Linearly reductive algebraic groups in characteristic zero

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1 Introduction

The main result in these notes is the equivalence of the properties of reductivity and linear reductivity for algebraic groups over a field of characteristic 0. The main tool in the proof is the Lie algebra of an algebraic group, and several results relating these two objects and their representations.

We start by giving a survey of the basic definitions and properties of algebraic groups in Section 2. Then we introduce reductive groups. The theory of reductive algebraic groups and their representations is very rich. For instance, it is possible to classify split reductive groups and their irreducible representations using root systems (see [4], Chapters 21 and 22), and the theory even extends to group schemes over a general base (see [3]). However, in these notes we do not go any deeper into the theory of reductive groups, and we hope that the reader is already be convinced of their importance or will take it for granted.

We then introduce the Lie algebra of an algebraic group and prove some of its basic properties. In Section 6, we prove further results relating representations of a group and of its Lie algebra, culminating in the proof that reductive group are linearly reductive in characteristic 0.

As a demonstration of the usefulness of the notion of linear reductivity, we prove in Section 7 that the ring of invariants of a dual action of a linearly reductive group on a finite type k-algebra is again of finite type.

In an attempt to compensate the lack of examples throughout these notes, we have included a section on a simple class of algebraic groups, that of diagonalisable groups.

None of the material presented here is original.

Contents

2 Foundations

The choice of foundations for these notes is that of schemes, functors and sheaves. When dealing with algebraically closed characteristic 0 base fields, other foundations like the *espaces algébriques de Serre* suffice, but the author wanted to get familiar with the more flexible language of schemes, that is optimal to deal with non-algebraically closed or positive characteristic fields, and eventually group schemes over a more general base. In this section there are almost no proofs, limiting ourselves to give precise statements that will be referred to in the rest of the notes.

2.1 Definitions

Let k be a field. We will denote $\mathbf{Alg}_k$ the category of k-algebras. By an *algebraic scheme* over k we mean a scheme of finite type over k. Any algebraic scheme X defines a functor $\mathbf{Alg}_k \to \mathbf{Set}\colon R \mapsto X(R) = \mathrm{Hom}_k(\mathrm{Spec}(R), X)$, its functor of points; and the functor from the category of algebraic schemes to the functor category $[\mathbf{Alg}_k, \mathbf{Set}]$ that associates an algebraic scheme over k with its functor of points is fully faithful. We will thus adopt the functorial point of view and identify an algebraic scheme with its functor of points.

We define the category of *algebraic group functors* over k to be the functor category $[\mathbf{Alg}_k, \mathbf{Grp}]$, where $\mathbf{Grp}$ is the category of groups. Morphisms in this category are also called *homomorphisms.* We define the category of *algebraic groups* over k to be the full subcategory of $[\mathbf{Alg}_k, \mathbf{Grp}]$ consisting of those functors $G\colon \mathbf{Alg}_k \to \mathbf{Grp}$ such that the composition of G with the forgetful functor $\mathbf{Grp} \to \mathbf{Set}$ is an algebraic scheme. In that case we will sometimes say that the functor G is representable by a scheme.

Alternatively, due to the Yoneda lemma, one may define an algebraic group to be an algebraic scheme G endowed with maps $\mu\colon G \times G \to G$ (the multiplication), $e\colon \mathrm{Spec}(k) \to G$ (the identity element) and $\beta\colon G \to G$ (the inversion) such that the usual diagrams of associativity, inverses and identity element commute. Then one may also define the notion of a homomorphism by asking the suitable square to commute. If G is affine, this structure can be defined "reversing all arrows" in the category of k-algebras, and one gets the notion of *Hopf algebra.*

The scheme $\mathrm{Spec}(k)$ admits a unique structure of algebraic group, called the trivial group and denoted $\mathbf{1}$. For any other algebraic group G, there is a unique homomorphism $\mathbf{1} \to G$, whose image is the k-rational point e, *the identity element.*

We will say that an algebraic group G is *commutative* if for all k-algebras R, the group $G(R)$ is commutative.

If k' is a field extension of k, and G is an algebraic group over k, the base change $G_{k'} = G \times_k \mathrm{Spec}(k')$ has a natural structure of algebraic group over k'. We will sometimes need to base change to the algebraic closure $\overline{k}$ of k to define some concepts or prove some properties.

2.2 Subgroups and kernels

We start by defining subgroups.

Definition 2.1. Let $\varphi\colon H \to G$ be a homomorphism of algebraic groups over a field k. If φ is a closed immersion, then we say that H is a *subgroup* of G (via φ) if φ is a closed immersion.

Subgroups of G form a subcategory of the category of algebraic groups over G that is a preorder. As usual, we will tacitly identify two isomorphic objects of the preorder of subgroups of G.

Note that if G is an algebraic group and $\varphi\colon X \to G$ is a monomorphism of algebraic schemes over k, then there is at most one algebraic group structure on X that makes φ a homomorphism.

Now we define kernels.

Definition 2.2. Let $\varphi\colon H \to G$ be a homomorphism of algebraic groups over a field k. The *kernel* $\ker(\varphi)$ of φ is the subfunctor $R \mapsto \ker(H(R) \to G(R))$ of H. It is an algebraic subgroup of H because it is represented by the fibre product of H and $\mathbf{1}$ over G, and $\mathbf{1} \to G$ is a closed immersion.

We have the following characterisation of subgroups ([4], 5.31, 5.34):

Proposition 2.3. *Let $\varphi\colon H \to G$ be a homomorphism of algebraic groups over a field k. Then the following conditions on φ are equivalent:*

1. *φ is an immersion,*
2. *φ is a closed immersion,*
3. *for all k-algebras R, the map $H(R) \to G(R)$ is injective,*
4. *$\ker(\varphi) = \mathbf{1}$.*

If these conditions hold we say that φ is injective*.*

A subgroup N of G is said to be *normal* if for all k-algebras R, $N(R)$ is normal in $G(R)$. From the definition, it follows that the kernel of a homomorphism is a normal subgroup.

Sometimes it can be checked whether two subgroups are equal on points:

Proposition 2.4. *Let G be an algebraic group over a field k and let H and H' be smooth subgroups of G. If there is a separably closed field extension k' of k such that $H(k') = H'(k)$, then $H = H'$.*

For a proof, see [4], 1.18, 1.44.

2.3 Group sheaves

It is not trivial to define what a quotient (i.e. surjective in an adequate sense) homomorphism of algebraic groups should be, nor is it to state and prove the analogue for algebraic groups of the isomorphism theorems for abstract groups. A too naïve idea would be to adopt the functorial point of view and to do everything for abstract groups. A quotient map $G \to Q$ would then be one such that $G(R) \to Q(R)$ is surjective for all k-algebras R. Unfortunately, many group homomorphisms that we would like to consider as quotient maps do not satisfy this condition (for example $\mathbb{G}_m \to \mathbb{G}_m$ given by $a \mapsto a^n$ on k-algebras, where the multiplicative group G_m is defined to be $\mathbb{G}_m(R) = R^\times$, the group of units, on k-algebras R). The functorial point of view is still key, but one has to observe that representable functors are sheaves for suitable Grothendieck topologies on $\mathbf{Aff}_k^{\mathrm{op}}$. The most convenient Grothendieck topology to establish the isomorphism theorems for algebraic groups turns out to be the fppf topology. See [8], 2.55, for a proof that the fppf topology is subcanonical, i.e. that schemes (when seen as functors) are sheaves. These observations bring our attention to sheaves of groups for a moment.

Let $\mathcal{C}$ be a site with associated topos τ. We are interested in the category $\mathbf{Grp}_\tau$ of group objects in τ, and in proving that the isomorphism theorems hold in this category. The category $\mathbf{Grp}_\tau$ is just the full subcategory of the functor category $[\mathcal{C}^{\mathrm{op}}, \mathbf{Grp}]$ whose objects are sheaves. We will call the objects of $\mathbf{Grp}_\tau$ *group sheaves.* In $\mathbf{Grp}_\tau$ there is the trivial group $\mathbf{1}$, and for every two group sheaves H and G there is a unique homomorphism $H \to G$ that factors through $\mathbf{1}$, called the *trivial homomorphism* and denoted e.

A map $f\colon A \to B$ in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is said to be *sheaf-surjective* if for every $U \in \mathrm{ob}\mathcal{C}$ and every $b \in B(U)$, there is a covering sieve $\mathcal{S}$ of U such that for every $V \to U$ in $\mathcal{S}$, the restriction $b|_V$ of b lies in the image of $f_V\colon A(V) \to B(V)$. If both A and B are sheaves, f is sheaf-surjective if and only if it is an epimorphism in τ.

Definition 2.5. Let $f\colon H \to G$ be a homomorphism of group sheaves

1. The homomorphism f is said to be a *quotient map* if it is *sheaf-surjective.*

2. The homomorphism f is said to be *injective* if for all objects U of $\mathcal{C}$, the homomorphism $f_U\colon H(U) \to G(U)$ is injective. We will also say that H is a *subgroup* of G.

3. If f is injective, then H is *normal* in G (through f) if for all U in $\mathcal{C}$, $H(U)$ is normal in $G(H)$.

4. The *presheaf image* $\mathrm{im}_{\mathrm{pre}} f$ of f is the presheaf $U \mapsto \mathrm{im}(f_U)$.

5. the *image* $\mathrm{im}\, f$ of f is the sheafification of $\mathrm{im}_{\mathrm{pre}} f$. It comes with a morphism $\mathrm{im}\, f \to G$.

6. The *kernel* $\ker f$ of f is the equaliser $\mathrm{eq}(f, e)$ in $\mathbf{Grp}_\tau$ of f and the trivial homomorphism $e\colon H \to G$. It is a sheaf and it comes with an injective morphism $\ker f \to H$ through which $\ker f$ is normal in H.

7. If f is injective and H is normal in G, the *presheaf quotient* $(G/H)_{\mathrm{pre}}$ of G by H is the presheaf $U \mapsto G(U)/H(U)$.

8. If f is injective and H is normal in G, the *quotient* G/H of G by H is the sheafification $(G/H)^{\#}_{\mathrm{pre}}$ of the presheaf quotient. It comes with a morphism $G \to G/H$.

With these definitions, we have:

Proposition 2.6. *Let $f\colon H \to G$ be a homomorphism of group sheaves. Then*

1. *The homomorphism f is injective if and only if* $\ker f = \mathbf{1}$.

2. *If f is injective and H is normal in G, then the quotient of G by H is the coequaliser $(G \to G/H) = \mathit{coeq}(f, e)$ in $\mathbf{Grp}_\tau$.*

3. *The morphism* $\operatorname{im} f \to G$ *is injective and* f *factors uniquely as* $H \to \operatorname{im} f \to G$. *Moreover,* $H \to \operatorname{im} f$ *is a quotient map.*

4. *If* f *is a quotient map, then the induced morphism* $H/\ker f \to G$ *is an isomorphism.*

5. *The induced morphism* $H/\ker(f) \to \operatorname{im} f$ *is an isomorphism.*

Proof. Points 1, 2 and 3 follow from the results for presheaves and some properties of the sheafification, such as that it is exact and preserves colimits. Point 5 follows from point 4. For point 4, we have a homomorphism $(H/\ker f)_{\text{pre}} \to G$ that is injective and sheaf-surjective. When sheafifying, we get that $H/\ker f \to G$ is still injective and sheaf-surjective, thus it is an isomorphism because every topos is well-balanced (a morphism that is both an epimorphism and a monomorphism is an isomorphism). □

The rest of isomorphism theorems for group sheaves follow easily from the proposition or just from the definitions and properties of the sheafification. We discuss briefly the case of product of subgroups, that we will use later in these notes. Let G be a group sheaf and let H and N be subgroups of G with N normal. We can define the presheaf group $(NH)_{\text{pre}} \colon U \mapsto N(U)H(U)$, and its sheafification $NH = ((NH)_{\text{pre}})^{\#}$, which is a subgroup of G. The intersection $N \cap H$ is a group sheaf (because finite limits of sheaves are sheaves) which is a normal subgroup of H. The inclusion $H \to G$ factors through $(NH)_{\text{pre}}$, so it also factors as $H \to NH$. Moreover, the induced map $H \to (NH/N)_{\text{pre}}$ is sheaf-surjective, so $u \colon H \to NH/N$ is a quotient map. As sheafification is exact, the kernel of u equals the sheafification of the kernel of $H \to (NH/N)_{\text{pre}}$, which equals the sheafification of the kernel of $H \to ((NH)_{\text{pre}}/N)_{\text{pre}}$, which is $N \cap H$ (already a sheaf). Therefore we have an isomorphism $N/N \cap H \cong NH/N$ by the proposition.

Proposition 2.6 allows us to define a short exact sequence of group sheaves in two ways. If

$$\mathbf{1} \longrightarrow N \xrightarrow{a} G \xrightarrow{b} Q \longrightarrow \mathbf{1}$$

is a sequence of group sheaves, the following are equivalent:

1. a is injective, N is normal in G and $Q \cong G/N$ under G,

2. b is a quotient map and $N \cong \ker b$ over G.

In that case we say that the sequence is *exact*.

2.4 Quotient maps and isomorphism theorems

The justification of our previous work with group sheaves is the following theorem. In this section we will work with the site $\mathbf{Alg}_k^{\text{op}}$ endowed with the fppf topology.

Theorem 2.7. *Let k be a field, let G be an algebraic group over k and let N be a normal subgroup of G. Then the quotient group sheaf G/N is an algebraic group and the morphism $G \to G/N$ is faithfully flat. Moreover, if G is affine, so is G/N.*

The proof of this theorem is difficult. Very roughly, the idea is that G/H is a separated algebraic space, and thus it has a nonempty dense schematic locus that, by homogeneity, can be moved around to cover the whole of G/H. For more on this, we reffer to [6]. For an elementary proof of the theorem, see [4], Appendix B. We will at some points need the following generalisation of the above theorem (see [4], 5.25, 5.28):

Theorem 2.8. *Let k be a field, let G be an algebraic group over k and let H be a subgroup of G. Let G/H be the (fppf) sheafification of the presheaf $R \to G(R)/H(R)$ on k-algebras R. Then the sheaf G/H is representable by an algebraic scheme and the morphism $G \to G/H$ is faithfully flat.*

Now we can define the notion of quotient map of algebraic groups.

Proposition 2.9. *Let k be a field and let $\varphi\colon G \to Q$ be a homomorphism of algebraic groups over k. The the following are equivalent:*

1. *φ is faithfully flat,*
2. *φ is a quotient map of (fppf) group sheaves.*

Theorem 2.7, together with our work on group sheaves makes the usual Noether isomorphism theorems work for algebraic groups. For example, if $f\colon G \to H$ is a homomorphism of algebraic groups, then $\operatorname{im} f$, which in principle is just a group sheaf, is an algebraic group, as $\operatorname{im} f \cong G/\ker f$ and this sheaf is representable, by 2.7.

We define a short exact sequence of algebraic groups to be one of fppf group sheaves in which all objects are algebraic groups.

Under certain smoothness hypothesis, it can be checked that a morphism is a quotient map on $\overline{k}$-points. More precisely, we have (see [4], 5.47):

Proposition 2.10. *Let $\varphi\colon G \to H$ be a homomorphism of algebraic groups over a field k. Then*

1. *If φ is a quotient map, then for all algebraically closed field extensions k' of k, the induced map $G(k') \to H(k')$ is surjective.*
2. *If H is smooth and there is a separably closed field extension k' of k such that the induced map $G(k') \to H(k')$ is surjective, then φ is a quotient map.*

2.5 Dimension

Dimension is especially well-behaved for algebraic groups ([4], page 17):

Proposition 2.11. *Let G be an algebraic group over a field k and let C be an irreducible component of G. Then* $\dim G = \dim C = \dim \mathcal{O}_{G,e}$.

The proposition means in particular that algebraic groups are equidimensional.

Dimension behaves well with respect to taking quotients ([4], 5.23):

Proposition 2.12. *Let G be an algebraic group over a field k and let H be a subgroup of G (not necessarily normal). Then* $\dim G = \dim G/H + \dim H$.

2.6 Smoothness and reducedness

An algebraic group G is always separated ([4], 1.22). Algebraic groups need not be smooth nor reduced. For example, the algebraic group $\boldsymbol{\mu}_p$ defined by the functor $\boldsymbol{\mu}_n(R) = \{a \in R \colon a^p = 1\}$ on k-algebras R over a field k of characteristic p is represented by $\mathrm{Spec}(k[X]/(X^p - 1)) \cong \mathrm{Spec}(k[X]/(X^p))$, which is not reduced and not smooth. However, there is the following result ([4], 1.26 and 1.28):

Proposition 2.13. *Let k be a field and let G be an algebraic group over k. The following are equivalent:*

1. *G is smooth,*
2. *the local ring $\mathcal{O}_{G,e}$ is regular,*
3. *G is geometrically reduced.*
4. $\dim \mathit{Lie}(G) = \dim(G)$

If the field k is perfect, the previous conditions are equivalent to

4. *G is reduced.*

In the previous proposition, $\mathrm{Lie}(G)$ is the k-vector space

$$\mathrm{Lie}(G) = \mathrm{Hom}_k(m_{G,e}/m_{G,e}^2, k)$$

which is just the tangent space of G at e. Later on (4) we will define a Lie algebra structure on $\mathrm{Lie}(G)$.

In characteristic 0, one doesn't have to worry about smoothness, as we have ([4], 8.39):

Theorem 2.14. *Any algebraic group over a field of characteristic* 0 *is smooth.*

2.7 Connectedness

We have the following characterisation of connected algebraic groups ([4], 1.36).

Proposition 2.15. *Let G be an algebraic group over a field k. Then the following are equivalent:*

1. *G is connected,*
2. *G is geometrically connected,*
3. *G is irreducible.*

The connected component G° of G containing e is called the *identity component of G*. We have ([4], 1.34, 1.52)

Proposition 2.16. *The identity component G° of G is a normal subgroup of G.*

The quotient of G by G° is denoted $\pi_0(G)$ and called the *group of connected components* of G. We have the following characterisation of G° and $\pi_0(G)$([4], 5.58).

Proposition 2.17. *Let G be an algebraic group over a field k. Then*

$$\mathbf{1} \longrightarrow G^\circ \longrightarrow G \longrightarrow \pi_0(G) \longrightarrow \mathbf{1}$$

is the unique short exact sequence with G° connected and $\pi_0(G)$ étale (over $\operatorname{Spec}(k)$*). It is called the* connected-étale exact sequence*.*

The following lemma will be used in these notes.

Proposition 2.18. *Let k be a field, let G be a smooth connected algebraic group over k and let H be a smooth subgroup of G such that* $\dim H = \dim G$*. Then $H = G$.*

Remark 2.19. In the above proposition, as G and H are smooth, the condition $\dim_k \operatorname{Lie}(H) = \dim_k \operatorname{Lie}(G)$ is equivalent to $\dim H = \dim G$.

Proof. By [4], 5.23, the dimension of the scheme G/H is 0. The map $G \to G/H$ is surjective, so $G_{\overline{k}} \to (G/H)_{\overline{k}}$ is surjective. As G is geometrically connected ([4], 1.32), $(G/H)_{\overline{k}} \cong \operatorname{Spec}(\overline{k})$. This forces $H(\overline{k}) \to G(\overline{k})$ to be a bijection and thus $G = H$ by 2.4 $\square$

2.8 Representations

For a vector space V over k, we define the *group functor* $\mathrm{GL}_V\colon \mathbf{Alg}_k \to \mathbf{Grp}\colon R \mapsto \mathrm{Aut}_{\mathbf{Mod}(R)}(R \otimes_k V)$. If V is finite dimensional, GL_V is representable by $\mathrm{Spec}(\mathrm{Sym}(V^* \otimes V)[d^{-1}])$, where $\mathrm{Sym}(V^* \otimes V)$ is the symmetric algebra on $V^* \otimes V$ and d is the element $d = \det(v_i^* \otimes v_j)$, where (v_i) is a basis of V and (v_i^*) is its dual basis. The definition of d doesn't depend on the chosen basis. Therefore, GL_V is an algebraic group if V is finite dimensional.

A *representation* of an algebraic group G (or G-representation) is a pair (V, r), where V is a vector space over k and $r\colon G \to \mathrm{GL}_V$ is a group functor homomorphism. We say that the representation is finite dimensional if V is finite dimensional and that it is *trivial* if r factors through $\mathbf{1}$. If (W, s) is another representation, then a morphism of representations from V to W is a linear map $V \to W$ such that for all k-algebras R, the induced R-linear map $R\otimes V \to R\otimes W$ is a morphism of representations of the abstract group $G(R)$. Given a G-representation (V, r) and a vector subspace $W \leq V$, there is at most one G-representation structure on W such that $W \to V$ is a morphism of representations. If this structure exists, we say that W is G-invariant, or that W is a subrepresentation (or G-subrepresentation if we want to specify the group) of V. For an arbitrary representation V of G, there exists a unique maximal trivial subrepresentation of V, denoted V^G. A G-representation (V, r) is said to be *simple* (or *irreducible*) if it has exactly two subrepresentations. A G-representation (V, r) is said to be *semisimple* if it is a direct sum of simple subrepresentations. The *category* $\mathbf{Rep}(G)$ *of finite dimensional representations* of G is an abelian category, and the forgetful functor $\mathbf{Rep}(G) \to \mathbf{Mod}(k)$ is exact and faithful. The same is true for the category of all representations of G. We say that an abelian category $\mathcal{A}$ is *semisimple* if every short exact sequence in $\mathcal{A}$ splits. It is checked easily that the category $\mathbf{Rep}(G)$ of finite dimensional representations of an algebraic group G is semisimple if and only if every finite dimensional representation of G is semisimple.

We can now formulate one of the important definitions of these notes:

Definition 2.20. Let k be a field and let G be an algebraic group over k. The group G is said to be *linearly reductive* if it is affine, smooth, connected and every finite dimensional representation of G is semisimple.

If the algebraic group G is affine, giving a representation (V, r) of G is equivalent to giving a k-linear map $\widehat{r}\colon V \to \mathcal{O}(G) \otimes_k V$, the *dual action*, satisfying $\mathrm{id}_V = (\widehat{e} \otimes \mathrm{id}_V) \circ \widehat{r}$ and $(\mathrm{id}_{\mathcal{O}(G)} \otimes \widehat{r}) \circ \widehat{r} = (\widehat{\mu} \otimes \mathrm{id}_V) \circ \widehat{r}$, where $\widehat{e}$ and $\widehat{\mu}$ are respetively the counit $\widehat{e}\colon \mathcal{O}(G) \to k$ and the comultiplication $\widehat{\mu}\colon \mathcal{O}(G) \to \mathcal{O}(G) \otimes_k \mathcal{O}(G)$ of G.

It is an elementary but important fact that any representation of an affine algebraic group is the union of its finite dimensiona subrepresentations. This contrasts with the case of Lie algebras, where there are simple representations of infinite dimension. A precise statement is:

Lemma 2.21. *Let G be an affine algebraic group over k, let V be any representation of G and let $v \in V$. Then there is a finite dimensional subrepresentation $W \leq V$ with $v \in W$.*

For a proof, see [5], Lemma in page 25. A consequence of this lemma is

Proposition 2.22. *Let k be a field and let G be an algebraic group over k. Then G is a affine if and only if there is a finite dimensional k-vector space V and an injective homomorphism $G \to GL_V$.*

The result justifies that affine algebraic groups are many times referred to as *linear algebraic groups.*

3 Solvable, unipotent, semisimple and reductive groups

We start by defining the notions of solvable and nipotent groups.

Definition 3.1. An algebraic group G over a field k is said to be *solvable* if there is a sequence of subgroups

$$\mathbf{1} = G_0 \leq G_1 \leq \cdots \leq G_n = G$$

such that for each $i \in [\![0, n-1]\!]$, G_i is normal in G_{i+1} and the quotient G_{i+1}/G_i is abelian.

An algebraic group G is said to be *unipotent* if every nonzero representation V of G contains a nonzero trivial subrepresentation (i.e. V^G is nonzero).

In order to define the solvable and unipotent radicals of a group we need the following lemma.

Lemma 3.2. *Let $\mathcal{P}$ be a class of algebraic groups, closed under isomorphism. We will say that $\mathcal{P}$ is* stable *if it satisfies*

1. *if N is a normal subroup of G and both N and G/N are in $\mathcal{P}$, then G is in $\mathcal{P}$,*

2. *if N is a normal subroup of G and G is in $\mathcal{P}$, then so is G/N.*

If $\mathcal{P}$ is a stable class of algebraic groups, then every algebraic group G contains a largest smooth connected normal subgroup in $\mathcal{P}$.

Proof. Let say that a subgroup of G is *pink* if it is smooth, connected, normal, and it is in $\mathcal{P}$. Among pink subgroups of G pick one, H, of maximal dimension. If H' is pink and $H \subset H'$, then by 2.18 and maximality of dimension, we have $H = H'$. Hence H is maximal among pink subgroups. If N is another maximal pink subgroup, then NH is connected, normal and $NH/H \cong N/N \cap H$, so NH is pink and by maximality $NH = N = H$. As any pink subgroup is contained in a maximal dimension pink subgroup, and hence in a maximal pink subgroup, H is the unique largest pink subgroup. □

By ordinary manipulations with series of subgroups, the class of solvable groups is stable. The class of unipotent groups is also stable. Indeed, if G has a unipotent normal subgroup N with unipotent quotient G/N, we have that V^N is naturally a representation of G/N and that $V^G = (V^N)^{G/N}$. On the other hand, if G is unipotent and Q is a quotient of G, then any representation V of Q can be regarded as a representation of G, and $V^Q = V^G$.

Definition 3.3. Let G be an algebraic group.

The unique largest connected normal solvable subgroup of G is called the *radical* of G and denoted $R(G)$. The group G is said to be *semisimple* if it is affine, smooth, connected and $R(G_{\overline{k}}) = \mathbf{1}$.

The unique largest connected normal unipotent subgroup of G is called the *unipotent radical* of G and denoted $R_u(G)$. The group G is said to be *reductive* if it is affine, smooth, connected and $R_u(G_{\overline{k}}) = \mathbf{1}$.

The assumptions of affinness and connectedness in the above definitions may vary in the literature.

In characteristic 0, the condition $R_u(G_{\overline{k}}) = \mathbf{1}$ can be weaken (see see [4], 19.11):

Proposition 3.4. *Let k be a perfect field and let G be an affine connected algebraic group over k. Then G is reductive if and only if $R_u(G) = \mathbf{1}$.*

4 The Lie algebra of an algebraic group

In this section we introduce the Lie algebra of an algebraic group. We will first define it as a vector space, and later we will define the Lie algebra structure on it. The Lie algebra of an algebraic group is especially useful in characteristic 0. The main illustration in this section is Theorem 4.12 that the functor "taking Lie algebra" is faithful, whose proof we only sketch.

4.1 Definition and basic properties

Definition 4.1. Let G be an algebraic group over a field k. The *Lie algebra* $\mathrm{Lie}(G)$ of G is the vector space $\mathrm{Lie}(G) = \mathrm{Hom}_{\mathbf{Mod}(k)}(m_{G,e}/m_{G,e}^2, k)$. Here,

e is identity point of G, which is k-rational, and $m_{G,e}$ is the maximal ideal of the stalk $\mathcal{O}_{G,e}$ of the structure sheaf of G at e. A homomorphism $f\colon G \to H$ induces a map of local rings $\mathcal{O}_{H,e} \to \mathcal{O}_{G,e}$, which in turn induces a linear map $df\colon \mathrm{Lie}(G) \to \mathrm{Lie}(H)$. We get a functor Lie from the category of algebraic groups over k to the category of k-vector spaces.

The Lie algebra $\mathrm{Lie}(G)$ is just the Zariski tangent space, and G is smooth if and only if $\dim_k \mathrm{Lie}(G) = \dim G$.

If V is a vector space over k, it is possible to see V as a group functor by setting $V(R) = R \otimes_k V$ for a k-algebra R. If V is finite dimensional, its group functor is represented by $\mathrm{Spec}(\mathrm{Sym}_k(V^*))$ and it is thus an algebraic group. The Lie algebra of V is V itself. More interestingly, the Lie algebra of GL_V is $\mathrm{Lie}(\mathrm{GL}_V) = \mathfrak{gl}(V)$, as we will later see.

We will denote by ε a variable such that $\varepsilon^2 = 0$, i.e., for a ring R, $R[\varepsilon]$ is short for $R[X]/(X^2)$. For an algebraic group G, we set TG to be the group functor $R \mapsto G(R[\varepsilon])$ (TG is actually an algebraic group, for example as a consequence of the following proposition, that implies $TG = Lie(G) \times G$, but we will not need that). For a k-algebra R we have the homomorphism $R[\varepsilon] \to R\colon \varepsilon \mapsto 0$. This induces a homomorphism $TG \to G$ of group functors.

Proposition 4.2. *There is a split short exact sequence (described in the proof), natural in G*

$$\mathbf{1} \longrightarrow Lie(G) \xrightarrow{e^{\varepsilon\cdot}} TG \longrightarrow G \longrightarrow \mathbf{1}$$

Moreover, the action of TG on $Lie(G)$ by conjugation, via *the second arrow above, is k-linear.*

Proof. There is a homomorphism $G \to TG$ given by the morphism $R \to R[\varepsilon]$ on k-algebras. As the composition $R \to R[\varepsilon] \to R$ is the identity on R, the composition $G \to TG \to G$ is the identity on G. This gives exactness at G and proves that the short exact sequence is split if it exists.

Let K be the kernel of the map $TG \to G$. An element $f \in K(R)$ is an arrow $\mathrm{Spec}(R[\varepsilon]) \to G$ such that the square

$$\begin{tikzcd} \mathrm{Spec}(R[\varepsilon]) \arrow[r, "f"] & G \\ \mathrm{Spec}(R) \arrow[u, "u"] \arrow[r] & \mathrm{Spec}(k) \arrow[u, "e"'] \end{tikzcd}$$

commutes. As $|u|$ is a homeomorphism (the bars denote the map of underlying topological spaces), $|f|$ is the constant map with value e, the identity element. Thus f is given by a sheaf morphism $|f|^{-1}\mathcal{O}_G \to R[\varepsilon]$. As $|f|^{-1}\mathcal{O}_G$ is the sheafification of the constant presheaf of value $\mathcal{O}_{G,e}$, f is given by a map of k-algebras $r\colon \mathcal{O}_{G,e} \to R[\varepsilon]$. The commutativity of the above square amounts to the commutativity of

$$\begin{array}{ccc} \mathcal{O}_{G,e} & \xrightarrow{r} & R[\varepsilon] \\ {\scriptstyle\gamma}\big\downarrow & & \big\downarrow \\ k & \longrightarrow & R \end{array}$$

where γ is induced by the identity $e\colon \mathrm{Spec}(k) \to G$. Using that $\mathcal{O}_{G,e} = k \oplus m_{G,e}$ via the split short exact sequence

$$0 \longrightarrow m_{G,e} \longrightarrow \mathcal{O}_{G,e} \xrightarrow{\gamma} k \longrightarrow 0$$

we see that the last square is commutative if and only if $r(m_{G,e}) \subset R\varepsilon$. As $\varepsilon^2 = 0$, using $R\varepsilon \cong R$, this gives an element of

$$\mathrm{Lie}(G)(R) = \mathrm{Hom}_{\mathbf{Mod}(k)}(m_{G,e}/m_{G,e}^2, R)$$

Conversely, given an element of $\mathrm{Lie}(G)(R)$, we get a map of k-modules $m_{G,e} \to R\varepsilon$, and direct summing it with the structure homomorphism $k \to R$, we get a map of k-modules $\mathcal{O}_{G,e} \to R[\varepsilon]$ that is actually a k-algebra homomorphism. This gives an isomorphism $K \cong \mathrm{Lie}(G)$ (naturality on R is straigthforward).

We now check that the isomorphism $K \cong \mathrm{Lie}(G)$ is a group homomorphism. The multiplication $G \times G \to G$ induces on stalks, as e is k-rational, a local k-algebra homomorphism $\mathcal{O}_{G,e} \to \mathcal{O}_{G,e} \otimes_k \mathcal{O}_{G,e}$. Moreover, the compositions

$$O_{G,e} \xrightarrow{m_e} \mathcal{O}_{G,e} \otimes_k \mathcal{O}_{G,e} \xrightarrow{\mathrm{id}\otimes\mu_e} \mathcal{O}_{G,e} \otimes_k k \cong \mathcal{O}_{G,e}$$

and

$$O_{G,e} \xrightarrow{m_e} \mathcal{O}_{G,e} \otimes_k \mathcal{O}_{G,e} \xrightarrow{\mu_e\otimes\mathrm{id}} k \otimes_k \mathcal{O}_{G,e} \cong \mathcal{O}_{G,e}$$

are both the idenditiy on $\mathcal{O}_{G,e}$. Using the isomorphism $\mathcal{O}_{G,e} \otimes_k \mathcal{O}_{G,e} = k \oplus (k \otimes m_{G,e}) \oplus (m_{G,e} \otimes k) \oplus (m_{G,e} \otimes m_{G,e})$, this gives that $m_e(a+b) = a + 1 \otimes b + b \otimes 1 + o(b)$, for $a \in k$ and $b \in m_{G,e}$, where $o(b) \in m_{G,e} \otimes m_{G,e}$. If $f, g \in K(R)$, f, g, fg correspond as before to k-algebra maps $r(f), r(g), r(fg)\colon \mathcal{O}_{G,e} \to R[\varepsilon]$ such that the image of $m_{G,e}$ is contained in $R\varepsilon$. As fg is the composition

$$\mathrm{Spec}(R[\varepsilon]) \xrightarrow{(f,g)} G \times G \xrightarrow{m} G$$

we have that $r(fg)$ is the composition

$$\mathcal{O}_{G,e} \xrightarrow{m_e} \mathcal{O}_{G,e} \otimes_k \mathcal{O}_{G,e} \xrightarrow{r(f) \sqcup r(g)} R[\varepsilon]$$

From the above observation about m_e we get that $r(fg)|_{m_{G,e}} = r(f)|_{m_{G,e}} + r(g)|_{m_{G,e}}$ and thus the isomorphism $K(R) \to \mathrm{Lie}(G)(R)$ is a group homomorphism. This gives the short exact sequence stated in the lemma. We omit the proof that it is natural in G, but it is easy.

It is left to prove that the action by conjugation of TG on $\mathrm{Lie}(G)$ is k-linear. As the action of $\mathrm{Lie}(G)$ on itself by conjugation is trivial, $\mathrm{Lie}(G)$ being commutative, the action of TG descends to an action of G on $\mathrm{Lie}(G)$ and it is enough to prove that this action is k-linear. Let $g \in G(R)$. Conjugation by g induces a group automorphism $G_R \to G_R$, where $G_R = G \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R)$. This induces a map of stalks at the identity that in turn induces an R-linear map $T\colon R \otimes_k m_{G,e} \to R \otimes_k m_{G,e}$. If $f \in \mathrm{Lie}(G)(R)$, extending schalars, f gives $R \otimes_k m_{G,e} \to R$, precomposing this map with T and then restricting scalars we obtain an element of $\mathrm{Lie}(G)(R)$. As T is R-linear, the map $\mathrm{Lie}(G)(R) \to \mathrm{Lie}(G)(R)$ induced by the action of g is R-linear.

$\square$

We will now introduce a convenient notation. Let R be a k-algebra, let S be an R-algebra and let $\beta \in S$ be an element with $\beta^2 = 0$. There is a unique map of R-algebras $R[\varepsilon] \to S$ sending ε to β. If $X \in \mathrm{Lie}(G)(R)$, we define $e^{\beta X} \in G(S)$ to be the image of X via the composition

$$\mathrm{Lie}(G)(R) \longrightarrow G(R[\varepsilon]) \longrightarrow G(S)$$

As the maps involved are group homomorphisms, we have $e^{\beta(X+Y)} = e^{\beta X} e^{\beta Y}$. Now, let $a \in R$ and let $u_a\colon R[\varepsilon] \to R[\varepsilon]$ be the unique map of R-algebras sending $\varepsilon \mapsto a\varepsilon$. We get a commutative diagram

$$\begin{tikzcd}
\mathrm{Lie}(G)(R) \arrow[r] \arrow[d, "\cdot a"] & G(R[\varepsilon]) \arrow[r, "\varepsilon \mapsto a\beta"] \arrow[d, "u_a"] & G(S) \arrow[d, "\mathrm{id}"] \\
\mathrm{Lie}(G)(R) \arrow[r] & G(R[\varepsilon]) \arrow[r, "\varepsilon \mapsto \beta"] & G(S)
\end{tikzcd}$$

where the proof of 4.2 shows that the left vertical arrow is multiplication by a. Thus we get the equality $e^{\beta(aX)} = e^{(\beta a)X}$. A similar argument shows that if $\alpha, \beta \in S$ have zero square, then $e^{\alpha(\beta X)} = e^{(\alpha\beta)X}$.

Definition 4.3. The linear action of G on $\mathrm{Lie}(G)$ gives a representation

$$\mathrm{Ad}\colon G \to \mathrm{GL}_{\mathrm{Lie}(G)}$$

called the *adjoint representation*. Taking Lie, we get a k-linear map

$$\mathrm{ad}\colon \mathrm{Lie}(G) \to \mathfrak{gl}(\mathrm{Lie}(G))$$

For $x, y \in \mathrm{Lie}(G)$, we define the bracket

$$[x, y] = \mathrm{ad}(x)(y)$$

By 4.2 and by definition, this bracket is k-bilinear

Remark 4.4. By definition of Ad, we have that, if R is a k-algebra, $x \in G(R)$ and $X \in \mathrm{Lie}(G)(R)$, then we have the equality $e^{\varepsilon \mathrm{Ad}(x)(X)} = xe^{\varepsilon X}x^{-1}$.

Proposition 4.5 (Functoriality of Ad)**.** *If $f\colon G \to H$ is a group homomorphism, then the induced map Lie$(G) \to$ Lie(H) is a morphism of G-representations, where Lie(H) is a G-representation via $G \to H$.*

Proof. Let $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{h} = \mathrm{Lie}(H)$. We have to see commutativity of

$$\begin{tikzcd} G \times \mathfrak{g} \arrow[r, "\mathrm{Ad}"] \arrow[d, "f \times df"'] & \mathfrak{g} \arrow[d, "df"] \\ H \times \mathfrak{h} \arrow[r, "\mathrm{Ad}"] & \mathfrak{h} \end{tikzcd}$$

If $x \in G(R)$ and $X \in \mathfrak{g}(R)$, the top arrow sends (x, X) to $df(\mathrm{Ad}(x)(X))$ whereas the bottom arrow sends it to $\mathrm{Ad}(f(x))(df(X))$. Applying $e^{\varepsilon \cdot}$, which is injective, we get

$$e^{\varepsilon df(\mathrm{Ad}(x)(X))} = f(e^{\varepsilon \mathrm{Ad}(x)(X)}) = f(xe^{\varepsilon X}x^{-1}) = f(x)e^{\varepsilon df(X)}f(x)^{-1} =$$
$$= e^{\varepsilon \mathrm{Ad}(f(x))(df(X))}$$

as required. □

Recall that for a k-vector space V, the Lie algebra $\mathfrak{gl}(V)$ is the k-vector space $\mathrm{End}_k(V)$ endowed with the commutator bracket.

Proposition 4.6. *Let V be a finite dimensional k-vector space. Then Lie$(GL_V) = \mathfrak{gl}(V)$ and the adjoint representation*

$$Ad\colon GL_V \to GL_{\mathfrak{gl}(V)}$$

is given by conjugation. Moreover,

$$ad\colon \mathfrak{gl}(V) \to \mathfrak{gl}(\mathfrak{gl}(V))$$

is the usual adjoint representation for the Lie algebra $\mathfrak{gl}(V)$.

Proof. For a k-algebra R, $\mathrm{GL}_V(R) \subset \mathrm{End}_R(V_R)$, where $V_R = V \otimes_k R$. The map $R \to R[\varepsilon]$ induces an injection $\mathrm{End}_R(V_R) \subset \mathrm{End}_{R[\varepsilon]}(V_{R[\varepsilon]})$ and we have the equality $\mathrm{End}_R(V_R) \oplus \varepsilon \mathrm{End}_R(V_R) = \mathrm{End}_{R[\varepsilon]}(V_{R[\varepsilon]})$. The equality $\mathrm{GL}_V(R[\varepsilon]) = \mathrm{GL}_V(R) \oplus \varepsilon \mathrm{End}_R(V_R)$ inside $\mathrm{End}_{R[\varepsilon]}(V_{R[\varepsilon})$ (where here $\oplus$ should really be $\times$) is readily checked.

The short exact sequence

$$\mathbf{1} \longrightarrow \mathrm{End}_R(V_R) = \mathfrak{gl}(V)(R) \xrightarrow{u} \mathrm{GL}_V(R[\varepsilon]) \xrightarrow{v} \mathrm{GL}_V(R) \longrightarrow \mathbf{1}$$

where $u\colon A \mapsto I_V + \varepsilon A$ and $v\colon B + \varepsilon C \mapsto B$, identifies $\mathfrak{gl}(V) \cong \mathrm{Lie}(\mathrm{GL}_V)$ (actually it should be checked that this isomorphism is also k-linear, not just a group homomorphism). Once we have this, the fact that Ad is given by conjugation is tautological from its definition.

Let us compute ad. By 4.2, there is a commutative square

$$\begin{tikzcd}
\mathfrak{gl}(V) \arrow[r, "e^{\varepsilon \cdot}"] \arrow[d, "\mathrm{ad}"'] & T\,\mathrm{GL}_V \arrow[d, "T\,\mathrm{Ad}"] \\
\mathfrak{gl}(\mathfrak{gl}(V)) \arrow[r, "e^{\varepsilon \cdot}"] & T\,\mathrm{GL}_{\mathfrak{gl}(V)}
\end{tikzcd}$$

If R is a k-algebra and $A \in \mathfrak{gl}(V)(R)$, the top arrow of the square sends A to

$$\begin{aligned}
T\,\mathrm{Ad}(e^{\epsilon A}) &= \mathrm{conj}_{I_V+\varepsilon A} = (B \mapsto (I_V + \varepsilon A)B(I_V + \varepsilon A)^{-1}) = \\
&= (B \mapsto (I_V + \varepsilon A)B(I_V - \varepsilon A)) = (B \mapsto B + \varepsilon(AB - BA)) = \\
&= I_{\mathfrak{gl}(V)} + \varepsilon(L_A - R_A) = e^{\varepsilon(L_A - R_A)}
\end{aligned}$$

where $L_A \in \mathfrak{gl}(V)$ is left multiplication by A and R_A is right multiplication by A. The bottom arrow of the square is $e^{\varepsilon\,\mathrm{ad}(A)}$ so, $e^{\varepsilon \cdot}$ being injective, we have that $\mathrm{ad}(A) = L_A - R_A$, as desired.

□

Remark 4.7. Suppose that $r\colon G \to \mathrm{GL}_V$ is a reprsentation and call $\mathfrak{g} = \mathrm{Lie}(G)$. Then if $v \in V_R$ for a k-algebra R and if $X \in \mathfrak{g}(R)$, then we have the equality

$$v + \varepsilon dr(X)(v) = e^{\varepsilon dr(X)}(v) = r(e^{\varepsilon X})(v)$$

in $V_{R[\varepsilon]}$. Indeed, $I + \varepsilon dr(X) = e^{\varepsilon dr(X)}$ by the proof of the above proposition.

If we apply this to $r = \mathrm{Ad}$, we get that for $X, Y \in \mathfrak{g}(R)$, we have the equality $Y + \varepsilon[X, Y] = \mathrm{Ad}(e^{\varepsilon X})(Y)$ inside $\mathfrak{g}(R[\varepsilon])$.

Proposition 4.8. *For an algebraic group G, the bracket on Lie(G) as defined above gives Lie(G) the structure of a Lie algebra over k. Moreover, if $f\colon G \to H$ is a group homomorphism, the induced map Lie$(G) \to$ Lie(H) is a morphism of Lie algebras.*

Proof. Call $\mathrm{Lie}(G) = \mathfrak{g}$. We first show the second statement, namely that

$$df([X, Y]) = [df(X), df(Y)]$$

for $X, Y \in \mathfrak{g}$. The equality holds if

$$df(Y) + \varepsilon df[X,Y] = df(Y) + \varepsilon[df(X), df(Y)]$$

inside $(k[\varepsilon])$. The left hand side is $df(Y + \varepsilon[X,Y]) = df(\mathrm{Ad}(e^{\varepsilon X})(Y))$ by the previous remark, whereas the right hand side is $\mathrm{Ad}(e^{\varepsilon df(X)})(df(Y))$, and both are equal by 4.5.

The above fact applied to $\mathrm{Ad}\colon G \to \mathrm{GL}_{\mathfrak{g}}$ implies that $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ preserves the bracket, i.e. that the Jacobi identity holds. To prove that $\mathfrak{g}$ is a Lie algebra, it is only left to show that $[X, X] = 0$ for $X \in \mathfrak{g}$. We give the proof assuming G is affine. By 2.22, take $r\colon G \to \mathrm{GL}_V$ injective, where V is some finite dimensional k-vector space. Then, by 4.10, $dr\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is injective too, so $dr[X, X] = [dr(X), dr(X)] = 0$ (by 4.6), and thus $[X, X] = 0$.

For the case where G is not affine, we have the left translation action $G \to \mathrm{Aut}(G)$, where $\mathrm{Aut}(G)$ is the group functor $\mathrm{Aut}(G)(R) = \mathrm{Aut}_R(G_R)$. There is a general formalism for Lie algebras of group functors (defined by the short exact sequence of 4.2), and one can compute that $\mathrm{Lie}(\mathrm{Aut}(G)) = \mathrm{Der}(\mathcal{O}_G)$, the Lie algebra of the structure sheaf of G (that *is* a Lie algebra). Then the fact that $\mathrm{Lie}(G) \to \mathrm{Lie}(\mathrm{Aut}(G))$ is injective gives the result. For the details of this argument, see [2], II, §4, n$^{\mathrm{o}}$4, 4.5, and for a beautiful formalism for Lie algebras of group functors and algebraic groups, see the whole section II, §4, *Calcul différentiel sur les schémas en groupes.* □

Remark 4.9. For an algebraic group G, let $\mathbf{Rep}(G)$ be the category of finite dimensional representations of G, and for a finite dimensional Lie algebra L over k, let $\mathbf{Rep}(L)$ be the category of finite dimensional representations of L. If (V, r) is a finite dimensional representation of G, applying Lie to the map $G \to \mathrm{GL}_V$ we get Lie algebra homomorphism $dr\colon \mathrm{Lie}\, G \to \mathfrak{gl}(V)$ thus obtaining a Lie algebra representation (V, dr) of $\mathrm{Lie}(G)$ on V. If (W, s) is another G-representation and $f\colon V \to W$ is a map of G-representations, then it is a map of $\mathrm{Lie}(G)$-representations as well. Indeed, for a k-algebra R, $v \in V_R$ and $x \in G(R)$, we have $r(x)f(v) = f(s(x)v)$, and we want to prove that if $X \in \mathrm{Lie}(G)$ then $dr(X)(f(v)) = f(ds(X)(v))$. This will follow from the equality

$$f(v) + \varepsilon dr(X)(f(v)) = f(v) + \varepsilon f(ds(X)(v))$$

in $W_{R[\varepsilon]}$. The left hand side is $e^{\varepsilon dr(X)}(f(v)) = r(e^{\varepsilon X})(f(v)) = f(s(e^{\varepsilon X})(v)) = f(v + \varepsilon ds(X)(v))$, which equals the right hand side. Here we used 4.7.

We thus get a functor $\mathbf{Rep}(G) \to \mathbf{Rep}(\mathrm{Lie}(G))$ that relates representations of the group G and of its Lie algebra. Understanding properties of this functor will sometimes allow us to reduce problems of algebraic groups to problems of Lie algebras.

Proposition 4.10. *Let k be a field. The functor Lie from the category of algebraic groups over k to Lie algebras over k preserve limits. In particular it preserves kernels, which precisely means that if $\varphi\colon G \to H$ is a homomorphism of algebraic groups over k, then the induced morphism of Lie algebras $Lie(\ker\varphi) \to \ker(Lie\,\varphi)$ is an isomorphism.*

The idea of the proof is that the Lie algebra is the kernel of some homomorphism, and kernels commute with limits. For details, see [4], 10.14. The fact that the functor Lie preserves limits makes us suspect that it may have a left adjoint. This is indeed the case if we asume that the characteristic of k is 0 and we restrict to semisimple algebraic groups and semisimple Lie algebras. This is essentially the content of [4], 23.70, that uses Tannaka reconstruction theorem to construct an algebraic group from the category of finite dimensional representations of a Lie algebra. I don't know what happens in general, and I ask

Question 4.11. Does the functor Lie defined in 4.10 has a left adjoint in general? In that case, can we identify its essential immage with the simply connected groups?

The following result in characteristic 0 is very powerful.

Proposition 4.12. *Let k be a field of characteristic 0. Then the functor $G \mapsto Lie(G)$ from the category of affine connected algebraic groups over k to the category of Lie algebras over k is faithful.*

Proof sketch. As G is smooth over $\mathrm{Spec}(k)$, the irreducible components of G are disjoint open subschemes, so that G being connected implies that G is irreducible. As G is also reduced, it is integral. This implies that the functor $G \mapsto \mathcal{O}_{G,e}$ is faithful. As $\mathcal{O}_{G,e}$ is a Noetherian local domain, it injects into its completion $\widehat{\mathcal{O}}_{G,e}$, so that the functor $G \mapsto \widehat{\mathcal{O}}_{G,e}$ is also faithful. It turns out that $\widehat{\mathcal{O}}_{G,e}$ has the additional structure of a *formal group*, and that in fact we have a faithful functor $G \mapsto \widehat{\mathcal{O}}_{G,e}$ from the category of affine connected algebraic groups over k to the category of formal groups over k. There is the notion of the Lie algebra of a formal group, and the Lie algebra of $\widehat{\mathcal{O}}_{G,e}$ is just $\mathrm{Lie}(G)$. The result follows because, over a field of characteristic 0, the functor that sends a formal group to its Lie algebra is an equivalence of categories (see [7], page 146, Theorem 3. See also sections 6 and 7 of Chapter IV of the same book for the definition of a formal group, and the Wikipedia page on formal groups for a nice survey). □

4.2 Internal Hom for representations

Let G be an algebraic group over a field k and let (V, v) and (W, w) be representations of G with V finite dimensional. We can define a representation $(\mathrm{Hom}_k(V, W), o)$ of G functorially on k-algebras R, sending $g \in G(R)$ to the automorphism of $\mathrm{Hom}_k(V,W)\otimes_k R \cong \mathrm{Hom}_R(V\otimes_k R, W\otimes_k R)$ (note that this holds because V is finite dimensional) that sends f to $w(g)\circ f\circ v(g)^{-1}$.

Proposition 4.13. *In the set up of the above paragraph, we have*

1. *If $f \in Hom_k(V,W)$, then f is a morphism of representations of G if and only if $k\langle f\rangle$ is a trivial subrepresentation of $Hom_k(V,W)$.*
2. *The induced representation $(Hom_k(V,W), do)$ of $Lie(G)$ is given by $Lie(G)\to \mathfrak{gl}(Hom_k(V,W))\colon x\mapsto (f\mapsto dw(x)\circ f - f\circ dv(x))$.*
3. *If $f\in Hom_k(V,W)$, then f is a morphism of representations of $Lie(G)$ if and only if $k\langle f\rangle$ is a trivial $Lie(G)$-subrepresentation of $Hom_k(V,W)$.*

Proof. The first statement follows from the definition of u and the third follows from the second. To prove the second statement, we use the naturality of the short exact sequence of 4.2, which gives the commutative square

$$\begin{array}{ccc} \mathrm{Lie}(G) & \xrightarrow{do} & \mathfrak{gl}(\mathrm{Hom}_k(V,W)) \\ {\scriptstyle e^{\varepsilon\cdot}}\downarrow & & \downarrow{\scriptstyle e^{\varepsilon\cdot}} \\ TG & \xrightarrow{To} & T(\mathrm{GL}_{\mathrm{Hom}_k(V,W)}) \end{array}$$

Evaluating on a k-algebra R, and taking $x\in \mathrm{Lie}(G)\otimes_k R$, we get

$$e^{\varepsilon do(x)} = f\mapsto Tw(e^{\varepsilon x})\circ f\circ (Tv(e^{\varepsilon x}))^{-1} = f\mapsto e^{\varepsilon dw(x)}\circ f\circ \left(e^{\varepsilon dv(x)}\right)^{-1} =$$

$$= f\mapsto (I+\varepsilon dw(x))\circ f\circ (I-\varepsilon dv(x)) = f\mapsto f+\varepsilon\left(dw(x)\circ f - f\circ dv(x)\right) =$$

$$= I+\varepsilon\left(L_{dw(x)} - R_{dv(x)}\right) = e^{\varepsilon\left(L_{dw(x)}-R_{dv(x)}\right)}$$

where I denotes each time the identity on the corresponding space and L and R denote left and right multiplication. As $e^{\varepsilon\cdot}$ is injective, this gives the result. $\square$

5 Diagonalisable groups

The aim of this section is to introduce one of the simplest class of algebraic groups, diagonalisable algebraic groups, and to prove that they are linearly reductive. We fix a field k.

Recall that a character of an affine algebraic group G over k is a homomorphism $\chi\colon G\to \mathbb{G}_m$. It corresponds to a k-algebra homomorphism $\varphi\colon k[T,T^{-1}]\to \mathcal{O}(G)$, which in turn corresponds to an element $\varphi(T)=a\in \mathcal{O}(G)^\times$. Let $\widehat{\mu}$ denote the comultiplication of G. The homomorphism φ determines a group homomorphism if and only if the square

$$
\begin{array}{ccc}
k[T, T^{-1}] & \xrightarrow{\qquad\varphi\qquad} & \mathcal{O}(G) \\
\Big\downarrow & & \Big\downarrow \widehat{\mu} \\
k[T, T^{-1}] \otimes_k k[T, T^{-1}] & \xrightarrow{\varphi \otimes \varphi} & \mathcal{O}(G) \otimes_k \mathcal{O}(G)
\end{array}
$$

commutes, where the first vertical arrow sends $T \mapsto T \otimes T$. This happens if and only if $\widehat{\mu}(a) = a \otimes a$, and an element a of $\mathcal{O}(G)^\times$ with this property is said to be *group-like*. Thus the characters $X(G)$ of G are in bijection with group-like elements of $\mathcal{O}(G)$. The set of group-like elements forms a subgroup of $\mathcal{O}(G)^\times$, so $X(G)$ inherits an abelian group structure. Functorially, the sum of two characters χ and χ' is described by the property that for all k-algebras R and for all $g \in G(R)$, $(\chi + \chi')(g) = \chi(g)\chi'(g)$. We shall denote by $a(\chi)$ the group-like element associated with a character χ.

Now, let M be a finite type abelian group, written multiplicatively. The functor

$$
D(M)\colon \mathbf{Alg}_k \to \mathbf{Grp}\colon R \mapsto \mathrm{Hom}_{\mathbf{Mod}(\mathbb{Z})}(M, R^\times)
$$

is representable by the k-algebra $k[M]$ that has M as a vector space basis and where multiplication is induced by the group law on M. The k-algebra $k[M]$ is of finite type because M is an abelian group of finite type. Hence $D(M)$ is an algebraic group. An algebraic group is said to be *diagonalisable* if it is isomorphic to $D(M)$ for some finite type abelian group M. We have the following

Proposition 5.1. *The group-like elements of an affine algebraic group G are k-linearly independent. Moreover, the group G is diagonalisable if and only if the group-like elements form a k-vector space basis of $\mathcal{O}(G)$.*

Proof. We prove by induction thatj $n+1$ distinct group-like elements of $\mathcal{O}(G)$ are linearly independent. For $n = 0$, a group-like element is a unit, so not 0. For $n > 0$, suppose that e and $e_1, \ldots, e_n$ are distinct group-like elements and $e = \sum_i c_i e_i$, with the $c_i \in k$ non-zero. The equality $\widehat{\mu}(e) = e \otimes e$ gives

$$
\sum_i c_i e_i \otimes e_i = \sum_i c_i c_j e_i \otimes e_j
$$

Using that the e_i are k-linearly independent by induction hypothesis we get that $c_i c_i = c_i$ for all i, so all $c_i = 1$. But we also have $c_i c_j = 0$ for $i \neq j$, which gives $n = 1$ and $e = e_1$, a contradiction.

Suppose that the abelian group M of group-like elements generate is a basis of $\mathcal{O}(G)$. As $\mathcal{O}(G)$ is a finite type k-algebra, there is a finite number of elements $a_1, \ldots, a_n$ of M that generate $\mathcal{O}(G)$ as a k-algebra. Using the linear independence of elements of M, this implies that M is finitely generated as $\mathbb{Z}$-module. As a k-algebra, $k[M] = \mathcal{O}(G)$, and the fact that $\widehat{\mu}(e) = e \otimes e$ for $e \in M$ determines the Hopf algebra structure of $\mathcal{O}(G)$ and gives $D(M) \cong G$. $\square$

The characters of diagonalisable groups are easy to understand, from the following proposition we deduce that the character group $X(D(M))$ of the diagonalisable group $D(M)$ is isomorphic to M.

Proposition 5.2. *The group-like elements of $k[M]$ are the elements of M.*

Proof. From the functorial definition of $D(M)$, one finds that the comultiplication is given by $\widehat{\mu}(m) = m \times m$ for $m \in M$, so the elements of M are group-like. Suppose that $a = \sum_{m \in M} c_m m$ ($c_m \in k$) is group-like. Then

$$\widehat{\mu}(a) = \sum_{m \in M} c_m m \otimes m = \sum_{m,n \in M} c_m c_n m \otimes n$$

so that $c_m c_n = \delta_{mn} c_m$ for all $m, n \in M$. As $a \neq 0$, we find that one of the c_m should be 1 and the others should all be 0. $\square$

Our definition of $D(M)$ immediately gives that $D(M_1 \oplus M_2) \cong D(M_1) \times D(M_2)$. From this and the classification of finite type abelian group we see that any diagonalisable algebraic group is a product of copies of $D(\mathbb{Z}) = \mathbb{G}_m$ and $D(\mathbb{Z}/n\mathbb{Z}) = \boldsymbol{\mu}_n$. Exploiting this fact and with some more work one can prove

Theorem 5.3. *The functor $D\colon M \mapsto D(M)$ from the opposite category of finite type abelian groups to the category of algebraic groups over k is fully faithful and faithfully exact. Quotients and subgroups of diagonalisable algebraic groups are again diagonalisable. Commutative extensions of diagonalisable algebraic groups are again diagonalisable.*

For the proof, see [4], 12.9.

From the theorem we deduce that the only short exacts sequences with $\mathbb{G}_m$ as middle term are isomorphic to

$$\mathbf{1} \longrightarrow \boldsymbol{\mu}_n \longrightarrow \mathbb{G}_m \xrightarrow{r \mapsto r^n} \mathbb{G}_m \longrightarrow \mathbf{1}$$

for some $n \in \mathbb{N}$.

We now study representations of diagonalisable algebraic groups, but first let us define some new terminology. If G is an affine algebraic group, $\chi\colon G \to \mathbb{G}_m$ is a character of G and V is a representation of G with dual action $\widehat{\sigma}$, then G is said to act on V *through* χ if the corresponding map $G \to \mathrm{GL}_V$ factors as

$$G \xrightarrow{\chi} \mathbb{G}_m \longrightarrow \mathrm{GL}_V$$

where the second map corresponds to the structure $\mathbb{G}_m$-action on V. This happens if and only if for all $v \in V$, $\widehat{\sigma}(v) = a(\chi) \otimes v$. Note that if V is one-dimensional, as $\mathrm{GL}_V \cong \mathbb{G}_m$, G acts on V by some uniquely determined character. More generally, if G doesn't act through χ on V, we can still consider the subspace $V_\chi = \{v \in V \colon \widehat{\sigma}(v) = a(\chi) \otimes v\}$ of V, which is the biggest subrepresentation of V on which G acts through χ. We have

Proposition 5.4. *Let G be an affine algebraic group over k, and let V be a representation of G. Then the subrepresentations $(V_\chi)_{\chi \in X(G)}$ have direct sum.*

Proof. Let $(\chi_i)_{i=1}^n$ be different characters of G, and let $v_i \in V_{\chi_i}$ such that $\sum_i v_i = 0$. Applying the dual action we find that $\sum_i a(\chi_i) \otimes v_i = 0$. As the $a(\chi_i)$ are linearly independent (5.1), this forces $v_i = 0$ for all i. □

The main result on the representations of diagonalisable algebraic groups is

Theorem 5.5. *Let G be an affine algebraic group over k. The following are equivalent:*

1. *G is diagonalisable,*
2. *for every representation V of G we have $V = \bigoplus_{\chi \in X(G)} V_\chi$,*
3. *every representation of G is sum of one-dimensional representations.*

As a consecuence, every diagonalisable group is linearly reductive.

Proof.

1. *1 implies 2.* We already know that the V_χ have direct sum. Let M be the set of group-like elements of $\mathcal{O}(G)$ and let $v \in V$. Then $\widehat{\sigma}(v) = \sum_{m \in M} m \otimes v_m$ for some $v_m \in V$, where $\widehat{\sigma}$ is the dual action. If e is the unit of G, then $(\widehat{e} \otimes 1_V) \circ \widehat{\sigma} = 1_V$, so $v = \sum_m v_m$. On the other hand $(\widehat{\mu} \otimes 1_V) \circ \widehat{\sigma} = (1_{\mathcal{O}(G)} \otimes \widehat{\sigma}) \circ \widehat{\sigma}$, so that $\sum_m m \otimes m \otimes v_m = \sum_m m \otimes \widehat{\sigma}(v_m)$, which implies $\widehat{\sigma}(v_m) = m \otimes v_m$ by linear independence of the m. Thus $v_m \in V_{\chi_m}$ if χ_m is the character corresponding to the group-like element m, and $v \in \bigoplus_\chi V_\chi$.

2. *2 implies 3.* Indeed, the V_χ are sum of one-dimensional representations.

3. *3 implies 2.* Indeed, each one-dimensional subrepresentation of V is contained in one of the V_χ.

4. *2 implies 1.* Let us apply this fact to the regular representation of G, in which the underlying verctor space is $\mathcal{O}(G)$ and the dual action is the comultiplication $\widehat{\mu}$. Let M be the set of group-like elements. It is enough to prove that the elements of M span $\mathcal{O}(G)$ (see 5.1). Let $r \in \mathcal{O}(G)_\chi$ and let $m = a(\chi)$. Then $\widehat{\mu}(r) = m \otimes r$ and by the unit identity $r = \widehat{e}(r)m$. Thus $\mathcal{O}(G)_\chi = k\langle m \rangle$ and the elements of M span $\mathcal{O}(G)$.

As every representation of a diagonalisable group is sum of simple representations, every representation is semisimple, so every diagonalisable group is linearly reductive. □

The class of diagonalisable algebraic groups can be enlarged a little bit:

Definition 5.6. An affine algebraic group G over k is said to be *of multiplicative type* if there is a field extension k' of k such that the base change $G_{k'}$ is a diagonalisable algebraic group over k'.

Among groups of multiplicative type, those that become isomorphic to a finite product of copies of $\mathbb{G}_m$ after a field base change are called *torus*. A *split torus* is a group isomorphic to a finite product of copies of $\mathbb{G}_m$. For a representation V of a torus T, the characters χ of T such that $V_\chi \neq 0$ are called the *weights* of the representation, and the corresponding V_χ's are the *weight spaces.*

Using Galois descent and little more, one can prove:

Theorem 5.7. *Algebraic groups of multiplicative type are linearly reductive.*

For a proof, see [4], 12.30.

A final comment on diagonalisable algebraic groups over a field of characteristic 0: if G is one such group, then $\mathrm{Lie}(G)$ is the abelian Lie algebra of dimension $\dim G$. This follows from the fact that diagonalisable groups are of the form $\boldsymbol{\mu}_{n_1} \times \cdots \boldsymbol{\mu}_{n_k} \times \mathbb{G}_m \times \cdots \times \mathbb{G}_m$, that the $\boldsymbol{\mu}_{n_i}$ are discrete (in characteristic 0) and that taking the Lie algebra behaves well with products of groups, together with the fact that $\mathrm{Lie}(\mathbb{G}_m) = \mathrm{Lie}(\mathrm{GL}_k) = \mathfrak{gl}(k)$. Note however that representations of abelian Lie algebras need not be semisimple, so we have found a family of examples of linearly reductive groups G for which the functor $\mathbf{Rep}(G) \to \mathbf{Rep}(\mathrm{Lie}(G))$ is not essentially surjective.

6 Semisimplicity

The main result in this section is Theorem 6.6 that allows us to compare categories of representations of a connected algebraic group and its Lie algebra in characteristic 0. The main application we will give is a proof of the fact (6.10) that reductivity and linear reductivity coincide in characteristic 0. Along the way, we also prove that the Lie algebra of a semisimple group is semisimple (6.5). We start with several technical results that will be necessary for our goals.

6.1 Technical preliminaries

Proposition 6.1. *Let G be an affine algebraic group over a field k, let V be a representation of G and let W be a vector subspace of V. Define the group functor*

$$\mathit{Stab}_G(W)\colon R \mapsto \{g \in G(R) \colon g(W_R) \subset W_R\}$$

where $W_R = W \otimes_k R$. Then $\mathit{Stab}_G(W)$ is representable, and it is thus an algebraic subgroup of G. Moreover, via the induced inclusion $\mathit{Lie}(\mathit{Stab}_G(W)) \to$

Lie(G) *we have that Lie*$(Stab_G(W)) = Stab_{Lie(G)}(W)$*, where* $Stab_{Lie(G)}(W)$ *is the stabiliser of* W *for the induced representation of Lie*(G) *on* V*.*

Proof. Let $A = \mathcal{O}_G(G)$ and let $\rho\colon V \to V \otimes_k A$ be the coaction. Let $(e_i)_{i\in I\sqcup J}$ be a basis of V such that $(e_i)_{i\in I}$ is a basis of W. Let $g \in G(R) = \mathrm{Hom}_{\mathbf{Alg}_k}(A, R)$. There are (a_{ij}) in A such that $\rho(e_i) = \sum_j e_j \otimes a_{ji}$ for every $i \in I \sqcup J$. The homomorphism g induces

$$\overline{g}\colon W \to V \otimes R$$

and we want to see when $\overline{g}(W) \subset W \otimes R$. We have

$$\forall i \in I,\ \overline{g}(e_i) = \sum_{j\in I\sqcup J} e_j \otimes g(a_{ji}) \in W \otimes R \iff \forall i \in I,\ \forall j \in J,\ g(a_{ji}) = 0$$

Therefore, $g \in \mathrm{Stab}_G(W)(R)$ if and only if $g\colon A \to R$ factors through $A \to A/I$, where I is the ideal of A generated by the a_{ij} with $i \in I$ and $j \in J$. Thus $\mathrm{Stab}_G(W)$ is represented by $\mathrm{Spec}(A/I)$.

Let us now prove the equality $\mathrm{Lie}(\mathrm{Stab}_G(W)) = \mathrm{Stab}_{\mathrm{Lie}(G)}(W)$. Just looking at the definition of $\mathrm{Stab}_G(W)$, we see that

$$\mathrm{Stab}_G(W) = G \times_{\mathrm{GL}_V} \mathrm{Stab}_{\mathrm{GL}_W}(W)$$

As taking the Lie algebra preserves limits (4.10), and assuming that the result is true for GL_V, we get that

$$\mathrm{Lie}(\mathrm{Stab}_G(W)) = \mathrm{Lie}(G) \times_{\mathfrak{gl}(V)} \mathrm{Lie}(\mathrm{Stab}_{\mathrm{GL}_V}(W)) =$$

$$= \mathrm{Lie}(G) \times_{\mathfrak{gl}(V)} \mathrm{Stab}_{\mathfrak{gl}(V)}(W) = \mathrm{Stab}_{\mathrm{Lie}(G)}(W)$$

and thus we can assume that $G = \mathrm{GL}_V$ and we have the standard representation on V. We have a diagram of short exact sequences

$$\begin{tikzcd}
\mathbf{1} \arrow[r] & \mathrm{Lie}(\mathrm{Stab}_{\mathrm{GL}_V}(W)) \arrow[r] \arrow[d] & T\,\mathrm{Stab}_{\mathrm{GL}_V}(W) \arrow[r] \arrow[d] & \mathrm{Stab}_{\mathrm{GL}_V}(W) \arrow[r] \arrow[d] & \mathbf{1} \\
\mathbf{1} \arrow[r] & \mathfrak{gl}(V) \arrow[r] & T\,\mathrm{GL}_V \arrow[r] & \mathrm{GL}_V \arrow[r] & \mathbf{1}
\end{tikzcd}$$

where the vertical arrows are injective and thus the square on the left is cartesian. For a k-algebra R and $a \in \mathfrak{gl}(V)(R)$, we have

$$a \in \mathrm{Lie}(\mathrm{Stab}_{GL_V}(W))(R) \iff 1_V + \varepsilon a \in T\,\mathrm{Stab}_{\mathrm{GL}_V}(W)(R) =$$

$$= \mathrm{Stab}_{\mathrm{GL}_V}(W)(R[\varepsilon]) \iff (1_V + \varepsilon a)(W_R \oplus \varepsilon W_R) \subset W_R \oplus \varepsilon W_R \iff$$

$$\iff a(W_R) \subset W_R \iff a \in \mathrm{Stab}_{\mathfrak{gl}(V)}(W)(R)$$

as desired. $\square$

Corollary 6.2. *Let G be an affine connected algebraic group over a field k of characteristic 0 and let V be a finite dimensional representation of G. Let $W \leq V$ be a subspace. Then W is a G-subrepresentation of V if and only if it is a $Lie(G)$-subrepresentation of V (for the induced representation of $Lie(G)$ on V).*

Proof. The subspace W is G invariant if and only if $\mathrm{Stab}_G(W) = G$. By 2.18, this happens if and only if $\dim \mathrm{Lie}(\mathrm{Stab}_G(W)) = \dim \mathrm{Lie}(G)$ and as $\mathrm{Lie}(\mathrm{Stab}_G(W)) = \mathrm{Stab}_{\mathrm{Lie}(G)}(W)$ by the previous proposition, this is equivalent to $\mathrm{Stab}_{\mathrm{Lie}(G)}(W) = \mathrm{Lie}(G)$, i.e. to W being $\mathrm{Lie}(G)$-invariant. □

Proposition 6.3. *Let G be an affine connected algebraic group over a field k and assume either that G is reductive or that $char(k) = 0$. Then $Z(G) = \ker(Ad)$, where $Z(G)$ is the centre of G and Ad is the adjoint representation $Ad\colon G \to GL_{Lie(G)}$. In particular, under the assumptions, the Lie algebra $Lie(Z(G))$ of the centre of G equals the centre $Z(Lie(G))$ of the Lie algebra of G.*

Proof. The centre of G is the subgroup functor $Z(G)(R) = \{g \in G(R) \colon \forall R' \in \mathrm{ob}\mathbf{Alg}_R,\ g_{R'} \in Z(G(R'))\}$. It is representable by a scheme ([4] 1.92), so it is an algebraic subgroup of G. To prove $Z(G) = \ker \mathrm{Ad}$ it is enough ([4], 1.44) to see that $Z(G)(\overline{k}) = \ker \mathrm{Ad}(\overline{k})$ and we may as well assume that k is algebraically closed, as $Z(G_{\overline{k}}) = Z(G)_{\overline{k}}$ and the same holds for $\ker \mathrm{Ad}$. Let $g \in G(k)$. Then $g \in Z(G)(k)$ if and only if the conjugation by g homomorphism $\mathrm{conj}_g \colon G \to G$ is the identity. By 4.12, $\mathrm{conj}_g = \mathrm{id}_G$ if and only if $\mathrm{Lie}(\mathrm{conj}_g) = \mathrm{Ad}(g) = \mathrm{id}_{\mathrm{Lie}(G)}$, which precisely means $g \in \ker \mathrm{Ad}(k)$. □

Proposition 6.4. *Let G be an affine connected algebraic group over a field k of characteristic 0 and let $H \leq G$ be a subgroup. Suppose that $Lie(H)$ is an ideal of $Lie(G)$. Then the identity component H° of H is normal in G.*

Proof. As $\mathrm{Lie}(H) = \mathrm{Lie}(H^\circ)$ we may assume that H is connected. We may define the normaliser $N_G(H)$ of H in G functorially as $N_G(H)(R) = \{g \in G(R) \colon \forall R' \in \mathrm{ob}\mathbf{Alg}_R,\ g_{R'} H(R) g_{R'}^{-1} \subset H(R)\}$. It is representable by [4], 1.83. Moreover, by [4], 10.48, we have the equality $\mathrm{Lie}(N_G(H))/\mathrm{Lie}(H) = (\mathrm{Lie}(G)/\mathrm{Lie}(H))^H$, where the last. As H is connected, $(\mathrm{Lie}(G)/\mathrm{Lie}(H))^H = (\mathrm{Lie}(G)/\mathrm{Lie}(H))^{\mathrm{Lie}(H)}$, the biggest $\mathrm{Lie}(H)$-invariant subspace of $\mathrm{Lie}(G)/\mathrm{Lie}(H)$ where $\mathrm{Lie}(H)$ acts via the 0 homomorphism. Indeed, call $V = \mathrm{Lie}(G)/\mathrm{Lie}(H)$. A subspace W of V is H-invariant if and only if it is $\mathrm{Lie}(H)$-invariant by 6.2. Moreover, in that case the induced $H \to \mathrm{GL}_W$ is trivial if and only if $\mathrm{Lie}(H) \to \mathfrak{gl}(W)$ is trivial, by 4.12.

Now, the subalgebra $\mathrm{Lie}(H)$ of $\mathrm{Lie}(G)$ is an ideal if and only if

$$\mathrm{Lie}(N_G(H))/\mathrm{Lie}(H) = (\mathrm{Lie}(G)/\mathrm{Lie}(H))^{\mathrm{Lie}(H)} = \mathrm{Lie}(G)/\mathrm{Lie}(H)$$

which happens if and only if $\mathrm{Lie}(N_G(H)) = \mathrm{Lie}(G)$, which is true (2.18) if and only if $N_G(H) = G$, i.e. if H is normal in G.

□

6.2 Main results

Proposition 6.5. *Let G be a semisimple algebraic group over a field k of characteristic 0. Then $Lie(G)$ is a semisimple Lie algebra.*

Proof. It is enough to prove that any commutative ideal $\mathfrak{n} \lhd \mathrm{Lie}(G)$ is 0. As G is connected and $\mathfrak{n}$ is an ideal of $\mathrm{Lie}(G)$, $\mathfrak{n}$ is stable under the adjoint action of G on $\mathrm{Lie}(G)$ by 6.2. We thus have a representation $(\mathfrak{n}, \rho)$ of G, where $\rho\colon G \to \mathrm{GL}_{\mathfrak{n}}$, and the induced representation $(\mathrm{Lie}(G), d\rho)$ of $\mathrm{Lie}(G)$ on $\mathfrak{n}$, which is given by the Lie bracket. Let $H = \ker \rho$. Then

$$\mathrm{Lie}(H) = \ker d\rho = \{h \in \mathrm{Lie}(G) \colon [h, \mathfrak{n}] = 0\}$$

by 4.10. Note that $\mathrm{Lie}(H)$ is an ideal of $\mathrm{Lie}(G)$, as if $x \in \mathrm{Lie}(G)$, $h \in \mathrm{Lie}(H)$ and $n \in \mathfrak{n}$, then

$$[[x, h], n] = [x, [h, n]] + [h, [n, x]] = [x, 0] + 0 = 0$$

By 6.4, this implies that $H^\circ \lhd G$ is normal. Thus the center $Z(H^\circ) \lhd G$ is normal and abelian, and it is therefore finite by semisimplicity of G. Therefore $0 = \mathrm{Lie}(Z(H^\circ)) = Z(\mathrm{Lie}(H^\circ)) = Z(\mathrm{Lie}(H)) \supset \mathfrak{n}$ by 6.3, and thus $\mathfrak{n} = 0$. ☐

Theorem 6.6. *Let k be a field of characteristic 0 and let G be an affine connected algebraic group over k. Then the functor*

$$\mathbf{Rep}(G) \to \mathbf{Rep}(Lie(G))$$

from 4.9 is fully faithful and, identifying $\mathbf{Rep}(G)$ with its essential image in $\mathbf{Rep}(Lie(G))$, the following property holds: if

$$0 \longrightarrow W \longrightarrow V \longrightarrow U \longrightarrow 0$$

is a short exact sequence in $\mathbf{Rep}(Lie(G))$ and V is in $\mathbf{Rep}(G)$, then W and U are also in $\mathbf{Rep}(G)$.

Proof. Let us first prove that the functor is fully faithful. Let (V, v) and (W, w) be representations of G and let $f \in \mathrm{Hom}_k(V, W)$. Recall that there is a representation $(\mathrm{Hom}_k(V, W), o)$ of G described in 4.13. We have to prove that f is a morphism of G-representations if and only if f is a morphism of the induced $\mathrm{Lie}(G)$-representations. By 6.2, $k\langle f\rangle$ is a G-subrepresentation of $(\mathrm{Hom}_k(V, W), o)$ if and only if $k\langle f\rangle$ is a $\mathrm{Lie}(G)$-subrepresentation of $(\mathrm{Hom}_k(V, W), do)$. Moreover, in that case, 4.12 implies that $k\langle f\rangle$ is trivial as G-representation if and only if $k\langle f\rangle$ is trivial as $\mathrm{Lie}(G)$-representation. Then 4.13 implies that f is a morphism of G-representations if and only if f is a morphism of $\mathrm{Lie}(G)$-representations. This proves fully faithfulness.

Now suppose that

$$0 \longrightarrow W \xrightarrow{a} V \xrightarrow{b} U \longrightarrow 0$$

is a short exact sequence in $\mathbf{Rep}(\mathrm{Lie}(G))$ and V is in $\mathbf{Rep}(G)$. Note that there is a unique $\mathrm{Lie}(G)$-representation structure on W (resp. U) such that a (reps. b) is a morphism, because a (resp. b) is injective (resp. surjective). The vector space W is a $\mathrm{Lie}(G)$-invariant subspace of V, so it is also G-invariant by 6.2 and thus inherits a unique G-representation structure such that a is a morphism of G-representations. This also gives U a structure of G-representation such that b is a morphism, and we thus get a short exact sequence of G-representations. By the uniqueness of structure of $\mathrm{Lie}(G)$-representations mentioned before, the induced short exact sequence of $\mathrm{Lie}(G)$-representations must be the original one, and we are done. $\square$

Corollary 6.7. *Let k be a field of characteristic 0, let G be an affine connected algebraic group over k and let (V, r) be a finite dimensional representation of G. Then (V, r) is semisimple if and only if the induced representation (V, dr) of $\mathrm{Lie}(G)$ is semisimple.*

In particular, if G is semisimple, then every finite dimensional representation of G is semisimple (i.e. G is linearly reductive).

Proof. The first part follows trivially from 6.6. For the second part we have that $\mathrm{Lie}(G)$ is semisimple if G is, by 6.5, and that by Weyl's theorem ([7], Theorem in page 46), every finite representation of $\mathrm{Lie}(G)$ is semisimple. This, together with the first part, implies the result. $\square$

Question 6.8. Can we relax hypothesis on the group and characteristic? In particular, does it hold for arbitrary semisimple groups?

Question 6.9. Are there linearly reductive groups that admit infinite dimensional representations that are not semisimple?

The following is one of the most important result in these pages, that characterises reductive groups over a field of characteristic 0 in terms of representations.

Theorem 6.10. *Let k be a field of characteristic 0, and let G algebraic group over k. Then G is reductive if and only if G is linearly reductive.*

Proof. First, suppose that G is reductive and let V be a finite dimensional representation of G. By 6.6, we only have to prove that the induced representation (V, ρ) of $\mathrm{Lie}(G)$ on V is semisimple. By [4], Proposition 21.60, there is an exact sequence

$$\mathbf{1} \longrightarrow F \longrightarrow T \times S \longrightarrow G \longrightarrow \mathbf{1}$$

where T is a (non-necessarily split) torus and S is semisimple. It is thus enough to prove the result for $G = T \times S$. In this case, $\mathrm{Lie}(G) = \mathrm{Lie}(T) \times \mathrm{Lie}(S)$ by 4.10. Note that T is connected, because after a base change of field it is a finite product of schemes of the form $\mathrm{Spec}(k[X, X^{-1}])$, connected. We know 5.7 that V is semisimple as a representation of T. By 6.6, this implies that V is semisimple as a representation of $\mathrm{Lie}(T)$. The fact that T is abelian implies that $\mathrm{Lie}(T)$ is abelian (indeed, Ad is trivial, so ad is trivial). Take $x_1, \ldots, x_l \in \mathrm{Lie}(T)$ a basis of $\mathrm{Lie}(T)$. If $\lambda_{i1}, \ldots, \lambda_{im_i}$ are the eigenvalues of x_i, then V decomposes as direct sum of the subspaces $V(a_1, \ldots, a_l)$, $a_i \in [\![1, m_i]\!]$, defined by the property that $\rho(x_i)$ acts centrally on $V(a_1, \ldots, a_l)$ with eigenvalue λ_{ia_i}. If $y \in \mathrm{Lie}(G)$ and $v \in V(a_1, \ldots, a_l)$, then $\rho(x_i)\rho(y)v = \rho(y)\rho(x_i)v = \lambda_{ia_i}\rho(y)v$, so the $V(a_1, \ldots, a_l)$ are $\mathrm{Lie}(G)$-invariant. Therefore, changing V by each of the $V(a_1, \ldots, a_l)$, we may assume that each $\rho(x_i)$ act centrally on V with eigenvalue β_i. In particular, every subspace of V is a $\mathrm{Lie}(T)$-subrepresentation of V. As $\mathrm{Lie}(S)$ is semisimple (see 6.5), there is a direct sum decomposition $V = \bigoplus_{i=1}^n V_i$ of V where each V_i is an irreducible $\mathrm{Lie}(S)$-subrepresentation of V. As the V_i are also $\mathrm{Lie}(T)$-invariant, they are $\mathrm{Lie}(G)$-invariant. Being irreducible as a representation of $\mathrm{Lie}(S)$, each V_i is also irreducible as a representation of $\mathrm{Lie}(G)$. Thus V is a semisimple representation of $\mathrm{Lie}(G)$. This proves one implication.

Now suppose that G is linearly reductive. It is enough to prove (3.4) that the unipotent radical $N = R_u(G)$ of G is trivial. Let (V, r) be a nonzero faithful finite dimensional representation of G. As N is normal in G, the subspace $V^N = \{v \in V : \forall R \in \mathrm{ob}\mathbf{Alg}_k, \forall h \in N(R), h(v \otimes 1) = v \otimes 1\}$ of V is G-invariant. As G is linearly reductive, there exists a G-invariant subspace $W \subset V$ such that $V = V^N \oplus W$. As $W^N \subset V^N$, we have $W^N = 0$, and as N is unipotent, this implies $W = 0$. Therefore $V^N = V$, so that $N \subset \ker r = \mathbf{1}$, and therefore $N = \mathbf{1}$. □

Remark 6.11. A less elementary proof of 6.10 can be found in [2], page 507, Proposition 3.3. Milne's proof ([4], Theorem 22.42) seems to be incorrect.

Let us finish this section with a tangential question.

Question 6.12. Let G be an algebraic group over a field k, and let k' be a separable extension of k. Suppose that $G_{k'}$ is linearly reductive. Can we conclude that G is linearly reductive?

7 Finite generation of the invariants

In this section we prove one of the main consequences of linear reductivity of an algebraic group G, namely that the ring of invariants R_0 of a dual action of G on a finite type k-algebra R is again a finite type k-algebra. This is true more generally when G is *reductive* (which, we recall, is only equivalent to *linearly reductive* in characteristic 0), but the proof is more dificult and we will content ourselves with the linearly reductive case. This finite generation result is the starting point of Geometric Invariant Theory (see [5]). In this section, k is a field.

An immediate but important consequence of 2.21 is the following

Lemma 7.1. *Let G be an affine algebraic group over k and let $\widehat{\sigma}\colon R \to O \otimes R$ be a dual action of G on a finite type k-algebra R, where O is the ring of G. Then there is a finite dimensional invariant vector subspace V of R generating R as a k-algebra.*

We now are ready to define the Reynolds operator, starting by defining the notion of invariants.

Definition 7.2. Let G be an affine algebraic group over k. Let V be a representation of G, with dual action $\widehat{\sigma}\colon V \to O \otimes V$. An element $v \in V$ is said to be an *invariant* if $\widehat{\sigma}(v) = 1 \otimes v$. We will denote V_0 the set of invariant elements elements of V.

The set V_0 is actually a subrepresentation of V, and it is the biggest trivial subrepresentation of V. If R is a k-algebra and $\widehat{\sigma}\colon R \to O \otimes R$ is a dual action, then R_0 is actually a subring of R: it is called the *ring of invariants* of the dual action.

Proposition 7.3. *Let G be a linearly reductive affine algebraic group over k, and let $\mathcal{A}$ be the category of (arbitrary dimensional) representations of G. There is a unique natural endomorphism E of the identity functor $1_{\mathcal{A}}$ such that*

1. *$E^2 = E$,*
2. *for every representation V, $\operatorname{im} E_V = V_0$, the set of invariants.*

The endomorphism E of $1_{\mathcal{A}}$ is called the *Reynolds* operator.

Proof. Let V be a finite dimensional representation. By linear reductivity, there is a subrepresentation V_1 of V such that $V = V_0 \oplus V_1$, where V_0 is the invariant subrepresentation. Let S be any nontrivial simple subrepresentation of V, then the map $S \to V/V_1 \cong V_0$ has to be 0, as any subrepresentation of V_0 is trivial. Thus $S \subset V_1$. At the same time, V_1 is sum of simple subrepresentation, by semisimplicity. Therefore V_1 is the sum of all simple nontrivial subrepresentations of V, so it is uniquely determined. If $f\colon V \to W$ is a map of finite dimensional representations, and S is a simple subrep of V, then $f(S)$ is trivial if S is, and if S is nontrivial, then $f(S)$ is nontrivial or 0. This proves that $f(V_0) \subset W_0$ and $f(V_1) \subset W_1$ (let us call this property *naturality of the splitting*).

If V has arbitrary dimension, we define V_1 to be the sum of all the W_1, for finite dimensional subrepresentations W of V. Then by Lemma 2.21 we still have $V = V_0 \oplus V_1$ and the splitting is natural. We define E_V to be the composition $V \to V/V_1 \cong V_0 \to V$ (i.e. the projection onto V_0). Naturality of E follows from naturality of the splitting, and conditions 1 and 2 are obvious from the definition. Uniqueness of E follows from uniqueness in the finite dimensional case proven above, Lemma 2.21 and naturality. $\square$

Proposition 7.4 (Reynolds identity)**.** *Let G be a linearly reductive affine algebraic group over k, let R be a k-algebra and let $\widehat{\sigma}\colon R \to O \otimes R$ be a dual action. Then the associated Reynolds operator (regarding R just as a linear representation) $E_R\colon R \to R$ is R_0-linear. In particular, $R_1 = \ker E_R$ is an R_0-module.*

Proof. We have the direct sum $R = R_0 \oplus R_1$, so it suffices to prove that $R_0 \cdot R_1 \subset R_1$, so that R_1 is an R_0-module. Let $r \in R_0$. Multiplication by r induces a morphism of linear representations $R \to R$, as for all $x \in R$ we have $\widehat{\sigma}(rx) = \widehat{\sigma}(r)\widehat{\sigma}(x) = (1 \otimes r)\widehat{\sigma}(x)$. If S is a simple nontrivial subrepresentation of R, then $r \cdot S$ is thus either 0 or nontrivial, so $r \cdot S \subset R_1$. As R_1 is the sum of all simple nontrivial subrepresentations of R, we are done. $\square$

We are ready to prove the most important result of this section.

Theorem 7.5. *Let G be a linearly reductive affine algebraic group over k, let R be a k-algebra and let $\widehat{\sigma}\colon R \to O \otimes R$ be a dual action. Suppose that R is of finite type as k-algebra, then the ring of invariants R_0 is also a finite type k-algebra.*

Proof. The result is a consequence of a series general statements.

1. *If R is Noetherian, then R_0 is Noetherian.*

   Let I be an ideal of R_0. The extended ideal is $RI = (R_0 \oplus R_1)I = I \oplus R_1 I$, and the last sum is direct because $R_1 I \subset R_1$, R_1 being an R_0-module. Thus $RI \cap R_0 = I$ and therefore the poset of ideals of R_0 injects into the poset of ideals of R, from which the conclusion follows.

2. *If R is of finite type and there is an $\mathbb{N}$-grading on R, $R = \bigoplus_{n \in \mathbb{N}} R^{(n)}$, such that $R^{(0)} = k$ and the grading is compatible with the action of G (i.e. for all $n \in \mathbb{N}$, $\widehat{\sigma}(R^{(n)}) \subset O \otimes R^{(n)}$), then R_0 is of finite type (say that R is* respected *if it admits such a grading).*

   By the previous statement, R_0 is Noetherian. It also admits an $\mathbb{N}$-grading with $R_0^{(0)} = k$, as a consequence of R being respected. It follows that R_0 is of finite type (see [1], Proposition 10.7).

3. *If R is of finite type, there is a finite type k-algebra R' endowed with a dual action of G and an equivariang surjection $R' \to R$ such that R' is respected.*

Let V be a finite dimensional invariant vector subspace of R containing a set of generators. There is a unique dual (k-algebra) action of G on $\mathrm{Sym}(V)$ such that $V \to \mathrm{Sym}(V)$ is equivariant, and this action is readily seen to be respected. The induced map $\mathrm{Sym}(V) \to R$ is equivariant and surjective.

4. *If R' is another k-algebra equipped with a dual action of G and $f\colon R' \to R$ is an equivariant surjective k-algebra homomorphism, then the induced map $f_0\colon R'_0 \to R_0$ is surjective.*

   By naturality of the Reynolds operator E, we have $R_0 = E(R) = E(f(R')) = f(E(R')) = f(R'_0)$.

Starting with R of finite type, statement 3 gives as a surjection $R' \to R$ with R' of finite type and respected. By statement 4, we get a surjection $R'_0 \to R_0$, and by statement 2, R'_0 is of finite type. This implies that R_0 is of finite type. □

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