Lie Groups: Fall, 2024
Lecture X:
The Affine Weyl Chambers and the Fundamental Group

December 14, 2024

We keep our running notation: G is a compact connected Lie group, T is a maximal torus of G; $\mathfrak{t}$ is its Lie algebra, W is the Weyl group with its natural action on T and $\mathfrak{t}$. There is a positive definite, symmetric inner product on $\mathfrak{t}$ leading to an isomorphism $\mathfrak{t}^* \cong \mathfrak{t}$ and for each root α, we have an element $x_\alpha \in \mathfrak{t}$, the element corresponding identified with α under this isomorphism. That is to say, $x_\alpha \in \ker(\alpha)^\perp$ and $\alpha(x_\alpha) = \langle \alpha, \alpha \rangle$. The lattice $\Lambda \subset \mathfrak{t}$ is the fundamental lattice, meaning that $T = \mathfrak{t}/\Lambda$; the lattice Λ_0 is the translation lattice of the affine Weyl group $W_{\rm aff}$, generated by the λ_α; and Λ_R^* is the dual to the root lattice. We have $\Lambda_0 \subset \Lambda \subset \Lambda_R^*$.

The main result in this section is the following:

Theorem 0.1. *Let G be a compact Lie group. Then Λ and Λ_0 are, as before, the fundamental lattice of G and the translation lattice for the affine Weyl group of G, respectively. Then the map $T \to G$ induces a surjection $\Lambda \to \pi_1(G)$ whose kernel is Λ_0. Thus, the inclusion $T \to G$ induces an isomorphism $\Lambda/\Lambda_0 \cong \pi_1(G)$.*

1 The Regular Elements $G_{\rm reg} \subset G$

1.1 The Definition and Description of $G_{\rm reg}$

Definition 1.1. An element $g \in G$ is said to be *regular* if it is contained in a unique maximal torus. We define by $G_{\rm reg} \subset G$ be subspace of regular elements. For T a maximal torus, we denote by $T_{\rm reg} \subset T$ is the set of regular element of G contained in T. These are the elements that are contained in no maximal torus except T.

We define a map $\widetilde{c}\colon G \times T \to G$ by $(g,t) \mapsto gtg^{-1}$. Since T is abelian, $\widetilde{c}$ induces a map

$$c\colon G/T \times T \to G$$

sending $(gT,t) \mapsto gtg^{-1}$. This map restricts to give a map $c_{\text{reg}}\colon G/T \times T_{\text{reg}} \to G_{\text{reg}}$.

Proposition 1.2. *The map $c_{\text{reg}}\colon G/T \times T_{\text{reg}} \to G_{\text{reg}}$ is a local diffeomorphism onto G_{reg}.*

Proof. It is clear that c is a smooth map. Any conjugate of an element of T_{reg} is an element of G_{rg}, Any element $g \in G_{\text{reg}}$ is conjugate to an element of $t' \in T$ and clearly $t' \in T_{\text{reg}}$. This shows that the image of c_{reg} is G_{reg}.

Let us first compute the differential of c_{reg} at points of the form (eT,t_0). Let $g(s)$ be a smooth path with $g(0)=e$ and let $t_0 \in T_{\text{reg}}$. We compute

$$\begin{aligned}(g(s)t_0(g(s)^{-1})'(0) &= g'(0)t_0 - t_0 g'(0)\\ &= [g'(0) - t_0 g'(0) t_0^{-1}] \cdot t_0\\ &= \big[(1-\mathrm{ad}(t_0))(g'(0)\big] \cdot t_0.\end{aligned}$$

Since t_0 is a regular element, the kernel of $1-\mathrm{ad}(t_0)$ is $\mathfrak{t}$ and the image of this map is the sum of the root spaces. Thus, the kernel of $\big[(1-\mathrm{ad}(t_0))(g'(0))\big] \cdot t_0$ is $\mathfrak{t}$ and its image is $\left(\oplus_{\alpha_i \in R} V_{\alpha_i}\right) \cdot t_0$. This shows that the differential of the map $G/T \times \{t_0\}$ at (eT,t_0) is injective with image $\left(\oplus_{\alpha_i \in R} V_{\alpha_i}\right) \cdot t_0$. Clearly, the map $\{e\} \times T_{\text{reg}} \to G$ given by $(eT,t) \mapsto t$ has injective differential at t_0 whose image is $\mathfrak{t} \cdot t_0$, which is a complementary space to the image of the differential of $G/T \times \{t_0\}$. This proves that the differential of map $G/G \times T_{\text{reg}} \to G_{\text{reg}}$ at (eT,t_0) is an isomorphism for every $t_0 \in T_{\text{reg}}$.

We have a commutative diagram:

$$\begin{array}{ccc} G/T \times T_{\text{reg}} & \xrightarrow{c_{\text{reg}}} & G_{\text{reg}} \\ {\scriptstyle (g\cdot)\times \mathrm{Id}}\downarrow & & \downarrow {\scriptstyle c_g} \\ G/T \times T_{\text{reg}} & \xrightarrow{c_{\text{reg}}} & G_{\text{reg}} \end{array}$$

where c_g is conjugation by g. The vertical maps are diffeomorphisms. It follows that the differential of c_{reg} is an isomorphism at every $(gT,t_0) \in G/T \times T_{\text{reg}}$.

Hence, the map a local diffeomorphism. In particular, G_{reg} is an open subset of G. □

This map is not a diffeomorphism. But after dividing out by the natural action of the Weyl group it becomes one.

Lemma 1.3. *Define an action of the Weyl group W on $G/T \times T$ by*

$$w * (gT, t) = (gw^{-1}T, wtw^{-1}).$$

These formulas give a well-defined left action of W stabilizing $G/T \times T_{\mathrm{reg}}$. The map $c\colon G/T \times T \to G$ is Weyl invariant and the quotient map

$$\overline{c}_{\mathrm{reg}}\colon ((G/T) \times T_{\mathrm{reg}})/W \to G_{\mathrm{reg}}$$

is a diffeomorphism

Proof. Let $w \in W$ and let $\widetilde{w}, \widetilde{w}' \in N(T)$ be two lifts of w. Then $g\widetilde{w}^{-1}T = g(\widetilde{w}')^{-1}T$ This shows that the formulas lead to a well-defined map. It is clearly a left action of W on $(G/T) \times T$ that stabilizes $(G/T) \times T_{\mathrm{reg}}$.

We claim that the W-action on G/T is a free action. For, if $g\widetilde{w}^{-1}T = gT$, then $\widetilde{w}^{-1}T = T$ implying that $\widetilde{w} \in T$. Thus, $\widetilde{w}$ represents the trivial element in W. This means that the action of W on $G/T \times T$ is a free action and hence $(G/T) \times T)_{\mathrm{reg}} \to ((G/T) \times T_{\mathrm{reg}})/W$ is covering projection.

Since for $g \in G$ and $t \in T$, we have $gtg^{-1} = g\widetilde{w}^{-1}(\widetilde{w}t\widetilde{w}^{-1})\widetilde{w}g^{-1}$, it follows that the map $c\colon (G/T) \times T \to G$ is W-invariant so that there is an induced map

$$\overline{c}\colon ((G/T) \times T)/W \to G.$$

The restricting we have a map

$$\overline{c}_{\mathrm{reg}}\colon ((G/T) \times T_{\mathrm{reg}})/W \to G_{\mathrm{reg}}$$

which is a local diffeomorphism, since before dividing out by the free W-action it is a local diffeomorphism.

The last thing to check is that this quotient map $\overline{c}_{\mathrm{rg}}$ is one-to-one. Suppose that t_0, t_0' are elements of T_{reg} and $g, g' \in G$ and we have $gt_0g^{-1} = g't_0'(g')^{-1}$. Then $g^{-1}g'$ conjugates t_0' to t_0. Since these are elements of T_{reg} it follows that $g^{-1}g' \in N(T)$. Let w be the element of the Weyl group represented by $g^{-1}g'$. Then $wt_0'w^{-1} = t_0$ and $wT = g^{-1}gT$ so that $gwT = g'T$. This proves that $w * (g'T, t_0') = (gT, t_0)$. □

Lemma 1.4. *The complement $G \setminus G_{\mathrm{reg}}$ is a finite union of the images in G of smooth maps from manifolds of dimension 3 less than the dimension of G.*

Proof. The regular elements in maximal torus are the complement of $\cup_\alpha \hat{U}_\alpha$. Thus, the complement of G_{reg} in G is the union over the roots α of the conjugates of $\hat{U}_\alpha$.

Let n be the dimension of G and k its rank. Then $\hat{U}_\alpha$ is a $(k-1)$-manifold and its conjugates are the image of $\big(G/N(\hat{U}_\alpha)\big) \times \hat{U}_\alpha \to G$ induced by $(g,u) \mapsto gug^{-1}$. The normalizer $N(\hat{U}_\alpha)$ has dimension $k+2$ so that $\big(G/N(\hat{U}_\alpha)\big) \times \hat{U}_\alpha$ has dimension $n-3$. □

Corollary 1.5. *For any $g \in G_{\mathrm{reg}}$, the inclusion $G_{\mathrm{reg}} \subset G$ induces an isomorphism $\pi_1(G_{\mathrm{reg}}, g) \to \pi_1(G, g)$.*

Proof. Since the complement of G_{reg} in G has codimension 3, any loop in G based at g deforms, relative to the base point, into G_{reg} and if a map of the 2-disk into G has the image of the boundary contained in G_{reg} then the map deforms relative to its boundary into G_{reg}. □

1.2 G/T is simply connected

Proposition 1.6. *Let T be a maximal torus of G. Then $\pi_1(G/T) = \{1\}$.*

Corollary 1.7. *The inclusion $T \to G$ induces a surjection $\pi_1(T) \to \pi_1(G)$.*

Proof. (of the corollary) Since we have a fibration $T \to G \to G/T$, there is a long exact sequence of homotopy groups ending in

$$\pi_1(T) \to \pi_1(G) \to \pi_1(G/T).$$

But we have just seen that $\pi_1(G/T) = \{e\}$. The result follows. □

Proof. (of the proposition) The inclusion of $G_{\mathrm{reg}} \to G$ induces an isomorphism on fundamental groups. The map $(G/T) \times T_{\mathrm{reg}} \to G_{\mathrm{reg}}$ is a covering space and hence injective on fundamental groups. Fix $t_0 \in T_{\mathrm{reg}}$. Then the inclusion $G/T \times \{t_0\} \to G/T \times T_{\mathrm{reg}}$ is injective on the fundamental group based at (eT, t_0). Thus, the composed map $G/T \times \{t_0\} \to G$ is injective on fundamental groups based at t_0.

But fixing a path in T from t_0 to e, we see that the map $G/T \to G$ given by $gT \mapsto gt_0g^{-1}$ is homotopic to the map $G/T \to G$ given by $gT \mapsto geg^{-1} = e$. Thus, any map $S^1 \to (G/T) \times \{t_0\} \to G$ and bounds a disk mapping to G. Together with the fact that the map $\pi_1((G/T \times \{t_0\}) \to \pi_1(G)$ is injective, this proves the result. □

2 The Case of Finite Center

2.1 The Results

Proposition 2.1. *Let G be a compact, connected Lie group with finite center. Then $\pi_1(G)$ is finite, and thus the universal covering group of G is also compact.*

Proof. The Lie algebra of the center of G is the subspace of $\mathfrak{t}$ on which all the roots vanish. Since the center of G is finite, this subspace is trivial. That is to say, the roots span $\mathfrak{t}^*$ over $\mathbb{R}$. Consequently the elements x_α identified with roots α under the isomorphism $\mathfrak{t}^* \cong \mathfrak{t}$ span $\mathfrak{t}$ of $\mathbb{R}$. The same is true of the $\lambda_\alpha = 2x_\alpha/\langle\alpha, \alpha\rangle$ On the other hand, by Proposition 3.2 of the previous lecture, all these elements lie in the lattice Λ_0. Thus, Λ_0 is a finitely generated subgroup of $\mathfrak{t}$ and spans $\mathfrak{t}$ over $\mathbb{R}$; i.e., Λ_0 is a full lattice in $\mathfrak{t}$ in the sense that $\mathfrak{t}/\Lambda_0$ is a torus.

Since Λ is discrete and $\Lambda_0 \subset \Lambda$, it follows that Λ/Λ_0 is a finite group. Again by Proposition 3.2 of the previous lecture, the lattice Λ_0 is contained in the kernel of the map $\Lambda \to \pi_1(G)$. By Corollary 1.7 the map $\Lambda \to \pi_1(G)$ is surjective. Hence, $\pi_1(G)$ is a quotient of Λ/Λ_0 and hence is finite. □

In fact, the next theorem says that Theorem 0.1 holds when the center of G is finite. That is to say, the map $\Lambda/\Lambda_0 \to \pi_1(G)$ is an isomorphism in the special case.

Theorem 2.2. *Let G be a compact, connected Lie group with finite center. Let T be a maximal torus and $\Lambda \subset \mathfrak{t}$ the co-weight lattice. Then the inclusion $T \subset G$ induces an isomorphism $\Lambda/\Lambda_0 \cong \pi_1(G)$.*

Before beginning the proof proper of the theorem we need some preliminary results.

2.2 Preliminary Results

Claim 2.3. *Since the center of G is finite, the affine Weyl chambers have compact closure in $\mathfrak{t}$.*

Proof. Since the affine Weyl group acts transitively on the set of affine Weyl chambers, it suffices to prove this result for one affine Weyl chamber. Fix a fundamental Weyl chamber C_0 with walls given by $\{\alpha_1 = 0\}, \ldots, \{\alpha_k = 0\}$, with the $\alpha_i > 0$ on C_0. This gives a notion of positive roots and the α_i are the simple roots. Hence, the α_i are an $\mathbb{R}$-basis for the subspace spanned over $\mathbb{R}$ by the roots. Since the center is finite, this subspace is all of $\mathfrak{t}$. That is to say the $\{\alpha_i\}_i$ are an $\mathbb{R}$-basis for $\mathfrak{t}$. Consider the unique affine Weyl chamber X contained in C_0 whose closure contains 0. Then

$$\overline{X} \subset \cap_{i=1}^{k} \{0 \leq \alpha_i \leq 1\}.$$

Since, according to results in Lecture 8, the α_i form a basis for $\mathfrak{t}$, this is a compact set. □

Corollary 2.4. *The action of Λ_0 on the set of affine Weyl chambers is free.*

Proof. If some non-zero element $\lambda \in \Lambda_0$ stabilized a chamber, then all powers of λ would stabilize that chamber. This is impossible since the chamber is contained in a compact subset of $\mathfrak{t}$ and λ is a non-trivial translation of $\mathfrak{t}$. □

Claim 2.5. *Let $T' = \mathfrak{t}/\Lambda_0$. The images in T' of the affine walls in $\mathfrak{t}$ divide T' into* Weyl chambers*, each one of which is the isomorphic image of an affine Weyl chamber in $\mathfrak{t}$. The Weyl group action on $\mathfrak{t}$ descends to an action on T'. The action of W on T' is simply transitively on the set of Weyl chambers of T'.*

Proof. Since the action of Λ_0 on the set of Weyl chambers is a free action, the first statement is immediate. Since $\Lambda_0 \subset W_{\mathrm{aff}}$ is a normal subgroup with quotient W, the Weyl group action of $\mathfrak{t}$ descends to an action on T' stabilizing the set of Weyl chambers. Since W_{aff} acts simply transitively on the set of affine Weyl chambers, W acts simply transitively on the Weyl chambers of T'. □

2.3 The Action of W on Λ_R^*/Λ_0

Proposition 2.6. *Translation by Λ_R^*, the dual lattice to the root lattice, leaves invariant the set of affine walls and hence Λ_R^* acts freely on the set of affine Weyl chambers. The Weyl group action on $\mathfrak{t}$ stabilizes Λ_R^*. Thus there is a group $\Lambda_R^* \rtimes W$ stabilizing the set of affine Weyl chambers. The induced Weyl group action on Λ_R^*/Λ_0 is trivial. The group $\Lambda_R^* \rtimes W$ contains W_{aff} as a normal subgroup.*

The analogue of these results hold when we replace Λ_R^* by the fundamental lattice Λ

Proof. Let $x \in \Lambda_R^*$. Then for each root α we have $\alpha(x) \in \mathbb{Z}$. Thus, translation by x takes the wall $\{\alpha = k\}$ to the wall $\{\alpha = k + \alpha(x)\}$, and hence preserves the set of affine walls. Thus, Λ_R^* acts as a group of translations stabilizing the affine Weyl structure. Since the affine Weyl chambers have compact closure, this action is a free action on the set of affine Weyl chambers.

Claim 2.7. *The action of the Weyl group W on the quotient Λ_R^*/Λ_0 is trivial.*

Proof. For any $x \in \Lambda_r^*$ and any root α, we show that the image of x under reflection in the wall $\{\alpha = k\}$ is congruent modulo Λ_0 to x. The reflection is given by

$$x \mapsto x + (k - \alpha(x))\lambda_\alpha.$$

The difference of these two elements is $(k - \alpha(x))\lambda_\alpha$, which, for $x \in \Lambda_R^*$, is an integral multiple of λ_α and hence an element of Λ_0. □

Corollary 2.8. *The action of W on Λ/Λ_0 is trivial.*

Proof. $\Lambda \subset \Lambda_R^*$. □

Lastly, we show that W_{aff} is a normal subgroup of $\Lambda_R^* \rtimes W$ with quotient naturally isomorphic to Λ_R^*/Λ_0. To show that W_{aff} is a normal subgroup we need only show that conjugation by an element of Λ_R^* stabilizes $\mathcal{W}_{\text{aff}}$. Let $\lambda \in \Lambda_R^*$. Then

$$\lambda(\lambda_0 w)\lambda^{-1} = \lambda\lambda_0(\lambda^{-1})^w w = \lambda(\lambda^{-1})^w \lambda_0 w.$$

Since $(\lambda^{-1})^w \cong \lambda^{-1}$ modulo Λ_0, the result is immediate.

We define a map $(\Lambda_r^* \rtimes W)/W_{\text{aff}} \to \Lambda_R^*/\Lambda_0$ by $\lambda w \mapsto [\lambda] \in \Lambda_R^*/\Lambda_0$.

Since $\Lambda_0 \subset \Lambda \subset \Lambda_R^*$ and Λ is a Weyl invariant lattice, it is clear that these results restrict to give analogous results for Λ instead of Λ_R^*. □

2.4 The Chamber Structure for T

Now we pass from $T' = \mathfrak{t}/\Lambda_0$ to $T = \mathfrak{t}/\Lambda$. Here is the basic result.

Proposition 2.9. *Each component of T_{reg} is the diffeomorphic image of a Weyl chamber in $\mathfrak{t}$. The Weyl group action on T preserves T_{reg} and acts transitively on the components of T_{reg}. The stablilzer of each component of T_{reg} is isomorphic to Λ/Λ_0.*

Proof. As we have just seen, the set of affine walls projects to T' to give a wall structure and a chamber structure on which the Weyl group acts simply transitively on the chambers.

Claim 2.10. *Λ/Λ_0 acts on T' preserving its Weyl chamber structure and acting freely on the set of chambers.*

Proof. Since $\Lambda \subset \Lambda_R^*$, for every $\lambda \in \Lambda$ and every root α, the evaluation $\langle \alpha, \lambda \rangle \in \mathbb{Z}$. From this it is clear this that translation by λ stabilizes the set of affine walls and hence stabilizes the set of affine Weyl chambers and acts freely on the affine Weyl chambers. Thus, Λ/Λ_0 acts freely on the set of Weyl chambers of $\mathfrak{t}/\Lambda_0 = T'$. □

The Weyl chambers of T are isomorphic images of affine Weyl chambers of $\mathfrak{t}$ and their union is T_{reg}.

Since W acts trivially on Λ/Λ_0, the semi-direct product $(\Lambda/\Lambda_0) \rtimes W$ is in fact a direct product $(\Lambda/\Lambda_0) \times W$. Let C be a chamber of T' and let $S(C) \subset (\Lambda/\Lambda_0) \times W$ be the stabilizer of C.

Lemma 2.11. *The projection $S(C) \to (\Lambda/\Lambda_0)$ is an isomorphism. The projection of $S(C) \to W$ is an isomorphism onto the stabilizer in W of the chamber $\overline{C}$ that is the image in T of C.*

Proof. Since the stabilizer of C is a subgroup of the product, its projection to either factor is a homomorphism. Since (Λ/Λ_0) acts freely on the chambers of T', the function from $[\lambda] \in (\Lambda/\Lambda_0)$ to chambers in T' that sends $[\lambda]$ to λC is an injection. Thus, for each $(\lambda] \in (\Lambda/\Lambda_0)$, where is a unique $w \in W$ such that $w\lambda C = C$, and hence a unique element $([\lambda], w) \in (\Lambda/\Lambda_0) \times W$ in the stabilizler of C with first component $[\lambda]$. This proves that the projection from the stabilizer of C in $(\Lambda/\Lambda_0) \times W$ to (Λ/Λ_0) is an isomorphism.

Now consider the projection of $S(C) \to W$. Clearly, any element in the image of this homomorphism stabilizes the image chamber $\overline{C}$ of C. Conversely, suppose that $w\overline{C} = \overline{C}$. Then wC and C are in the same (Λ/Λ_0) orbit, and hence there is $\lambda \in \Lambda$ such that $\lambda w C = C$. The element $([\lambda], w) \in (\Lambda/\lambda_0) \times W$ stabilizes C. This proves that the image of $S(C)$ in W is the stabilizer of $\overline{C}$. On the other hand, since (Λ/Λ_0) acts freely on the chambers of T' no element $S(C)$ is contained in (Λ/Λ_0) and hence the homomorphism $S(C) \to W$ is injective and consequently is an isomorphism onto its image. □

This completes the proof of the proposition. □

Corollary 2.12. *Under the action of W, the stabilizer of any chamber of T_{reg} is isomorphic to Λ/Λ_0.*

2.5 Proof of Theorem 2.2

We have already established that the map induced by $T \subset G$ defines a surjection $\Lambda \to \pi_1(G)$, and that $\Lambda_0 \subset \Lambda$ is contained in the kernel of this map. It remain to show that Λ_0 is equal to the kernel.

Consider the isomorphism $G/T \times_W T_{\text{reg}} \cong G_{\text{reg}}$. The components of T_{reg} are isomorphic to convex subsets of $\mathfrak{t}$ and hence are contractible. The Weyl group acts transitively on the components of T_{reg} and the subgroup stabilizing any component is isomorphic to Λ/Λ_0. Thus, we have an isomorphism

$$(G/T) \times_{\Lambda/\Lambda_0} C_0 \to G_{\text{reg}}, \tag{2.1}$$

where C_0 is a component of T_{reg} and Λ/Λ_0 is the subgroup of W stabilizing this chamber.

Since the W action on G/T is free, we can write the space in Expression (2.1) as fiber bundle over $(G/T)/(\Lambda/\Lambda_0)$ with fiber C_0. Since C_0 is contractible, the fundamental group of this space is identified with $\pi_1((G/T)/\Lambda/\Lambda_0))$. Since G/T is simply connected and the action of Λ/Λ_0 on G/T is free, the fundamental group of the quotient is identified with Λ/Λ_0. This is the fundamental group of G_{reg} and hence by Proposition 2.1, the fundamental group of G. This completes the proof of Theorem 2.2.

Corollary 2.13. *Let G be a compact, connected Lie group. Then the fundamental lattice Λ is equal to the lattice Λ_0 generated by the λ_α as α ranges over the roots if and only if G is simply connected In this case, the center of G is Λ_R^*/Λ_0.*

More generally, for any compact Lie group with finite center, the center is identified with Λ_R^*/Λ and its fundamental group is identified with Λ/Λ_0.

3 The Case of General Compact Lie Group

Now we turn to proving Theorem 0.1 for general compact Lie groups.

3.1 The Decomposition

Proposition 3.1. *Let G be a compact, connected Lie group. Let $Z \subset G$ be the connected component of the center of G. There is a compact simply connected group H and a central subgroup $A \subset Z \times H$ and an isomorphism $Z \times_A H \to G$. The projection of $A \to H$ is an injection of A onto a central subgroup of H.*

Proof. Let $\mathfrak{z} \subset \mathfrak{t}$ be the Lie algebra of Z, let $\mathfrak{h}$ be the quotient Lie algebra $\mathfrak{g}/\mathfrak{z}$. Choose a G-invariant symmetric, positive definite inner product on $\mathfrak{g}$. The adjoint representation $G \times \mathfrak{g} \to \mathfrak{g}$ acts trivially on $\mathfrak{z}$ and hence stabilizes it. Thus, the adjoint action of G also stablilzes $\mathfrak{z}^\perp$. Thus, $[\mathfrak{g}, \mathfrak{z}^\perp] \subset \mathfrak{z}^\perp$, and in particular, $\mathfrak{z}^\perp$ is an ideal, and hence a sub Lie algebra of $\mathfrak{g}$.

In fact, this shows that $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{z}^\perp$ is a decomposition of $\mathfrak{g}$ as a direct sum of two Lie algebras. In particular, the Lie algebras $\mathfrak{z}^\perp$ and $\mathfrak{g}/\mathfrak{z}$ are identified. Let $\overline{H} = G/Z$. Since the intersection of the kernels of the roots $\alpha\colon \mathfrak{t} \to \mathbb{R}$ is $\mathfrak{z}$, it follows that the center of $\overline{H}$ is finite. Let H be the universal covering group. By Proposition 2.1 H is a compact, connected Lie group. The Lie algebra of H is $\mathfrak{g}/\mathfrak{z}$.

The inclusion of $\mathfrak{h} \to \mathfrak{g}$ by an isomorphism onto $\mathfrak{z}^\perp$ induces a map $H \to G$. We also have the inclusion $Z \subset G$. Since Z is central in G the product of these two homomorphism defines a map of Lie groups $Z \times H \to G$. By construction it induces an isomorphism on Lie algebras. Since G is connected, this map is onto with discrete kernel. Since $Z \times H$ is compact, the kernel is a finite, central subgroup $A \subset Z \times H$. Since the map restricted to Z is injective, $A \cap Z = \{0\}$, meaning that the kernel of the projection mapping $A \to H$ is injective. □

Theorem 3.2. *Let G be a compact Lie group. Then Λ and Λ_0 are, as before, the fundamental lattice of G and the translation lattice for the affine Weyl group of G. Then the map $T \to G$ induces a surjection $\Lambda(G) \to \pi_1(G)$ whose kernel is $\Lambda_0(G)$. Thus, the inclusion $T \to G$ induces an isomorphism $\pi_1(G) \cong \Lambda(G)/\Lambda_0(G)$.*

Proof. In the previous lecture we showed that the map $\Lambda \to \pi_1(G)$ is surjective and that $\Lambda_0(G)$ is contained in the kernel. It remains only to show that Λ_0 is equal to the kernel of this map.

First Case: $G = Z \times H$ with H having a finite center.

Let us consider the case when $G = Z \times H$ (i.e., when $A = \{e\}$). A maximal torus of G is of the form $Z \times T_H$ and its Lie algebra is $\mathfrak{z} \oplus \mathfrak{t}_H$. We denote by $\Lambda_0(G)$ and $\Lambda_0(H)$ the lattices in $\mathfrak{t}$ and $\mathfrak{t}_H$ that are the translation subgroups of the affine Weyl groups, $W_{\text{aff}}(G)$ and $W_{\text{aff}}(H)$, of G and H, respectively. Analogously let $\Lambda(G)$ $\Lambda(H)$, an $\Lambda(Z)$ be the fundamental lattices of G, H, and Z, respectively. In this case the roots of G are trivial on $\mathfrak{z}$ and are identified with the roots of H. Thus, $\Lambda_0(G) \subset \mathfrak{z}^\perp$ and is identified with $\Lambda_0(H)$. Also, we have $\Lambda(G) = \Lambda(Z) \times \Lambda(H)$. By Theorem 2.2 for the compact group H with finite center, $\Lambda(H)/\Lambda_0(H) = \pi_1(H)$ Thus,

$$\begin{aligned}\Lambda(G)/\Lambda_0(G) &= \Lambda(Z) \times \Lambda(H)/\{0\} \times \Lambda_0(H) \\ &= \Lambda(Z) \times \pi_1(H) = \pi_1(Z) \times \pi_1(H) = \pi_1(G).\end{aligned}$$

This establishes the Theorem 0.1 in the product case.

The General Case.

Now consider $G = Z \times_A H$, with H simply connected, where is a finite subgroup whose projection to H is an injection onto a central subgroup. Since $Z \times H \to G$ is a finite covering, it is an isomorphism on the Lie algebras. Since H is simply connected, $\Lambda(H) = \Lambda_0(H)$. Thus, we have

$$\Lambda_0(G) = \Lambda_0(H) = \Lambda(H).$$

We have a commutative diagram:

$$\begin{array}{ccccccccc}
 & & \{0\} & \longrightarrow & \{0\} & \longrightarrow & \{0\} & & \\
 & & \uparrow & & \uparrow & & \cdot\uparrow & & \\
\{0\} & \longrightarrow & \pi_1(Z \times H) & \longrightarrow & \pi_1(G) & \longrightarrow & A & \longrightarrow & \{0\} \\
 & & \uparrow & & \uparrow & & \uparrow{=} & & \\
\{0\} & \longrightarrow & \Lambda(Z) \times \Lambda(H) & \longrightarrow & \Lambda(G) & \longrightarrow & A & \longrightarrow & \{0\} \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\{0\} & \longrightarrow & \{0\} \times \Lambda(H) & \longrightarrow & \{0\} \times \Lambda(H) & \longrightarrow & \{0\} & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \{0\} & \longrightarrow & \{0\} & \longrightarrow & \{0\} & &
\end{array}$$

The rows are exact. The first column is exact by what we established in the product case. The last column is obviously exact. A simple diagram chase shows that in fact the second column is exact, too. Hence,

$$\pi_1(G) \cong \Lambda(G)/\Lambda(H) = \Lambda(G)/\Lambda_0(G).$$

This completes the proof of Theorem 0.1. □

3.2 Description of Λ_R^* and Λ_0 when the center is not finite

Let us begin with the product case: $G = Z \times H$ where Z is a central torus in G and H has finite center. Then:

- $\Lambda_0(G) = \{0\} \times \Lambda_0(H)$ is a co-compact attice in $\mathfrak{h} \subset \mathfrak{g}$.
- $\Lambda(G) = \Lambda(Z) \times \Lambda(H)$
- $\Lambda_R(G) = \Lambda_R(H)$ acts trivially in $\mathfrak{z}$. and is a full lattice in the subspace of $\mathfrak{g}^*$ of elements that annihilate $\mathfrak{z}$.

It follows that $\Lambda_R(G)^* = \Lambda_R(H)^* \times \mathfrak{z}$, is the product of a lattice in $\mathfrak{h}$ with $\mathfrak{z}$, and hence has positive dimension.

The quotient $\Lambda(G)/\Lambda_0(G)$ is identified with $\Lambda(Z) \times \Lambda(H)/\Lambda_0(H)$, ane hence $\Lambda(G)/\Lambda_0(G) \cong \pi_1(G)$. In this case, the Weyl chambers are no longer compact and the action of Λ on the set ha fixed points. The subgroup $\Lambda_Z \subset \Lambda_G$ stablilizes every Weyl chamber.

I will leave the general case $Z \times_A H$ as a homework exercise.