Lie Groups: Fall, 2024
Lecture XII:
Linear Representations of $SL(2, \mathbb{C})$

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The purpose of this lecture is to classify all finite dimensional complex linear representations of $SL(2, \mathbb{C})$ and of its Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. Recall that if $\mathfrak{g}$ is a complex Lie algebra, then a complex linear representation of $\mathfrak{g}$ on a finite dimensional complex vector space V is a complex-linear homomorphism of Lie algebras $\rho\colon \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(V)$. This means for each $X \in \mathfrak{g}$ we have a complex linear map $\rho(X)\colon V \to V$ satisfying; (i) ρ is a complex linear map, and (ii) $\rho([X, Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X)$.

1 Complete Reducibility

Here is the result about complete irreduciblity.

Theorem 1.1. *Any finite dimensional representation of* $\mathfrak{sl}(2, \mathbb{C})$ *is* completely reducible. *That is to say given* $\rho\colon \mathfrak{sl}(2, \mathbb{C}) \to \mathrm{End}_{\mathbb{C}}(V)$ *with* V *a finite dimensional complex vector space, there is a direct sum decomposition* $V \cong \oplus_i V_i$ *into complex subspaces such that:*

- *the action of* $\mathfrak{sl}(2, \mathbb{C})$ *on* V *stabilizes each of the* V_i*, and*
- *for each* i *there is no non-trivial subspace of* V_i *stabilized by* $\mathfrak{sl}(2, \mathbb{C})$.

Corollary 1.2. *Every finite dimensional complex linear representation of* $SL(2, \mathbb{C})$ *is* completely reducible*; i.e., for any such action* $\mu\colon SL(2, \mathbb{C}) \times V \to V$ *can be written as a direct sum of actions, each of which admits no non-trivial invariant subspace.*

Proof. (of the corollary) Let $\mu\colon SL(2, \mathbb{C}) \times V \to V$ be a finite dimensional complex representation. Equivalently, we can view μ as a homomorphism of Lie groups $\mu\colon SL(2,\mathbb{C}) \to GL(V)$. Let $\rho\colon \mathfrak{sl}(2,\mathbb{C}) \to \mathrm{End}_{\mathbb{C}}(V)$ be the induced map on Lie algebras. Using Theorem 1.1 we decompose V as $\oplus_i V_i$ with each V_i stabilized by $\rho(\mathfrak{sl}(2,\mathbb{C}))$ and irreducible as an $\mathfrak{sl}(2,\mathbb{C})$-module. That is to say ρ factors through $\oplus\mathrm{End}_C(V_i) \subset \mathrm{End}_{\mathbb{C}}(V)$. Invoking the theorem that any map of Lie algebras induces a map of the underlying Lie groups (provided that the domain Lie group is simply connected), we see that μ factors through a homomorphism $\overline{\mu}\colon SL(2,\mathbb{C}) \to \oplus_i GL(V_i) \subset GL(V)$. This proves that the $SL(2,\mathbb{C})$-module V decomposes as a direct sum of the $SL(2,\mathbb{C})$-modules V_i.

We also need to see that each V_i is irreducible as an $SL(2,\mathbb{C})$-module. If there is a non-trivial invariant subspace $W_i \subset V_i$ invariant under $SL(2,\mathbb{C})$, then W_i is invariant under $\rho(\mathfrak{sl}(2,\mathbb{C}))$, contrary to the fact that V_i is irreducible as a $\mathfrak{sl}(2,\mathbb{C})$ representation. □

The rest of this lecture is devoted to enumerating the irreducible $\mathfrak{sl}(2,\mathbb{C})$-representations and proving Theorem 1.1.

2 Classification of the Irreducible Representations

We define elements in $\mathfrak{sl}(2,\mathbb{C})$ as follows:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and

$$Y = \begin{pmatrix} 0 & 0 \\ 1 & 0. \end{pmatrix}$$

Then these three elements form a vector space basis for $\mathfrak{sl}(2,\mathbb{C})$

Then the basic bracket relations are $[H,X] = 2X$ and $[H,Y] = -2Y$ and $[X,Y] = H$.

Claim 2.1. *For any finite dimensional complex representation ρ of $\mathfrak{sl}(2,\mathbb{C})$ the element $\rho(H)$ is diagonalizable with integral eigenvalues.*

Proof. The representation ρ determines a complex representation $\mu\colon SL(2,\mathbb{C}) \to GL(V)$. Of course, $\mu(\exp(2\pi i H))$ is the identity. This means that $\rho(H)$ is diagonalizable with integer eigenvalues. □

Definition 2.2. Let $\mathfrak{sl}(2,\mathbb{C}) \times V \to V$ be a finite dimensional, complex of $\mathfrak{sl}(2,\mathbb{C})$. Then for every $r \in \mathbb{Z}$ by $E_r(V)$ we mean the character space of V on which H acts by multiplication by r.

Claim 2.3.

$$\rho(X)\colon E_k(V) \to E_{k+2}(V)$$

$$\rho(Y)\colon E_k(V) \to E_{k-2}(V).$$

Proof. Suppose $H(a) = ka$. Then $HX(a) - XH(a) = 2X(a)$, so that $HX(a) = (k+2)X(a)$. The other equation is proved similarly. □

Here is the main theorem of this section

Theorem 2.4. *For each integer $k \geq 0$, up to isomorphism, there is a unique irreducible $\mathfrak{sl}(2,\mathbb{C})$-representation of dimension $k+1$. Irreducible representations are characterized by the condition that $\ker(X)$ is one dimensional. The dimension of an irreducible representation is one larger than the character of H on $\ker(X)$.*

We begin by finding irreducible sub-representations of any finite dimensional representation.

Proposition 2.5. *Suppose that V is a finite dimensional, complex $\mathfrak{sl}(2,\mathbb{C})$ representation and suppose that $a \in E_k(V)$ is a non-zero element in the kernel of X. Then $a, Ya, Y^2a, \ldots Y^k(a)$ are linear independent in V and span an $\mathfrak{sl}(s,\mathbb{C})$-submodule W_k of V. Furthermore, W_k is an irreducible $\mathfrak{sl}(2,\mathbb{C})$-module of dimension $k+1$, and the kernel of $X\colon W_k \to W_k$ is $\mathbb{C}(a)$.*

Proof.

Claim 2.6. *The subspace generated by $\{Y^r(a)\}_{0\leq r<\infty}$ is stabilized by $\mathfrak{sl}(2,\mathbb{C})$ and*

$$X(Y^{r+1}a)) = \sum_{s=0}^{r}(k-2s)Y^r(a).$$

Proof. The proof of the equation is by induction on r. For $r = 0$ we have $ka = H(a) = XY(a) - YX(a)$ and by consruction $X(a) = 0$. Thus, $ka = XY(a)$, establishing the claim for $r = 0$.

Suppose by induction that the result holds for $r-1$ and let us establish it for r.

$$(k-2r)Y^r a = H(Y^r a) = XY^{r+1}(a) - YX(Y^r(a)) = XY^{r+1}(a) - Y(\sum_{s=0}^{r-1}(k-2s)Y^{r-1}(a)).$$

Thus,

$$\sum_{s=0}^{r}(k-2s)Y^r(a) = XY^{r+1}(a). \tag{2.1}$$

This shows that the subspace spanned by the $\{Y^r(a)\}_{r=0}^{\infty}$ is stabilized by X. It is obviously stabilized by Y, and since it is generated by elements in the various $E_{k-2r}(V)$, it is also stable under H. Hence, it is a $\mathfrak{sl}(2,\mathbb{C})$ submodule of V. We denote it W_k $\square$

Notice that the coefficient of $Y^r(a)$ in Expression 2.1 is non-zero unless $r=k$. It follows that for any $r\neq k$ if $Y^r(a)\neq 0$ then $Y^{r+1}(a)\neq 0$. Since $Y^0(a)=a$ is non-zero, it follows that $Y^r(a)\neq 0$ for $0\leq r\leq k$. By the same argument if $Y^{k+1}(a)\neq 0$, then $Y^r(a)\neq 0$ for all $r\geq 0$. This is impossible, since these elements lie in $E_{k-2r}(V)$ and V is finite dimensional. This proves that $Y^r(a)\neq 0$ exactly for $0\leq r\leq k$. Since $Y^r(a)\in E_{k-2r}(V)$ these elements are independent.

Thus, W_k is an $\mathfrak{sl}(2,\mathbb{C})$-representation of dimension $k+1$. If $b\in W_k$ is non-zero, then there is a smallest integer $\ell\geq 0$ such that $X^{\ell+1}b=0$. Then $X^{\ell}(v)$ is a non-zero element in the kernel of X. This means that $X^{\ell}(b)$ is a non-zero multiple of a. Hence, $\{Y^r(b)\}_{r=0}^{k}$ span all of W_k. This shows that W_k is irredudible. It follows from the Equation 2.1 that the kernel of $X\colon W_k\to W_k$ is $\mathbb{C}(a)$

This completes the proof of the proposition. $\square$

Corollary 2.7. *For each integer $k\geq 1$ there is, up to isomorphism, one irreducible $\mathfrak{sl}(2,\mathbb{C})$-module of dimension $k+1$.*

Proof. Let V be an irreducible $\mathfrak{sl}(2,\mathbb{C})$-module. Let k be the largest integer such that $E_k(V)\neq 0$. Then any $a\neq 0$ in $E_k(V)$ is in the kernel of X. By the previous proposition there is a submodule $W_k\subset V$ of dimension $k+1$ spanned over $\mathbb{C}$ by $\{a,Y(a),\ldots,Y^k(a)\}$. Since V is irreducible it must be equal to W_k. This proves that every irreducible $\mathfrak{sl}(2,\mathbb{C})$ is isomorphic to some W_k as constructed above.

Conversely, given $k\geq 0$ we let U be a graded vector space with $E_r(U)$ non-zero for all r with $-k\leq r\leq k$ and $r\equiv k \pmod 2$. These U_{k-2r} are one-dimensional and generated by a_{k-2r}. We define the action of $\mathfrak{sl}(2,\mathbb{C})$ by setting $H(a_{k-2r})=(k-2r)a_{k-2r}$, $Y(a_{k-2r})=a_{k-2(r+1)})$ and

$$X(a_{k-2(r-1)}) = \sum_{s=0}^{r}(k-2s)a_{k-2r}.$$

Then the computations leading to Equation 2.1 can be read in the other order to show that this is an $\mathfrak{s}l(2,\mathbb{C})$-representation. Thus, each W_k indeed exists as an irreducible $\mathfrak{s}l(2,\mathbb{C})$-module. $\square$

We have shown that an $\mathfrak{s}l(2,\mathbb{C})$-module V is irreducible if and only if the kernel of $X\colon V \to V$ is one-dimensional. Furthermore, the isomorphism class of irreducible $\mathfrak{s}l(2,\mathbb{C})$-modules is determined by the integer that is the eigenvalue of H on $\ker(X)$. The dimension of the irreducible representation is one larger than this eigenvalue. In particular, there is, up to isomorphism, a unique irreducible representation of dimension $k+1$ for each integer $k \geq 0$.

3 Complete Reducibility

Theorem 3.1. *Any finite dimensional complex representation V of $\mathfrak{s}l(2,\mathbb{C})$ is a direct sum of irreducible representations.*

Proof. The proof is by induction on the dimension of the representation. If the dimension is 1, then the action of $\mathfrak{s}l(2,\mathbb{C})$ is trivial and the representation is irreducible. Suppose V is of dimension n and we have established the result for all representations of dimension less than n.

Since V is finite dimensional there is a largest k such that $E_k(V) \neq 0$. Take $a \in E_k(V)$ a non-zero element. Then $X(a) \in E_{k+2}(V) = 0$. According to Proposition 2.5 there is an embedding $i\colon W_k \subset V$, with W_k an irreducible $\mathfrak{s}l(2,\mathbb{C})$-module of V.

Let $\overline{V} = V/i(W_k)$. Since the dimension of $\overline{V}$ is less than that of V it is a direct sum of irreducible $\mathfrak{s}l(2,\mathbb{C})$modules $\overline{V} \cong \oplus_i W_{k_i}$.

Claim 3.2. *The projection $V \to \overline{V}$ maps $\ker(X\colon V \to V)$ onto $\ker(X\colon \overline{V} \to \overline{V})$*

Proof. Clearly, if $a \in V$ is in $\ker(X\colon V \to V)$, then its image $\overline{a} \in \overline{V}$ is in the $\ker(X\colon \overline{V} \to \overline{V})$. In an irreducible $\mathfrak{s}l(2,\mathbb{C})$-module any homogeneous element b in $\ker(X)$ has $H(b) = rb$ for some $r \geq 0$.

Suppose that $\overline{a} \in \ker(X\colon \overline{V} \to \overline{V})$ is a homogeneous element. Lift $\overline{a}$ to a homogeneous element $a \in V$. Then $X(a) \in i(W_k)$. If this element is zero, then $a \in \ker(X)$ as required. Suppose it is non-zero. It is homogeneous of degree ≥ 1. Thus, $XYX(a)$ is a non-zero multiple, say t, of $X(a)$ and $a - (1/t)YX(a)$ is in the kernel of X and projects to $\overline{a}$ in $\overline{V}$. $\square$

By taking a direct sum of maps produced by Proposition 2.5, we define an $\mathfrak{s}l(2,\mathbb{C})$-map from $\overline{V}$ (which is a direct sum of irreducible representations) to V, say $\psi\colon \overline{V} \to V$, with the property that the map ψ induces from $\ker(X\colon \overline{V} \to \overline{V})$ to $\ker(X\colon V \to V)$ splits the map induced by the projection $\ker(X\colon V \to V) \to \ker(X\colon \overline{V} \to \overline{V})$.

The $\mathfrak{s}l(2,\mathbb{C})$-map

$$W_k \oplus \overline{V} \xrightarrow{i \oplus \psi} V$$

induces an isomorphism on the kernels of X. Thus, the kernel of this map is an $\mathfrak{s}l(2,\mathbb{C})$-module with the property that X acts without kernel. Such a module is automatically the trivial module. This implies that $i \oplus \psi$ is an injection. But the dimensions of the domain and range of this map are the same, so it is an isomorphism of $\mathfrak{s}l(2,\mathbb{C})$-representations. $\square$

Notice that what we have proved is the following. Associating to an $\mathfrak{s}l(2,\mathbb{C})$-representation the kernel of X is a functor from the category of finite dimensional complex $\mathfrak{s}l(2,\mathbb{C})$-representations to the category of finite dimensional graded complex vector spaces with non-trivial graded summands only in non-negative degrees, and this functor is an equivalence of categories.