha \in \mathrm{Hom}(E_{s(\alpha)}, E_{t(\alpha)})$. Since β induces a direct sum decomposition $E_i = \bigoplus_{c \in \mathbb{Q}} E_{i,c}$ for each $i \in Q_0$, each f_α can be written in coordinates as $f_\alpha = (f_{\alpha,c',c})_{c,c' \in \mathbb{Q}}$, with $f_{\alpha,c',c} \in \mathrm{Hom}(E_{s(\alpha),c}, E_{t(\alpha),c'})$.

The point y_1 is then $(f_{\alpha,c+1,c})_{\alpha \in Q_1, c \in \mathbb{Q}}$ in coordinates, corresponding to a representation E' of Q'. From these descriptions it is clear that $\mathbf{DF}(\mathcal{U}, u)_1 \to \mathbf{DF}(BL(\beta), 0)_1$ induces an injection of lattices $L_u = L_{E'} \hookrightarrow \mathrm{Grad}_{F^{\eta_0}}(L_x)$.

An element of $\mathrm{Grad}_{F^{\eta_0}}(L_x)$ is the data of a subspace $M_{i,c}$ of each $E_{i,c}$. The $M_{i,c}$ define a subrepresentation M of E' if and only if $f_{\alpha,c+1,c}(M_{s(\alpha),c}) \subset M_{t(\alpha),c+1}$ for all $\alpha \in Q_1$ and $c \in \mathbb{Q}$.

Let $M_{i,\geq c} = (\bigoplus_{c'>c} E_{i,c'}) \oplus M_{i,c}$ and $E_{i,\geq c} = \bigoplus_{c' \geq c} E_{i,c'}$. The $E_{i,\geq c}$ define a subrepresentation $E_{\geq c}$ of E just because β defines a filtration of x. Moreover, $E_{\geq c}/E_{>c}$ is semisimple, and actually null (Definition 7.2.1) because E is nilpotent. Therefore the $M_{i,\geq c}$ define a subrepresentation $M_{\geq c}$ of E.

The condition for the $M_{i,c}$ to define an element of $\Lambda(F^{\eta_0})$ is that $M_{\geq c}/M_{\geq c+1}$ is semisimple (or equivalently, since $M_{\geq c}$ is nilpotent, null) for all $c \in \mathbb{Q}$. Choose a splitting $E_{i,c} = M_{i,c} \oplus M'_{i,c}$ as vector spaces. Then we can write

$$M_{i,\geq c}/M_{i,\geq c+1} = M'_{i,c+1} \oplus \left(\bigoplus_{c<c'<c+1} E_{i,c'} \right) \oplus M_{i,c}.$$

Because F^{η_0} is paracomplemented, we have that $f_{\alpha,c_2,c_1} = 0$ for $c_1 \leq c_2 < c_1 + 1$. Therefore $M_{\geq c}/M_{\geq c+1}$ has coordinates $\left(f_{i,c+1,c}|_{M_i}^{M'_{i,c+1}} \right)$. These are all 0 if and only if $f_{i,c+1,c}(M_{i,c}) \subset M_{i,c+1}$ for all c and i. Therefore $L_u = L_{E'} = \Lambda(F^{\eta_0})$, as desired.

Now by [36, Lemma 5.5.11 and Lemma 5.5.12], we have

$$\mathbf{DF}(\mathcal{Y}_1, y_1)_1 = \{\lambda \in \mathbf{DF}(\mathcal{U}, u)_1 \mid \beta^\vee(\lambda) = 0\}.$$

Therefore

$$\begin{aligned} L_{y_1} = \{\lambda \in \mathbf{DF}(\mathcal{U}, u)_1 \mid \lambda \geq 0,\ \beta^\vee(\lambda) = 0,\ &\text{and if } \mu \in \mathbf{DF}(\mathcal{U}, u)_1 \\ &\text{satisfies } 0 \leq 2\mu \leq \lambda \text{ and } \beta^\vee(\mu) = 0, \text{ then } \mu = 0\}. \end{aligned}$$

Note that if $\mu \in \mathbf{DF}(\mathcal{U}, u)_1$ and $0 \leq 2\mu \leq \lambda$, then $0 \leq 2\beta^\vee(\mu) \leq \beta^\vee(\lambda)$, so $\beta^\vee(\lambda) = 0$ implies $\beta^\vee(\mu) = 0$. Therefore the map $\mathbf{DF}(\mathcal{Y}_1, y_1)_1 \to \mathbf{DF}(\mathcal{U}, u)_1$ sends L_{y_1} injectively into L_u, and $L_{y_1} = \{\lambda \in L_u \mid \beta^\vee(\lambda) = 0\}$. Note that $\beta^\vee(\lambda) = (F^{\eta_0})^\vee(\lambda)$ by Lemma 7.5.7, so we get $L_{y_1} = \Lambda(F^{\eta_0})^{\mathrm{ss}}$, as desired. □

Corollary 7.5.11. *In the setting of Theorem 7.5.9 we have that, under the canonical bijection* $\mathbb{Q}^\infty\text{-}\mathrm{Filt}(\mathcal{X}, x) = \mathbb{Q}^\infty\text{-}\mathrm{Filt}(L_x)$*, the iterated balanced filtration of x equals the iterated HKKP filtration of L_x.*

Proof. This follows directly from the definition of the iterated HKKP filtration (Definition 6.7.3), from Theorem 7.5.9 and from Theorem 4.3.4. □

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