Lie Groups: Fall, 2022
Lecture I

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1 Lie Groups

1.1 The Definition

We begin with the fundamental definitions for this course.

Definition 1.1. A *Lie group* is a smooth finite dimensional manifold G with two structure maps, which are required to be smooth maps, $m\colon G \times G \to G$ and $\iota\colon G \to G$, together with an element $e \in G$. These structure maps define a group structure on G with m as a product, e as the identity element, and ι as the map $g \mapsto g^{-1}$. A *map* $\varphi\colon G \to H$ of Lie groups is a smooth map from the manifold under lying G to that underlying H that is also a group homomorphism

If G is a complex manifold and the structure maps m and ι are holomorphic, then G is a *complex Lie Group*. A *map* between complex Lie groups is a holomorphic map that is a group homomorhism.

There is one technical issue in the definition of Lie groups and complex Lie groups; namely what we mean by a manifold. There are two conditions that are optional in the definition of a manifold: Hausdorff and 2^{nd} countable (which means that there is a countable basis for the topology). Usually, manifolds are assumed to be Hausdorff and second countable. We shall always require that the manifolds underlying Lie groups be Hausdorff. Normally, we shall implicitly assume that they are second countable as well, but it is not essential as the following lemma shows.

Lemma 1.2. *Let G be a connected Lie group. Then G is second countable.*

Proof. Since G is a finite dimensional manifold its topology is first countable, meaning that every point p has a countable collection of open sets $V_n(p)$ that form a basis for the topology at that point. That is to say that any open subset containing the point p contains one of the $V_n(p)$. To show that a first countable space is second countable, one only needs show that it has a countable dense set. Since G is a finite dimensional manifold, there is a neighborhood U of the identity that is homeomorphic to an open ball in $\mathbb{R}^n$ for some finite n. Consequently, U has a countable dense set. Consider the subset W of G all of elements that can be written as finite products $g_1 \cdots g_k$ where each $g_i \in U$. Then W is clearly an open subset of G. We claim that W is also a closed subset of G. For suppose that g is a limit point of W. Choose a sequence, h_k, converging to g so that each $h_k \in W$. Then $h_k^{-1}g$ converges to $e \in G$ and hence for some (indeed all) k sufficiently large $h_k^{-1}g \in U$. Since, $h_k \in W$ and $h_k^{-1}g$ is in U, it follows that $g \in W$. Hence, W is both open and closed in G and is non-empty (since it contains $e \in G$). Since G is connected, $W = G$.

Now consider all finite products of a countable dense subset S of U. Since every element of G is a finite product of elements in U, a standard diagonalization argument shows that the set of elements represented by finite products of elements in S is dense in G. $\square$

Corollary 1.3. *A Lie group is second countable if and only if it has at most countably many connected components.*

Definition 1.4. A *morphism* $\varphi\colon G \to G'$ from one Lie group to another is a smooth map that is a group homomorphism.

Obviously, these definitions define the category of Lie groups. Those that are second countable form a faithful subcategory.

1.2 Submanifolds and sub Lie groups

Definition 1.5. Let M be a smooth manifold. A *smooth submanifold* is a subset $N \subset$M with the property that for each $m \in M$ there is a local coordinate system $(x^1, \ldots, x^k)$ defined on an open set U containing the point m such that $N \cap U$ is given by the subset of U where the equations $\{x^{r+1} = \cdots = x^k = 0\}$ hold. Then N inherits a unique smooth structure such that the inclusion $N \to M$ is a smooth map. Such a map is called *a smooth embedding.* [Since we are working exclusively in the smooth category we shall drop the adjective *smooth* from the terminology both for submanifolds and embedding. It is implicit.]

Notice that if N is a submanifold of M then it is a closed subset of M. There is a converse to this. Suppose that $\varphi\colon N \to M$ is an immersion (injective differential at every point) and is a one-to-one map. Then the image $\varphi(N)$ is a submanifold of M if and only if it is a closed subset. An example showing that the image is not automatically closed is given by the map

$$\mathbb{R}^1 \xrightarrow{f} \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$$

where $f(t) = (t, \pi t)$.

Definition 1.6. Let G be a Lie group. A *Lie subgroup* $H \subset G$ is a smooth submanifold H of G that is closed under the product and inverses and contains the identity element[1].

The terminology is justified by the following lemma.

Proposition 1.7. *If $H \subset G$ is a sub-Lie group, then the restriction of the product and inverse of G to H give H the structure of a Lie group and the inclusion $H \subset G$ is a morphism of Lie groups. Furthermore, the space of left cosets $H\backslash G$ has the structure of a smooth manifold in such a way that the projection $G \to H\backslash G$ is a submersion (i.e., has surjective differential at every point) and is a locally trivial fiber bundle with fibers isomorphic to H.*

Proof. We consider the case when G is connected, and leave the generalization to non-connected groups to the reader. Since H is closed under product and inverses and contains the identity, the restriction of the group structure maps from G to H define the structure of a group on H. We need only see that the product and the inverse are smooth maps of H. But they are smooth maps of G and H is a smooth submanifold invariant under the maps. Hence, the restriction of the maps to H are smooth. The inclusion $H \subset G$ is a smooth map and a group homomorphism and hence, by definition a morphism of Lie groups.

Fix a positive definite inner product $\langle \cdot, \cdot \rangle_e$ on T_eG. There is a unique left-invariant Riemannian metric μ on G whose value at the tangent plane to the identity is $\langle \cdot, \cdot \rangle_e$. Of course, the left action of H on G preserves this metric. The distance d_μ between two points is the infimum of the length of any rectifiable curve joining the points.

Claim 1.8. *For $g \in G$ and $R < \infty$ the closed μ-ball, $\overline{B_\mu(g, R)}$, centered at g of radius R is compact.*

[1] In the literature one sometimes finds the more general notion of sub Lie group where the submanifold is not required to be closed, just to be one-to-one immersed.

Proof. By equivariance, it suffices to prove this for $g = e$. Choose a coordinate patch for G centered at the identity that is defined on the open ball of radius 2 at the origin in Euclidean space. Since the space of unit tangent vectors to the closed Euclidean ball of radius 1 is compact, there is a finite maximum to the μ-length of any of these unit vectors. In particular, for some $\epsilon > 0$, the closed μ-ball $\overline{B_\mu(e,\epsilon)}$ of radius ϵ is a closed subset of the closed unit ball with respect to the Euclidean metric. Hence, this closed μ-ball is compact.

Now suppose that $\overline{B_\mu(e,R)}$ is compact for some R. Then there is a finite set $\{g_i \in G\}_{i\in I}$ such that this ball is contained in $\cup_{i\in I}\{B_\mu(g_i,\epsilon/2)\}_i$. By the triangle inequality this implies that

$$\overline{B_\mu(e,R+\epsilon/3)} \subset \cup_{i\in I}\overline{B_\mu(g_i,\epsilon)}.$$

By the equivariance of the metric each $\overline{B_\mu(g_i,\epsilon)}$ is compact, and consequently so is $\overline{B_\mu(e,R+\epsilon/3)}$. The claim follows. □

Since H is a closed subspace of G, there is a ball B in a coordinate patch centered at the identity with coordinates $(x_1,\ldots,x_n)$ such that $U = H\cap B$ is the subspace $\{x_{k+1} = \cdots = x_n = 0\}\cap B$. Let $S = B\cap\{x_1 = \ldots = x_k = 0\}$. The product map $U\times S \to G$ has identity differential at the identity. Thus, possibly after replacing S and U by smaller neighborhoods of the identity, we can assume that the product map $\psi\colon U\times S\to G$ is a diffeomorphism onto an open subset of B. Now choose smaller neighborhoods $S' \subset S$ and $U' \subset U$ such that twice the diameter of $\psi(U',S')$ is less that the distance from $\psi(U',S')$ to the complement of $\psi(U,S)$ in G

Claim 1.9. *Let $\psi\colon H\times S\to G$ be the product map. Then the restriction $\psi\colon H\times S'\to G$ is a diffeomorphism onto an open neighborhood of H.*

Proof. Since $\psi\colon U\times S\to G$ is a local diffeomorphism onto an open subset of G, by equivariance the same is true for $\psi\colon H\times S\to G$ and *a fortiori* $\psi\colon H\times S'\to G$ is a local diffeomorphism whose image is an open neighborhood of H in G. It remains to show that this map is one-to-one. Suppose that we have $s, s_1 \in S'$ and $h, h_1 \in H$ such that $\psi(h,s) = \psi(h_1,s_1)$. We shall show that $(h,s) = (h_1,s_1)$ in $H\times S$

First notice that $\psi(h^{-1}h_1,s_1) = \psi(e,s)$, implying that $(h^{-1}h_1)\psi(U'\times S')\cap\psi(U'\times S') \neq \emptyset$. Since the metric is invariant under the action of H, this implies that the distance from $h^{-1}h_1 = \psi(h^{-1}h_1,e)$ to $e = \psi(e,e)$ is at most twice the diameter of $\psi(U',S')$, which by our choices, means that $h_1^{-1}h \in \psi(U,S)$. Since $\psi(U,S)\cap H = U$, we see that $h_1^{-1}h \in U$. Since

$\psi\colon U \times S \to G$ is an embedding, this implies that $(h_1^{-1}h, s_1) = (e, s)$ in $H \times S$. We conclude that $(h, s) = (h_1, s_1)$ in $H \times S$. □

Definition 1.10. We call S' as above a *slice for the action of H on G at the identity.* For any $g \in G$ a *slice for the action of H on G at g* is then the image of a slice for the action at the identity under right multiplication by g.

We have now shown that for a slice S the map $H \times S \to G$ is a diffeomorphism onto an open subset. Thus, S is a coordinate patch for $H\backslash G$ near the identity coset. Pushing these local coordinates around by $g \in G$ gives coordinate patches covering $H\backslash G$. It is clear that the overlap of two of these patches is smooth and the projection map $G \to H\backslash G$ is a smooth submersion. This also defines the local product structure for the fibration $G \to H\backslash G$.

Lastly, we need to show that $H\backslash G$ is a Hausdorff space. Let Hg and Hg' be distinct orbits and fix $x' \in Hg'$. Since Hg is a closed subset and the closed μ-balls are compact, there is an $x \in Hg$ that minimizes the μ-distance $d > 0$ from x' to Hg. By equivariance, the μ-distance from any point $y \in Hg'$ to Hg is d. We call this the distance from Hg' to Hg. The notion of distance between orbits is symmetric and satisfies the triangle inequality. Using the existence of slices, one sees that it is a continuous function $H\backslash G \times H\backslash G \to \mathbb{R}$. Hence, it is a metric on $H\backslash G$, proving that $H\backslash G$ is a Hausdorff space. □

Definition 1.11. A sub Lie group $K \subset G$ is said to be *normal* if K is a normal subgroup of G in the usual group-theoretic sense.

Lemma 1.12. *If $K \subset G$ is a normal Lie subgroup, then the space of left cosets $K\backslash G$ has the structure of a Lie group such that the projection $G \to K\backslash G$ is a homomorphism of Lie groups.*

Proof. Since K is a normal subgroup of G, the group structure on G induces a group structure on $K\backslash G$ in such a way that the projection $G \to K\backslash G$ is a group homomorphism. We have just seen that $K\backslash G$ is a smooth manifold and that the projection $G \to K\backslash G$ is a smooth map. It remains only to show that the structure maps for the group structure on G/K are smooth. Let us consider the multiplication map $\mu\colon K\backslash G \times K\backslash G \to K\backslash G$. Fix points $x, y \in K\backslash G$. Lift these to points $\widetilde{x}, \widetilde{y} \in G$ and let $S_{\widetilde{x}}, S_{\widetilde{y}}$ be slices from the projection mapping $G \to K\backslash G$ at $\widetilde{x}$ and $\widetilde{y}$, respectively. Let $S_{\widetilde{x}\widetilde{y}}$ be a slice for the projection $G \to K\backslash G$ at $\widetilde{x}\widetilde{y}$. Choosing $S_{\widetilde{x}}$ and $S_{\widetilde{y}}$ sufficiently small, we can assume that the image of the product $\mu(S_{\widetilde{x}} \times S_{\widetilde{y}})$ is contained in $K \times S_{\widetilde{x}\widetilde{y}} \subset G$. It is a smooth map. Thus, the composition

$$S_{\widetilde{x}} \times S_{\widetilde{y}} \xrightarrow{\mu} K \times S_{\widetilde{x}\widetilde{y}} \xrightarrow{\pi_2} S_{\widetilde{x}\widetilde{y}}$$

is also smooth. This is the restriction of the multiplication map for the quotient to $S_{\widetilde{x}} \times S_{\widetilde{y}}$.

The argument for the inverse is similar. □

There is an analogue of the first part of Lemma 1.7 for one-to-one immersed subgroups.

Lemma 1.13. *Let G be a Lie group. Suppose that H is a smooth manifold and $\varphi\colon H \to G$ is a one-to one smooth immersion whose image is a subgroup of G. Then there is a unique Lie group structure on H so that φ is a homomorphism of Lie groups.*

Proof. Since H is a smooth manifold and a group, we need only show that group multiplication and inverse are smooth maps. Let $(h, h') \in H \times H$. There there are neighborhoods U, U' and V of h, h' and hh', respectively, such that $\varphi\colon U \to G$ and $\varphi\colon U' \to G$ and $\varphi\colon V \to G$ are embeddings onto smooth (locally closed) submanifolds. Taking U and U' sufficiently small we can arrange that the product in G maps $U \times U' \to V$. Since the group multiplication of G is smooth the composition $U \times U' \to V \subset G$ is smooth, and since V is a locally closed smooth submanifold of G, this implies that $U \times U' \to V$ is smooth.

The argument for the inverse map is analogous. □

Notice that in this case there is no reasonable manifold structure on $H\backslash G$. Indeed, in general the quotient space is not Hausdorff.

2 Examples of Lie Groups

Groups naturally arise as symmetry groups of some mathematical structure, so they come with their defining action. Most Lie groups, complex Lie groups, or linear algebraic groups arise in this way.

Any discrete group is a Lie group. If we require, as one often does, that a manifold must be second countable, they only the countable discrete groups are Lie groups. Of particular interest are the finite groups.

Example 1. The symmetries of a square in the plane, meaning a Euclidean isometry of the square onto itself consists of rotations through multiples of

$\pi/2$ around the central point of the square, together with flips, either about a line bisecting two opposite sides or a line passing through two opposite vertices. These form a group of order 8 with a normal subgroup being the group of 4 rotations. Similarly, the Euclidean symmetries of a regular n-gon in the plane is a group of order $2n$ with a normal subgroup being the cyclic group consisting of the n rotational symmetries. The full group is the dihedral group because the action of the quotient group of order two acts on the rotations by sending every rotation to its inverse.

Example 2. The real line $\mathbb{R}$ with m being addition and $\iota(x) = -x$ is a Lie Group, the *additive group over* $\mathbb{R}$. The non-zero real numbers under multiplication form a Lie group. The unit circle in the complex plane with product being product of complex numbers and ι being inverse of complex numbers is a Lie group.

Example 3. Let V be a finite dimensional real vector space. Then the general linear group of V, denoted $GL(V)$, is a Lie group under matrix multiplication and matrix inverse. $SL(V)$ the subgroup of $GL(V)$ of matrices of determinant 1.

Example 4. Let Q be a non-degenerate quadratic form on a finite dimensional real vector space V. W define $O(Q)$, the *orthogonal group of* Q, to be the subgroup of $GL(V)$ that leaves Q invariant in the sense that $A \in GL(V)$ is in $O(Q)$ if and only if $Q(Av) = Q(v)$ for all $v \in V$. Check that $O(Q)$ is a smooth submanifold of $GL(V)$ that closed under the product and taking inverses and contains the identity. Applying the above lemma, we see that it is a sub-Lie group of $GL(V)$ and hence is a Lie Group in its own right.

The example $O(n)$ is the orthogonal group of the standard Euclidean inner product on $\mathbb{R}^n$. The group $SO(n)$ is the subgroup of $O(n)$ of matrices of determinant 1. Show that $SO(n)$ is the component of the identity of $O(n)$.

Example 5. If G_1 and G_2 are Lie groups, then the product smooth manifold $G_1 \times G_2$ is naturally a Lie group under the product operations. Notice that $G_1 \times \{e\}$ and $\{e\} \times G_2$ are sub Lie groups of $G_1 \times G_2$, and this is a categorical product in the category of Lie Groups.

2.1 Some Counter-Examples

Consider the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. The translation structure on $\mathbb{R}^2$ induces an Abelian group structure on T^2 that makes it a compact Lie group. Any sub-Lie group $H \subset T^2$ is a closed subset of T^2 and hence is compact. As a result every connected sub Lie groups of T^2 is isomorphic to one of T^2, S^1,$\{e\}$. If $\mathbb{R} \subset \mathbb{R}^2$ is a line through the origin in an irrational direction, then it induces an injective map $\mathbb{R}^1 \to T^2$ map of Lie groups whose image is not compact and hence not a sub Lie group.

Notice that there is a quotient space $T^2/\mathbb{R}^1$ inherits a group structure and also is is locally isomorphic to $\mathbb{R}^1$ with local coordinates in which the group structure is smooth. But the quotient is not a Lie group since it is not Hausdorff.

There are similar examples in higher dimensional tori of all possible codimensions ≥ 1.

The examples show that in general if $\rho\colon H \to G$ is a map of Lie groups, then the image is not necessarily a sub Lie group of G.

3 Examples of Complex Lie groups

As in the real case, we have:

Lemma 3.1. *If G is a complex Lie Group and $H \subset G$ is a complex submanifold containing the identity element of G and closed under the product operation and the inverse map, then H together with the restriction to H of these structure maps is a complex Lie Group.*

Definition 3.2. A *complex linear algebraic group* is a complex algebraic sub variety of $M(n \times n, \mathbb{C})$ contained in the Zariski open subset $GL(n, \mathbb{C})$ and closed under matrix multiplication and inverses.

There are complex analogues of all the real Lie Groups mentioned above. The additive group of complex numbers and the multiplicative group $\mathbb{C}^*$ of non-zero complex numbers are both linear algebraic groups over $\mathbb{C}$ and hence complex manifolds. If V is a finite dimensional complex vector space then its complex linear automorphisms form a complex Lie Group. Of course, we can assume that V is isomorphic to $\mathbb{C}^n$ for some $n \geq 0$. Thus, for some $n \geq 0$ the complex Lie group $GL(V)$ is isomorphic to the complex Lie Group $GL(n, \mathbb{C})$, the group of invertible $n \times n$ complex matrices. The product is matrix multiplication and the inverse is the matrix inverse. The group is an open subset of the complex vector space $M(n \times n, \mathbb{C})$ of complex $n \times n$ matrices. In fact, being the complement of the divisor where $\{\det = 0\}$, $GL(n, \mathbb{C})$ is a Zariski open set and is a linear algebraic group over $\mathbb{C}$. We also have $SL(n, \mathbb{C}) \subset GL(n, \mathbb{C})$ of matrices of determinant 1 also a linear algebraic group over $\mathbb{C}$, and hence a complex Lie Group. For any non-degenerate complex quadratic form Q on $\mathbb{C}^n$ we have its complex orthogonal group, defined as in the real case. This also is a linear algebraic group over $\mathbb{C}$ and hence a complex Lie group. Similarly, for a non-degenerate, skew symmetric, complex bilinear form on $\mathbb{C}^n$ we have the complex symplectic group, again a linear algebraic group over $\mathbb{C}$, and hence a complex Lie group.

Consider a maximal lattice $\Lambda \subset \mathbb{C}$. By definition Λ is generated by two elements that are linearly independent over $\mathbb{R}$. The quotient $\mathbb{C}/\Lambda$ is a compact complex curve diffeomorphic to $S^1 \times S^1$. Addition on $\mathbb{C}$ induces a group structure on $\mathbb{C}/\Lambda$ that makes it a complex Lie group. Show that it is not a complex linear algebraic group.