Lie Groups: Fall, 2024
Lecture II
Lie Algebras, the Adjoint Action, and the Exponential Mapping

September 9, 2024

1 Lie Algebras

1.1 The Basics

Definition 1.1. Fix a field K of characteristic 0. A *Lie algebra* over K is a K-vector space V together with a bilinear map $V \otimes_K V \to V$ denoted by $X \otimes Y \mapsto [X, Y]$, called the *bracket* or the *Lie bracket* required to satisfy the following two axioms:

1. $[X, Y] = -[Y, X]$.
2. $[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0$.

The second equation is called the *Jacobi Identity.* It can also be interpreted as saying that $[A, \cdot]$ is a derivation with respect to $[\cdot, \cdot]$, i.e.,

$$[A, [B, C]] = [[A, B], C] + [B, [A, C]].$$

Clearly, these algebraic equations make sense for vector spaces over any field K, though one often needs K to be of characteristic zero in many of the arguments. (Indeed, one can work with modules over a ring, defining what are called Lie rings, but this is beyond the scope of these lectures.) We are primarily interested in the case of real and complex Lie algebras that are finite dimensional.

We will explain in more detail how Lie groups and Lie Algebras are related and where the Jacobi identity comes from, but for now we content ourselves with giving some examples of Lie algebras.

Lemma 1.2. *Fix a field K. Suppose that $(L, [\cdot,\cdot])$ is a Lie algebra over K and $L' \subset L$ is a linear K-subspace that is closed under the bracket. Then L' with the induced bracket is a Lie algebra over K.*

Proof. Exercise. $\square$

Definition 1.3. With $L' \subset L$ as in the previous lemma, L' is a *sub Lie algebra* of L.

Example 4. The space $M(n \times n, K)$ of $n \times n$ matrices with entries in K is a Lie algebra where the Lie bracket is given by $[A, B] = AB - BA$. Obviously, this bilinear map is skew-symmetric. To establish the Jacobi identity, we compute:

$$[A,[B,C]] = A(BC-CB) - (BC-CB)A$$

$$[C,[A,B]] = C(AB-BA) - (AB-BA)C$$

$$[B,[C,A]] = B(CA-AC) - (CA-AC)B.$$

Using the associativity of matrix multiplication we cancel these terms in pairs.

Example 5. Let $\mathcal{A}$ be an associative algebra over K. Then the computation in Example 4, is valid in $\mathcal{A}$ and shows that defining $[A, B] = AB - BA$ for all $A, B \in \mathcal{A}$ defines a Lie algebra structure on $\mathcal{A}$. This is the Lie algebra *determined* by the associative algebra. In fact, we shall show in the next lecture the Poincaré-Birkhoff-Witt Theorem which says that associated to a Lie algebra L there is an associative algebra $U(L)$ called the *universal enveloping algebra of* L. There is a injective linear map from $L \to U(L)$ compatible with the Lie bracket of L and the $AB - BA$ bracket on $U(L)$. That is to say, the general Lie algebra L is a sub Lie algebra of the Lie algebra determined by an associate algebra. (The proof works over any field of characteristic 0.)

Example 6. Let M be a smooth manifold and denote by $Vect(M)$ the vector space of smooth vector fields on M. The action of $Vect(M)$ on $C^\infty(M)$ identifies this space with the space of $\mathbb{R}$-linear maps $D\colon C^\infty(M) \to C^\infty(M)$ that are derivations in the sense that $D(fg) = D(f)g + fD(g)$. This space of first-order operators generates an associative algebra $\mathcal{D}(M)$ of differential operators on $C^\infty(M)$, with product being composition. The Lie bracket of vector fields is then induced from the $AB - BA$ bracket on $\mathcal{D}(M)$ making it a Lie algebra over $\mathbb{R}$. For vector fields X and Y, the composition XY is a second order operator (and hence is not a vector field);.

Nevertheless, $XY - YX$ is a derivation (because the second-order terms cancel because of the equality of cross partial derivatives). Hence, $XY - YX$ is a vector field. This shows that the subspace of vector fields on M is a sub Lie algebra of the Lie algebra on $\mathcal{D}(M)$ defined from the associative multiplication on $\mathcal{D}(M)$. Indeed, $\mathcal{D}(M)$ is the universal enveloping algebra of the Lie algebra of vector fields.

2 The Adjoint Action and the Lie Algebra of a Lie Group

Let G be a real Lie Group. There is a natural action of G (the first copy) on itself (the second copy) by conjugation:

$$\mathrm{Ad}_G \colon G \times G \to G$$

defined by $\mathrm{Ad}_G(g, g') = gg'g^{-1}$. This is a left action of G on itself, called the *adjoint* action. When G is clear from the context we denote this adjoint map simply as Ad. The action is smooth, and in the case of a complex Lie group, the action is holomorphic.

The adjoint action fixes $e \in G$ and hence differentiating at the identity of the second variable gives an induced linear action $\mathrm{Ad}_G \colon G \times T_eG \to T_eG$. We use the standard notation and denote T_eG by $\mathfrak{g}$. The adjoint action of G on $\mathfrak{g}$ is a representation of G as linear automorphisms of $\mathfrak{g}$. That is to say we have a linear representation which is a morphism of Lie groups

$$G \xrightarrow{\mathrm{Ad}_G} GL(\mathfrak{g}).$$

In the case of a complex Lie group this is a complex linear representation of G on the complex vector space $\mathfrak{g}$, i.e., it determines a holomorphic map $G \to GL_C(\mathfrak{g})$.

In either case, we can differentiate this Lie group morphism at the identity of G and obtain a (real or complex) linear map from $\mathfrak{g}$ to the endomorphism ring of $\mathfrak{g}$

$$\mathrm{ad}_G \colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g}).$$

Proposition 2.1. *The adjoint action*

$$\mathrm{ad}_{GL(n,\mathbb{R})} \colon \mathfrak{gl}(n,\mathbb{R}) \otimes \mathfrak{gl}(n,\mathbb{R}) \to \mathfrak{gl}(n,\mathbb{R})$$

defines a Lie algebra structure on $\mathfrak{gl}(n,\mathbb{R})$*. Furthermore, identifying* $\mathfrak{gl}(n,\mathbb{R})$ *with* $M(n \times n, \mathbb{R})$ *we have*

$$\mathrm{ad}_{GL(n,\mathbb{R})}(X)(Y) = XY - YX.$$

Proof. Differentiating the conjugation action of G on itself at the identity (in the second variable) produces the usual conjugation action of G on $\mathfrak{gl}(n,\mathbb{R}) = M(n \times n,\mathbb{R})$. We compute the differential of this action at $e \in G$. Let $\gamma(t)$ be a one-parameter family in $GL(n,\mathbb{R})$ with $\gamma(0) = \mathrm{Id}$ and denote by $A \in M(n \times n,\mathbb{R})$ the derivative of this family at $t = 0$. Then $(\gamma^{-1})'(0) = -A$. Fix $B \in M(n \times n,\mathbb{R})$. Then we have

$$\frac{d(\gamma(t)B\gamma(t)^{-1})}{dt}\Big|_{t=0} = AB - BA.$$

□

Corollary 2.2. *The map*

$$\mathrm{ad}_{GL(n,\mathbb{R})} \colon \mathfrak{gl}(n,\mathbb{R}) \otimes \mathfrak{gl}(n,\mathbb{R}) \to \mathfrak{gl}(n,\mathbb{R})$$

determines a Lie algebra over $\mathbb{R}$. Analogously, the map

$$\mathrm{ad}_{GL(n,\mathbb{C})} \colon \mathfrak{gl}(n,\mathbb{C}) \otimes \mathfrak{gl}(n,\mathbb{C}) \to \mathfrak{gl}(n,\mathbb{C})$$

determines a Lie algebra over $\mathbb{C}$. In both cases, the identification of the Lie algebra with $n \times n$ matrices (over $\mathbb{R}$ or $\mathbb{C}$) identifies the Lie bracket of $\mathfrak{gl}(n,\mathbb{R})$ or $\mathfrak{gl}(n,\mathbb{C})$ coming from the adjoint representation with the usual bracket of matrices, i.e., the $AB - BA$ bracket.

Definition 2.3. The Lie algebra structure on $\mathfrak{gl}(n,\mathbb{R})$ (or $\mathfrak{gl}(n,\mathbb{C})$) given by $\mathrm{ad}_{GL(n,\mathbb{R})}$ (or $\mathrm{ad}_{GL(n,\mathbb{C})}$) is *THE Lie algebra* of $GL(n,\mathbb{R})$ (or $GL(n,\mathbb{C})$).

Corollary 2.4. *Suppose that $H \subset GL(n,\mathbb{R})$ is a sub-Lie group. Let $\mathfrak{h} \subset M(n \times n,\mathbb{R})$ be the tangent space to H at the identity. Then $\mathfrak{h}$ is closed under Lie bracket of matrices; and*

$$\mathrm{ad}_H \colon \mathfrak{h} \otimes \mathfrak{h} \to \mathfrak{h}$$

is given by

$$\mathrm{ad}_H(X)(Y) = XY - YX.$$

In particular, ad_H *induces a Lie algebra structure on* $\mathfrak{h}$.

Proof. The restriction of $\mathrm{Ad}_{GL(n,\mathbb{R})} \colon GL(n,\mathbb{R}) \times \mathfrak{gl}(n,\mathbb{R}) \to \mathfrak{gl}(n,\mathbb{R})$ to $H \subset GL(n,\mathbb{R})$ is $\mathrm{Ad}_H \colon \mathfrak{gl}(n,\mathbb{R}) \to \mathfrak{gl}(n,\mathbb{R})$. This restriction leaves $\mathfrak{h} \subset \mathfrak{gl}(n,\mathbb{R})$ invariant and this restriction is $\mathrm{Ad}_H \colon H \times \mathfrak{h} \to \mathfrak{h}$. Hence, the restriction of $\mathrm{ad}_{\mathfrak{gl}(n,\mathbb{R})} \colon \mathfrak{gl}(n,\mathbb{R}) \times \mathfrak{gl}(n,\mathbb{R}) \to \mathfrak{gl}(n,\mathbb{R})$ to $\mathfrak{h} \times \mathfrak{h}$ is $\mathrm{ad}_{\mathfrak{h}}$. □

Definition 2.5. The subspace $\mathfrak{h} \subset \mathfrak{g}l(n, \mathbb{R})$ together with the induced Lie bracket is *THE Lie algebra of H*.

For a general Lie group G the adjoint action of $\mathfrak{g}$ on itself is defined as above in terms of the adjoint action of G on $\mathfrak{g}$. What remains to show is that this defines a Lie group structure on $\mathfrak{g}$ in the case when G is not a subgroup of $GL(n, \mathbb{R})$ for some n.

2.1 The Lie Algebra of a General Lie Group

2.1.1 Vector Fields

Recall that the (infinite dimensional) space of smooth vector fields on a manifold has a Lie bracket. If X and Y are vector fields, then their bracket $[X, Y]$ is defined by giving its value on a general function f by $[X, Y](f) = X(Y(f)) - Y(X(f))$. As we checked in the last lecture by direct computation, the second order derivative terms in $X(Y(f))$ cancel those of $Y(X(f))$ (basically this is equality of cross partials) so that the bracket is again a vector field. Invoking the fact that the bracket is written $XY - YX$ in the associative algebra of all differential operators, we conclude that this bracket defines the structure of a Lie algebra on the (infinite dimensional) vector space of vector fields..

Definition 2.6. A vector field χ on G is *left-invariant* if for each $g, h \in G$, $D(g\cdot)(\chi(h)) = \chi(gh)$.

Lemma 2.7. *1. Given $X \in \mathfrak{g}$ there is a unique left-invariant vector field χ_X whose value at the identity is X.*
2. If X and Y are left-invariant vector fields, then so is $[X, Y]$.

Proof. If χ is a left-invariant vector field then $\chi(g) = D(g\cdot)\chi(e)$. This proves the uniqueness of a left-invariant vector field with a given value at the identity. Since the action $G \times TG \to TG$ given by defining the action of g to be $D(g\cdot)$ is a smooth map, for any $X \in \mathfrak{g}$, the formula $\chi(g) = D(g\cdot)X$ defines a smooth vector field, proving the existence.

Suppose that X and Y are left-invariant vector fields. Since $g\cdot$ is a diffeomorphism it commutes with the Lie bracket of vector fields. Thus, $D(g\cdot)[X, Y] = [D(g\cdot)X, D(g\cdot)Y]$. □

The left-invariant vector fields on a Lie group G form a finite dimensional Lie algebra. Associating to each such vector field its value at the identity element of the group gives a linear isomorphism between the left-invariant vector fields and $\mathfrak{g}$. Transferring the Lie algebra structure from the space of left invariant vector fields to $\mathfrak{g}$ defines a Lie algebra structure on $\mathfrak{g}$. This is *THE Lie algebra* of G. The symbol $\mathfrak{g}$ denotes this Lie algebra structure on T_eG. If G is a complex Lie group this process defines a complex Lie algebra structure on $\mathfrak{g}$.

It remains to show that Lie algebra structure on $\mathfrak{g}$ just defined, we have $[X, Y] = ad(X)(Y)$.

Proposition 2.8. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ as defined above. For $X, Y \in \mathfrak{g}$ we have $ad(X)(Y) = [X, Y]$, the bracket coming from the Lie bracket of the left-invariant extensions of X and Y. In particular, $X \otimes Y \mapsto \mathrm{ad}(X)(Y)$ defines the Lie algebra structure on $\mathfrak{g}$.*

Proof. Let X and Y be elements of $\mathfrak{g}$. If X and Y are linearly dependent, say $Y = aX$ then $[X, Y] = 0$ and $\mathrm{ad}_G(X)(Y) = 0$ since the one-parameter subgroups generated by X and Y commute with each other. Thus, it suffices to assume that X and Y are linearly independent in $\mathfrak{g}$.

Extend them to left-invariant vector fields on G, denoted $\widetilde{X}$ and $\widetilde{Y}$, respectively. Let $\xi(s)$ be the integral curve for $\widetilde{X}$ though e and let $\varphi(t)$ be the integral curve for $\widetilde{Y}$ through e. Then $\varphi'(t) = \varphi(t)Y$ and $\xi'(s) = \xi(s)X$. Let $U \subset \mathbb{R}^2$ be an open neighborhood of the origin and define $T \colon U \to \Sigma$ by $T(s, t) = \varphi(t)\xi(s)$. By the implicit function theorem, if U is sufficiently small, T is an embedding onto a smooth, locally closed surface $\Sigma \subset G$.

We view (s, t) as coordinates on this surface Σ in G. Then

$$(\partial/\partial s)_{(s,t)} = \varphi(t)\xi(s)X = T(s, t)X,$$

so that $\partial/\partial s$ is the restriction of $\widetilde{X}$ to the surface Σ. Since $Y_{(0,0)} = Y$, it follows that

$$\widetilde{Y}_{(0,0)}(\widetilde{X}_{(0,t)}) = (\partial^2/\partial t\partial s)_{(0,0)}.$$

On the other hand,

$$(\partial/\partial t)_{(s,t)} = \varphi(t)Y\xi(s),$$

and in particular,

$$(\partial/\partial t)_{(s,0)} = Y\xi(s).$$

Thus,

$$\widetilde{Y}(s, 0) = \xi(s)Y = \xi(s)\left((\partial/\partial t)_{(s,0)}\xi^{-1}(s)\right) = \mathrm{Ad}_G(\xi(s))\left((\partial/\partial t)_{(s,0)}\right).$$

This gives

$$\begin{aligned}
\widetilde{X}_{(0,0)}\widetilde{Y}_{(s,0)} &= (\partial/\partial s)_{s=0}(\widetilde{Y}_{(s,0)}) \\
&= \mathrm{ad}_{\mathfrak{g}}(X)\left((\partial/\partial t)_{(0,0)}\right) + (\partial/\partial s)_{s=0}\left((\partial/\partial_t)_{(s,0)}\right) \\
&= \mathrm{ad}_{\mathfrak{g}}(X)(Y) + (\partial^2/\partial s\partial t)_{(0,0)}.
\end{aligned}$$

The equality of cross partials yields

$$(\widetilde{X}\widetilde{Y} - \widetilde{Y}\widetilde{X})_{(0,0)} = \mathrm{ad}_{\mathfrak{g}}(X)(Y).$$

$\square$

For a complex Lie group G, its Lie algebra $\mathfrak{g}$ is a complex vector space and $\mathrm{Ad}_{\mathrm{G}} \colon G \times \mathfrak{g} \to \mathfrak{g}$ is a holomorphic map. The same arguments show that $\mathrm{ad}_G \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a complex bilinear pairing producing a complex Lie algebra structure on $\mathfrak{g}$ and in the case when $G \subset gl(n, \mathbb{C})$ this complex Lie algebra structure agrees with the one coming from Lie bracket of complex matrices.

2.2 Naturality of Lie Algebra of a Lie Group

Proposition 2.9. *Let $\varphi \colon H \to G$ be a Lie group homomorphism. Then its differential at the identity $d\varphi_e \colon \mathfrak{h} \to \mathfrak{g}$ is a map of Lie algebras, i.e., a linear map commuting with the Lie bracket operations.*

Proof. We have a commutative diagram

$$\begin{array}{ccc}
H \times H & \xrightarrow{\mathrm{Ad}_H} & H \\
{\scriptstyle \varphi\times\varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\
G \times G & \xrightarrow{\mathrm{Ad}_G} & G.
\end{array}$$

Differentiating at $e \in H$ and at $e \in G$ in the second factor produces a commutative diagram

$$\begin{array}{ccc}
H \times \mathfrak{h} & \xrightarrow{\mathrm{Ad}_H} & \mathfrak{h} \\
{\scriptstyle \varphi\times d\varphi_e}\downarrow & & \downarrow{\scriptstyle d\varphi_e} \\
G \times \mathfrak{g} & \xrightarrow{\mathrm{Ad}_G} & \mathfrak{g}.
\end{array}$$

Lastly, differentiating at the identity in the first variable gives a commutative diagram

$$\begin{array}{ccc} \mathfrak{h} \times \mathfrak{h} & \xrightarrow{\mathrm{ad}_H} & \mathfrak{h} \\ {\scriptstyle d\varphi_e \times d\varphi_e}\big\downarrow & & \big\downarrow{\scriptstyle d\varphi_e} \\ \mathfrak{g} \times \mathfrak{g} & \xrightarrow{\mathrm{ad}_G} & \mathfrak{g}. \end{array}$$

This diagram says that for $X, Y \in \mathfrak{h}$, we have

$$d\varphi_e(\mathrm{ad}_H(X)(Y)) = \mathrm{ad}_G(d\varphi_e(X), d\varphi_e(Y)).$$

By definition of the bracket, this translates to

$$d\varphi_e([X, Y]) = [d\varphi_e(X), d\varphi_e(Y)].$$

□

Proposition 2.10. *Let $\varphi\colon H \to G$ be a homomorphism of complex Lie groups. The $d\varphi_e\colon \mathfrak{h} \to \mathfrak{g}$ is a morphism of complex Lie algebras.*

Corollary 2.11. *Let V be a finite dimensional complex vector space, let G be a complex Lie group and let $G \times V \to V$ be a complex linear representation in the sense that $\rho\colon G \to GL(V)$ is a map of complex Lie groups. Then the differential of ρ at the identity, $d\rho_e\colon \mathfrak{g} \to \mathfrak{g}l(V)$ is a complex linear map sending the Lie bracket of $\mathfrak{g}$ to the bracket of complex linear endomorphisms given by $[A, B] = AB - BA$.*

3 The Exponential Mapping.

We have shown how to pass from a Lie group to its Lie algebra by differentiating at the identity element (twice) the conjugation map of G on itself. The basic construction passing from a Lie algebra $\mathfrak{g}$ to G is the exponential mapping. This mapping identifies a neighborhood of the origin in $\mathfrak{g}$ with a neighborhood of the identity in G.

3.1 The case of $GL_n(\mathbb{R})$

Since $GL(n, \mathbb{R})$ is an open subset of $M(n \times n, \mathbb{R})$, any $A \in M(n \times n, \mathbb{R})$ determines a tangent vector to $GL(n, \mathbb{R})$ at the identity element. This identifies $M(n \times n, \mathbb{R})$ with $\mathfrak{g}l(n, \mathbb{R})$. The power series

$$\exp(tA) = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$$

converges absolutely for all $t \in \mathbb{R}$ and hence defines a smooth curve $\gamma_A(t)$ in $M(n \times n, \mathbb{R})$. By construction it satisfies $\gamma_A(0) = \mathrm{Id}$ and $\gamma_A'(0) = A$. The usual power series manipulations show that for all $t_1, t_2 \in \mathbb{R}$ we have $\gamma_A(t_1)\gamma_A(t_2) = \gamma_A(t_1 + t_2)$. Since $\gamma_A(t) \in GL(n, \mathbb{R})$ for all $|t|$ sufficiently small, it follows that, for all $t \in \mathbb{R}$ the matrix $\gamma_A(t)$ is contained in $GL(n, \mathbb{R})$, and furthermore, γ_A is a homomorphism of Lie groups $(\mathbb{R}, +) \to GL(n, \mathbb{R})$. We define the exponential map

$$\exp\colon \mathcal{M}(n \times n, \mathbb{R}) \to GL(n, \mathbb{R})$$

by

$$\exp(A) = \gamma_A(1) = e^A.$$

This is a smooth map from $M(n \times n, \mathbb{R}) \to GL(n, \mathbb{R})$ whose differential at the origin is the identity. By the implicit function theorem there is a neighborhood U of $0 \in M(n \times n, \mathbb{R})$ that maps diffeomorphically onto an open subset $\exp(U)$ of the identity in $GL(n, \mathbb{R})$. The inverse map is the logarithm $\log\colon \exp(U) \to U$.

In the case of $GL(n, \mathbb{C})$ the exponential map (given by the same power series) associates to each $A \in M(n \times n, \mathbb{C})$ a homomorphism of Lie groups $\gamma_A\colon (\mathbb{C}, +) \to GL(n, \mathbb{C})$ with $\gamma_A'(0)\colon \mathbb{C} \to M(n \times n, \mathbb{C})$ the complex linear map sending $1 \in \mathbb{C}$ to A. We define a holomorphic map $\exp\colon \mathfrak{g}l(n, \mathbb{C}) \to GL(n, \mathbb{C})$ to send A to e^A. Analogously, the differential of this map at $0 \in \mathfrak{g}l(n, \mathbb{C})$ is the identity so that it is a local holomorphic isomorphism from some neighborhood of 0 in $\mathfrak{g}l(n, \mathbb{C})$ to an open neighborhood of the identity in $GL(n, \mathbb{C})$.

3.2 The Exponential Map for a General Lie Group

Theorem 3.1. *Let G be a Lie group. Then for every $A \in \mathfrak{g}$ there is a unique morphsim of Lie groups $\gamma_A\colon (\mathbb{R}, +) \to G$ with the property that $\gamma_A'(0) = A$.*

Proof. Fix $A \in \mathfrak{g}$. Let χ_A be the left-invariant vector field whose value at $g \in G$ is $g \cdot A$. By the existence and uniqueness results for ODEs, for some $\epsilon > 0$, there is a unique integral curve $\gamma_A\colon (-\epsilon, \epsilon) \to G$ for this vector field whose value at 0 is e.

Claim 3.2. *The maximal interval of definition for the integral curve γ_A is the entire real line.*

Proof. By the existence theorem for solutions to ODEs, there is $\epsilon > 0$ such that γ_A is defined on $(-\epsilon, \epsilon)$. By uniqueness of solutions to ODEs, if I and J

are intervals of definition for an integral curve of χ_A, both containing 0, then the integral curves defined on these two intervals agree on the intersection of the intervals and hence the two curves define an integral curve on $I \cup J$. From this it is easy to see that there is a maximal interval of definition for the integral curve γ_A. We must show that this is $\mathbb{R}$.

Let $I \subset \mathbb{R}$ be the maximal interval of definition for γ_A and suppose that I is bounded above. Fix t_0 within $\epsilon/2$ of the least upper bound of I. Consider the curve $\mu(t_0+t) = \gamma_A(t_0)\gamma_A(t)$ for $t \in (-\epsilon, \epsilon)$. Then $\mu'(t_0+t) = \gamma_A(t_0)\gamma'_A(t) = \gamma_A(t_0)\gamma_A(t) \cdot A$. This shows that μ is an integral curve for χ_A. Since it and γ_A agree at t_0, they agree on their common domain of definition. This is a contradiction since it allows us to extend the domain of definition beyond the least upper bound of I and I was assumed to be the maximal interval of definition for the integral curve. Consequently, the interval I has no upper bound. Symmetrically, I has no lower bound. The only interval with no upper and no lower bound is $\mathbb{R}$. □

Fix t and let s be variable. Consider the integral curve for χ_A given by $\mu(s) = \gamma_A(t)\gamma_A(s)$. The curve $\gamma_A(t+s)$ also is an integral curve for χ_A. Both these integral curves take the value $\gamma_A(t)$ at $s = 0$. Thus, $\gamma(t)\gamma(s) = \mu(s) = \gamma_A(t+s)$ for all s. Since this true for all t and all s and since $\gamma_A(0) = e$, the map $\gamma_A \colon \mathbb{R} \to G$ is a homomorphism from the additive Lie group of reals to G.

Claim 3.3. *Suppose that $\gamma \colon (\mathbb{R}, +) \to G$ is a homomorphism of Lie groups and suppose that $\gamma'(0) = A$. Then $\gamma(t) = \gamma_A(t)$ for all $t \in \mathbb{R}$.*

Proof. Since γ is a homomorphism, it follows that $\gamma'(t) = \gamma(t)\gamma'(0)$, and thus γ is an integral curve for χ_A whose value at $t = 0$ is the identity. There is only one such integral curve and it is γ_A. □

This completes the proof of Theorem 3.1 □

Definition 3.4. We define *the exponential map*, $\exp_G \colon \mathfrak{g} \to G$ by sending $A \in \mathfrak{g}$ to $\gamma_A(1)$ where γ_A is the one-parameter subgroup whose tangent vector at the identity is A.

The following is clear from the definition.

Proposition 3.5. *The exponential mapping is a smooth map whose differential at $0 \in \mathfrak{g}$ is the identity. Hence, there is a neighborhood $U \subset \mathfrak{g}$ of 0 such that $\exp_G$ is a diffeomorphism from U to an open neighborhood $\exp_G(U)$ of the identity in G. We denote the inverse by $\log \colon \exp(U) \to U$.*

Proof. The exponential is a smooth map since integral curves of a smooth family of vector fields with a smooth family of initial conditions vary smoothly. To compute the differential of exp, first note that $\gamma_A(st) = \gamma_{tA}(s)$ since they are both integral curves for $t\chi_A$ and take the value e at $s = 0$. Thus,

$$D(\exp)_e(A) = (\frac{d}{dt})|_{t=0}(\gamma_{tA}(1)) = (\frac{d}{dt})|_{t=0}(\gamma_A(t)) = A.$$

The local diffeomorphism then follows from the Inverse Function Theorem. $\square$

Corollary 3.6. *If $H \subset G$ is a Lie subgroup with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$, then $\exp_G|_{\mathfrak{h}} = \exp_H$. Any particular, any one-parameter subgroup tangent to H at the origin is contained in H.*

More generally, if $\varphi\colon H \to G$ is a map of Lie groups, then for any $A \in \mathfrak{h}$ the map of the additive one-parameter subgroup $\gamma_A\colon (\mathbb{R}, +) \to H$ tangent to A has image in G that is the map of the additive one-parameter subgroup tangent to $d\varphi_e(A)$.

Proof. For $H \subset G$, for any $A \in \mathfrak{h}$ the left-invariant vector field gA is tangent to H. Hence the integral curve γ_A that passes through e at $t = 0$ lies in H. The second statement is left as an exercise. $\square$

Theorem 3.7. *Let G and H be connected Lie groups and $\alpha, \beta\colon G \to H$ two Lie group homomorphisms. Then $\alpha = \beta$ if and only if α and β induce the same map $\mathfrak{g} \to \mathfrak{h}$.*

Proof. The 'only if'; direction of the implication is clear. We establish the 'if' direction. Suppose $D_e(\alpha) = D_e(\beta)\colon \mathfrak{g} \to \mathfrak{h}$. Fix $A \in G$ and consider the one-parameter subgroup $\exp(tA) \subset G$. Its image under α is the one-parameter subgroup $\exp(tD_e(\alpha))$ and analogously for β. It follows that α and β agree on all the images of all these one-parameter subgroups, i.e., on the image of exp. Since the exponential map is onto a neighborhood U of the identity, it follows that $\alpha|_U = \beta|_U$.

Claim 3.8. *Each element of G is a finite product of elements in U.*

Proof. Without loss of generality, we can assume that U is closed under $g \mapsto g^{-1}$ Clearly, the subset of elements of G that can be written as a finite product of elements of U is an open, non-empty subset of G. We claim that it is also closed. For suppose that $\lim_{n\mapsto\infty} g_n = g$ and each g_n is a finite product of elements in U. To prove that the subset of elements that can be written as a finite product of elements in U is closed, we prove that g is a product of finitely many element in U. But for some n the element $g_n \in gU$, i.e., $g^{-1}g_n \in U$, and hence $g_n^{-1}g \in U$. Since $g = g_n(g_n^{-1}g)$, and g_n is a finite product of elements in U, so is g.

Being open and closed and non-empty, the subset consisting of all finite products of elements in U is all of the connected space G. □

Since $\alpha|_U = \beta|_U$, it follows that $\alpha = \beta$. □

Remark 3.9. Given Lie groups G and H and a map between their Lie algebras $\psi\colon \mathfrak{g} \to \mathfrak{h}$ there may not be a map $G \to H$ extending the map ψ on the Lie algebras. For example, an isomorphism from the Lie algebra of S^1 to the Lie algebra of $\mathbb{R}$ does not extend to a map of Lie groups. In a later lecture we shall show that in the special case when G is simply connected any map of Lie algebras $\mathfrak{g} \to \mathfrak{h}$ extends (uniquely) to a map of Lie groups $G \to H$.

3.3 Important Technical Result

This technical result is both important and the first indication of the power of the exponential map.

Theorem 3.10. *Let G be a Lie group and suppose that $H \subset G$ is a topologically closed subgroup. Then H is a sub Lie group,*

Proof.

Claim 3.11. *Fix a positive definite symmetric inner product on $\mathfrak{g}$. Denote by $|h|$ for $h \in \mathfrak{g}$ the associated norm on $\mathfrak{g}$. Suppose that $h_n \in \mathfrak{g}$ is a sequence of elements converging to 0 with $\exp(h_n) \in H$ for all n. Suppose that as $n \mapsto \infty$ the sequence $\frac{h_n}{|h_n|}$ converges to a unit vector $v \in \mathfrak{g}$. Then $\exp(tv) \in H$ for all $t \in \mathbb{R}$.*

Proof. The result is clear for $t = 0$. Fix $t > 0$ in $\mathbb{R}$. Let m_n be the greatest integer less than $t/|h_n|$. Then $m_n h_n \mapsto tv$ as $n \mapsto \infty$. Since $\exp(m_n h_n) = \exp(h_n)^{m_n} \in H$ and H is closed, it follows that for all $t \geq 0$ we have $\exp(tv) \in H$. Since $\exp(-tv) = \exp(tv)^{-1}$, the result follows for all $t \in \mathbb{R}$. □

Claim 3.12. *Let $W \subset V$ be the set of w for which $\exp(tw) \in H$ for all $t \in \mathbb{R}$. Then W is a real linear subspace of V.*

Proof. By construction if $w \in W$ then $tw \in W$ for all $t \in \mathbb{R}$. Thus, to show that W is a real linear subspace, it suffices to show that if $w_1, w_2 \in W$ with $w_1 + w_2 \neq 0$, then $(w_1 + w_2) \in W$. For all t sufficient close to zero, we have $\exp(tw_1)\exp(tw_2)) = \exp(f(t))$ for a smooth function $f(t)$ with $f(0) = 0$ and $f'(0 = (\exp(tw_1)\exp(tw_2)'(0)) = w_1 + w_2$. In particular,

$$\lim_{t\mapsto 0} f(t)/t = w_1 + w_2.$$

Because $w_1, w_2 \in W$, from the definition of W and the fact that H is a group, $\exp(f(t)) \in H$ for all t sufficiently close to 0. The limit statement above shows for n sufficiently large, $f(1/n) = \frac{1}{n}(w_1 + w_2) + o(1/n)$. For all n sufficiently large, set $h_n = f(1/n)$. We see that $\frac{h_n}{|h_n|}$ converges to $\frac{w_1+w_2}{|w_1+w_2|}$ as $n \mapsto \infty$. The Claim 3.11 now implies that $w_1 + w_2 \in W$. $\square$

Claim 3.13. *A neighborhood of* 0 *in* W *maps via the exponential mapping isomorphically onto the intersection of* H *with a neighborhood* U *of* $e \in G$.

Proof. Let $W' \subset \mathfrak{g}$ be the orthogonal complement of W in $\mathfrak{g}$. We have $W \oplus W' = \mathfrak{g}$. Let $\psi(w, w') = \exp(w)\exp(w')$. The map $\rho = \psi^{-1}$ is a diffeomorphism from an open neighborhood $U \subset G$ of e to an open neighborhood $\rho(U)$ of $0 \in W \oplus W'$. If there is no smaller neighborhood $U' \subset U$ of the identity as claimed, then there is a sequence (w_n, w_n') tending to $(0, 0)$ such that for all n we have $\exp(w_n)\exp(w_n') \in H$ and $w_n' \neq 0$. Since H is a group and $\exp(w_n) \in H$, it follows that $\exp(w_n') \in H$ for all n. Choosing a subsequence, we can assume that $\frac{w_n'}{|w_n'|}$ converges to a unit vector in $v \in W'$. Applying the first claim, we see that $v \in W$. This contradiction shows that there is some neighborhood $U' \subset G$ of e and a diffeomorphism $U \to \rho(U) \subset W \oplus W'$ onto an open neighborhood of $(0, 0)$ such that $\rho(H \cap U)) = W \cap \rho(U)$.

This shows that $W \cap U$ is a smooth submanifold of U. $\square$

Now for any $h \in H$ we see that $h \cdot U$ is an open neighborhood of h in G and the intersection $H \cap hU$ maps via $\rho \circ h^{-1}$ to $W \cap \rho(U) \subset \rho(U)$. Since this holds for every $h \in H$ and since H is a closed subset of G , it follows that H is a smooth submanifold of G.

We have already seen that a subset of G that is a subgroup and a submanifold (in this strong sense) is a sub Lie group. $\square$