Lie Groups: Fall, 2024
Lecture IV: Lie's Theorems

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1 Lie sub algebras to Lie subgroups: The Statement

The first result is that a sub Lie algebra of the Lie algebra of a Lie group G integrates to a unique group (up to isomorphism) with an one-to-one immersion into G.

Theorem 1.1. *Let G be a Lie group and $\mathfrak{h} \subset \mathfrak{g}$ a sub Lie algebra. Then there is a connected Lie group H, a Lie group map $H \to G$ that is a one-to-one immersion whose differential at the identity identifies the Lie algebra of H with $\mathfrak{h}$. The image of H is G is the subgroup generated by the restriction of the exponential map to $\mathfrak{h}$.*

Corollary 1.2. *With the hypotheses and notation of Theorem 1.1 there is a local Lie subgroup of G whose Lie algebra is $\mathfrak{h}$. Any two such local Lie subgroups have the same germ.*

Proof. (Theorem 1.1 implies the corollary) Given $\mathfrak{h} \subset \mathfrak{g}$, according to Theorem 1.1 there is a Lie group H and a one-one immersion of $H \to G$ whose Lie algbra is $\mathfrak{h}$. Any local Lie subgroup of H given by a sufficiently small open subset of H is as stated in the corollary.

Conversely, given a connected local sub Lie group with Lie algebra $\mathfrak{h}$, we have shown it extends uniquely to a connected, one-one immersed sub group. Theorem 1.1 says that that sub group is the immersion $H \to G$ and hence the local Lie group is a sub local group of H. Thus, all local sub Lie groups of G with Lie algebra $\mathfrak{h}$ represent the germ of H in G.

□

2 Maps of Lie algebras to Maps of Lie Groups: The Statement

Theorem 2.1. *Let G_1 and G_2 be connected Lie groups with G_1 simply connected, and let $\varphi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ be a Lie algebra homomorphism.Then there is a unique homomorphism $\psi\colon G_1 \to G_2$ with $d_e\psi = \varphi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$.*

Theorem 2.2. *Let G_1 and G_2 be simply connected Lie groups. Suppose that $\varphi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is an isomorphism then there is a unique Lie group isomorphism $G_1 \to G_2$ that induces φ on their Lie algebras.*

Proof. (Theorem 2.1 impies Theorem 2.2) Applying Theorem 2.1, there is a map of Lie groups $\psi\colon G_1 \to G_2$ whose differential at the identity is φ. In particular ψ is local diffeomorphism at the identity. This implies that the kernel of ψ is a discrete normal subgroup K and ψ factors to give an injective Lie group map $\overline{\psi}\colon G_1/K \to G_2$. Since ψ is onto a neighborhood of the identity in G_2 and G_2 is connected, it follows that ψ is onto. Thus, $\overline{\psi}$ is onto. Also ψ is a local diffeomorphism and one-one. Thus, $\overline{\psi}$ is a diffeomorphism and a group isomorphism. That is to say $\overline{\psi}$ is a Lie group isomorphism. The last thing to note is that $\pi_1(G_1/K) \cong K$ and $\pi_1(G_2) = \{e\}$. Since $\overline{\psi}\colon G_1/K \to G_2$ is a diffeomorphism, this implies that $K = \{1\}$. This proves the existence of a map of Lie groups as required.

We turn to uniqueness. If $\rho\colon \mathbb{R} \to G_1$ is a one-parameter subgroup tangent to $X \in \mathfrak{g}_1$, then $\psi \circ \rho$ is the one-parameter subgroup in G_2 tangent to $\varphi(X)$. Thus, ψ is determined by φ on the image of the exponential map of G_1. This image generates G_1, hence ψ is determined by φ. □

3 Proof of Theorem 1.1

Before proving Theorem 1.1 we need to discuss distributions and foliations.

Definition 3.1. A distribution of dimension k in a smooth manifold M is a smoothly varying family of tangent k-planes a $D^k(x) \subset T_xM$ for every $x \in M$. Smooth variation means that in a neighborhood U each $x \in M$ there are local vector fields $\chi_1, \cdots, \chi_k$ such that for each $y \in U$ the $\chi_i(y)$ are contained in $D^k(y)$ and are linearly independent implying that they generate $D^k(y)$. An *integral submanifold* for a distribution is a k-dimensional submanifold $P \subset M$ such that $T_pP = D^k(p)$ for every $p \in P$. Not every distribution has integral submanifolds. There is an obvious necessary condition. Namely, the distribution must be what is called *involutive*.

Definition 3.2. A distribution D^k is *involutive* if for every pair of vector fields ξ, ζ tangent to the distribution meaning that $\xi(x), \zeta(x)$ are contained. in $D^k(x)$ for every x, the Lie bracket $[\xi, \zeta]$ must be contained in D^k.

If P is an integral submanifold and ξ and ζ are vector fields tangent to P, then since Lie bracket of vector fields is natural under smooth maps, it follows that $[\xi, \zeta]$ is also tangent to P. Thus, for D^k to have integral submanifolds through each point, the distribution must be involutive. A theorem of Frobenius states the converse.

Theorem 3.3. *(Frobenius) A distribution $\mathcal{D}$ in M has a (local) integral submanifold through every point of $p \in M$ if and only if it is involutive; i.e., if and only if the space of vector fields tangent to $\mathcal{D}$ is a Lie subalgebra of the space of all vector fields on M. In this case any two integral submanifolds through x coincide in a neighborhood of x.*

Definition 3.4. A k-dimensional *foliation* of M is a decomposition of M as a collection of connected k-dimensional submanifolds one-one immersed in M. These are the *leaves* of the foliation. Near each point $m \in M$ there is a *flow box*; i.e., coordinate system $U^k \times V^\ell$ such that each leaf of the foliation meets $U \times V$ is a subset of the form $U \times \Delta$ where Δ is an at most countable, totally disconnected subset of V. Thus, the slices $\{U \times \{v\}\}_{v \in V}$ are the *local leaves* of the foliation, meaning each slice is a component of the intersection of leaf of the foliation with the flow box. Each global leaf of the foliation is given the leaf topology (which is not in general the subspace topology). This is the topology generated by the open sets that a component of the intersection of the leaf with any flow box. With the leaf topology, each leaf inherits a smooth manifold structure from M and is a smooth manifold smoothly one-one immersed in M.

Theorem 3.5. *If D^k is an involutive distribution on M, then through every point there is a global integral submanifold. In fact, the collection of integral submanifolds foliates M.*

Theorems 3.3 and 3.5 are not a deep theorems. The problems associated to this lecture lead you through proofs of them.

Proof. (of Theorem 1.1) Now we apply this theory to the case of a sub Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ of the Lie algebra of a group G. The distribution we take is the left invariant distribution whose value at e is $\mathfrak{h}$.

Take a basis for $\{X_1, \ldots, X_k\}$ for $\mathfrak{h}$. They generate left invariant vector fields $\xi_1, \ldots, \xi_k$ that are a basis at each point for the distribution at every point. The bracket of $[\xi_i, \xi_j]$ is a left invariant vector field, and its value at the origin is $[X_i, X_j]$. Since $X_i, X_j \in \mathfrak{h}$ and $\mathfrak{h}$ is a sub Lie algebra, $[X_i.X_j] \in \mathfrak{h}$ and hence $[\xi_i, \xi_j]$ is tangent to the distribution. The general vector fields tangent to this distribution are of the form $\sum_i f_i\xi_i$ for some smooth functions $f_1, \ldots, f_k$. Then

$$[\sum_i f_i\xi_i, \sum_j g_j\xi_k j] = \sum_{i,j} f_i\xi_i(g_j)\xi_j - g_j\xi_j(f_i)\xi_i + f_ig_j[\xi_i, \xi_j].$$

The first two terms are visibly in D^k, being functions times the ξ_i and the last term is in D^k by what we just showed above.

Thus, this distribution integrates to a foliation. Let H be the (global) leaf of this foliation containing the origin. With its leaf topology it is a k-dimensional manifold smoothly one-one immersed in G. Its tangent space at the origin in $\mathfrak{h}$.

Since the foliation is invariant under left multiplication, if $h \in H$, then $h \cdot H \subset H$; that is to say H is closed under multiplication. For a neighborhood $U \subset H$ of $e \in H$ that is in the image of the exponential map, for every element $u \in U$, it is also true that $u^{-1} \in H$. Hence, the subset of $h \in H$ with $h^{-1} \in H$ is both an open and closed subset of H in the leaf topology. Since H is connected, it follows that H is closed under inverses. Thus, H is a subgroup of G and with the leaf topology it is a Lie group. The immersion is a map of Lie groups. Its Lie algebra is $\mathfrak{h}$.

The uniqueness statement follows from the uniqueness in Frobenius's theorem.. $\square$

4 Proof of Theorem 2.1

Let G_1 and G_2 be groups with G_1 simply connected, and let $\varphi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ be a Lie algebra homomorphism. Consider the product Lie group $G_1 \times G_2$. Its Lie algebra is $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ with the direct sum bracket. The graph of φ is a linear subspace $V \subset \mathfrak{g}_1 \oplus \mathfrak{g}_2$ whose projection onto the first factor is a linear isomorphism. Since φ is a Lie algebra homomorphism $V \subset \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a Lie subalgebra. According to Theorem 1.1, there is a connected Lie group H and a one-one immersion $H \to G_1 \times G_2$ that is a homomorphism of Lie groups. In addition the Lie algebra of H is V. The manifold H is a leaf of the left-invariant distribution whose tangent plane at the identity is V. Thus, the projection $G_1 \times G_2 \to G_1$ is an isomorphism on each tangent plane of the distribution. This means that the composition $H \to G_1 \times G_2 \to G_1$, denoted ρ, is a homomorphism of Lie groups that is local diffeomorphism.

Since G_1 is connected this implies that ρ is surjective with discrete kernel. Since G_1 is simply connected and H is connected, the kernel of ρ is trivial. That is to say the composite $\rho\colon H \to G_1$ is an isomorphism of Lie groups. Hence, the composite, $\psi = \pi_2 \circ \rho^{-1}\colon G_1 \to G_2$, is a Lie group homomorphism whose graph is $H \subset G_1 \times G_2$. This implies that the graph of $d_e(\psi)$ in $\mathfrak{g}_1 \times \mathfrak{g}_2$ is V which, recall, is the graph of $\varphi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$. It follow that $d_e(\psi) = \varphi$.

This completes the proof of Theorem 2.1.

5 Isogeny of Lie Groups

We begin by discussing covering groups of connected Lie groups.

Definition 5.1. A map of connected Lie groups $\varphi\colon G_1 \to G_2$ is a *covering Lie group* if the map is a covering projection and a homomorphism of Lie groups.

As an example, suppose that G is a connected Lie group and $K \subset G$ is a discrete, normal subgroup. Then $G \to G/K$ is a covering group.

Lemma 5.2. *A morphism of connected Lie groups $\varphi\colon G_1 \to G_2$ is a covering Lie group if and only if it induces an isomorphism of Lie algebras.*

Proof. If φ is a covering Lie group, then it is a local diffeomorphism and $d_e\varphi$ is a linear isomorphism. Since φ is a Lie group map, $d_e\varphi$ is a homomorphism of Lie algebras and hence an isomorphism of Lie algebras.

Conversely, if $d_e\varphi$ is an isomorphism, then φ is a local diffeomorphism at the identity. That is to say, there is a neighborhood U_1 of the identity in G_1 such that $\varphi|_{U_1}$ is a diffeomorphism onto an open subset U_2 of the identity in G_2. Since G_2 is connected it is generated by U_2. Thus, φ is onto.

Let $K \subset G_1$ be the kernel of φ. It is a normal subgroup. Since $\varphi|_{U_1}$ is injective, $K \cap U_1 = \{e\}$. Let $W \subset U_1$ be a smaller open neighborhood of the identity with the property that $W = W^{-1}$ and $W^2 \subset U$. We claim that $kW \cap k'W = \emptyset$ for all $k \neq k'$ elements of K. For if $w_0 \in kW \cap k'W$ then we have $kw = k'w'$ for some $w, w' \in W$. This implies that $k^{-1}k' = w(w')^{-1} \in W^2 \subset U$. Since $k^{-1}k' \in K$ and $K \cap U = \{e\}$, it follows that $k = k'$. This shows that K is a discrete subgroup of G_1.

Since $K \subset G_1$ is a normal group we have a Lie group homomorphism $\pi\colon G_1 \to G_1/K$. The above argument shows that π evenly covers the image, $\overline{W} \subset G_1/K$, of W. Clearly, $\overline{W}$ is an open neighborhood of the identity in G_1/K Now consider $\overline{g} \in G_1/K$. Left translation by $\overline{g}$ maps the image of $\overline{W}$ isomorphically to $\overline{g}\overline{W}$ a neighborhood of $\overline{g}$ in G_1/K. Multiplication by a lift

$g \in G_1$ of $\overline{g}$ and maps $\pi^{-1}(\overline{W})$ isomorphically to $\pi^{-1}(\overline{g}\overline{W}) = \coprod_{k \in K} gkW$. This shows that φ is also a covering map on $\overline{g}\overline{W}$ for every $\overline{g} \in G_1/K$. Thus, $G_1 \to G_1/K$ is a covering projection.

Since $K = \ker(\varphi)$, the map φ factors through $\pi\colon G_1 \to G_1/K$ to give a map $\overline{\varphi}\colon G_1/K \to G_2$. This map is a group homomorphism and smooth so it is a map of Lie groups with trivial kernel. Since G_2 is connected, it is generated by the neighborhood $\overline{\varphi}(\overline{W})$ of the identity. Hence $\overline{\varphi}$ is surjective. Thus, $\overline{\varphi}$ is a bijective, local diffeomorphsim and hence a diffeomorphism It is also a homomorphism of Lie groups, and hence an isomorphism of Lie groups. Hence, $\varphi\colon G_1 \to G_2$ is also a covering Lie group. □

5.1 The Universal Covering Group

Proposition 5.3. *Let G be a connected Lie group and let $\widetilde{G}$ be the universal covering of G and fix $\widetilde{e} \in \widetilde{G}$ a point above $e \in G$. Then there is a unique Lie group structure on $\widetilde{G}$ with the properties that (i) $\widetilde{e}$ is the identity element and (ii) the projection $\widetilde{G} \to G$ is a Lie group homomorphism. The kernel of this homomorphism is a discrete subgroup $K \subset \widetilde{G}$ and the covering projection induces a Lie group isomorphism $\widetilde{G}/K \to G$. In particular, the Lie algebras of $\widetilde{G}$ and G are canonically identified.*

Proof. Given $g_1, g_2 \in \widetilde{G}$, let $\omega_1(t)$ and $\omega_2(t)$ be paths defined on $[0,1]$ in $\widetilde{G}$, each beginning at $\widetilde{e}$ with $\omega_i(1) = g_i$. Let $\overline{\omega}_1(t)$ and $\overline{\omega}_2(t)$ be the images of these paths in G, and let $\overline{\mu}(t) = \overline{\omega}_1(t)\overline{\omega}_2(t)$. This is a path beginning at e. Using unique path lifting, lift $\overline{\mu}$ to a path $\mu(t)$ beginning at $\widetilde{e}$. We define $g_1 g_2 = \mu(1)$.

A standard argument with covering spaces shows that if we choose different paths $\omega_1'(t)$ and $\omega_2'(t)$ from $\widetilde{e}$ to g_1 and g_2, respectively, the two definitions of $g_1 g_2$ agree. [Show that as we vary the paths, the notion of $g_1 g_2$ is locally constant. Since $\widetilde{G}$ is simply connected two pairs of paths from $\widetilde{e}$ to g_1 and g_2 came be joined by a connected family of such pairs of paths. This and the local constancy of the resulting product, show that the product $g_1 g_2$ is well defined.] It is direct to see that $\widetilde{e}$ acts as a two-sided identity for this multiplication and that this multiplication is associative.

Given $g \in \widetilde{G}$, one defines g^{-1} by choosing a path ω from $\widetilde{e}$ to g, projecting ω to a path $\overline{\omega}$ in G, forming the path $\overline{\omega}^{-1}(t) = (\overline{\omega}(t))^{-1}$ and lifting $\overline{\omega}^{-1}$ to a path μ beginning at $\widetilde{e}$. We define $g^{-1} = \mu(1)$. It is clear from the definitions that $gg^{-1} = g^{-1}g = \widetilde{e}$. Thus, we have defined a group structure on $\widetilde{G}$ with $\widetilde{e}$ as the identity element. Clearly, the projection mapping is a homomorphism of groups

One defines the smooth structure on $\widetilde{G}$ by requiring the projection map to be a local diffeomorphism. One checks easily group multiplication and inverse are smooth mappings in this smooth structure. Thus, the projection is a smooth map and a group homomorphism; that is to say the projection is morphism of Lie groups. $\square$

5.2 All Covering Groups

Lemma 5.4. *Any discrete, normal subgroup of a connected Lie group is abelian and central.*

Proof. Let G be a connected Lie group and $K \subset G$ a discrete normal subgroup. Since K is normal, $gKg^{-1} = K$ for all $g \in G$. That is to say conjugation by G induces a map $G \to \mathrm{Auto}(K)$. But since K is discrete, so is $\mathrm{Auto}(K)$. But G is connected, so any map $G \to \mathrm{Auto}(K)$ is constant, meaning that the adjoint action of G on K is trivial. Thus, K is contained in the center of G and *a fortiori* is abelian. $\square$

Corollary 5.5. *Let G be a connected Lie group and M a connected manifold. Suppose that $\pi\colon M \to G$ is a covering projection. Then there is a Lie group structure on M such that $\pi\colon M \to G$ is a covering Lie group.*

Proof. Every connected covering of G corresponds to a subgroup of $\pi_1(G, e)$. The universal covering Lie group $\widetilde{G} \to G$ corresponds to the trivial subgroup The kernel of the projection mapping $\widetilde{G} \to G$ is a discrete normal subgroup K of $\widetilde{G}$ isomorphic to $\pi_1(G)$. By the previous lemma K is central in $\widetilde{G}$.

All other connected covering spaces of G are isomorphic to $\widetilde{G}/K'$ where K' is a subgroup of K. Since K is central, K' is also central, and *a fortiori* is a normal subgroup. Thus, $\widetilde{G}/K$ inherits the structure of a Lie group from $\widetilde{G}$. Clearly, then he projection $G/K \to G$ is a covering Lie group. $\square$

Definition 5.6. Two connected Lie groups G_1 and G_2 are *isogenous* if there is a Lie group G and Lie group maps $\varphi_i\colon G \to G_i$, for $i = 1, 2$, that are covering Lie groups.

Corollary 5.7. *Let G_1 and G_2 be connected Lie groups. Then the following are equivalent:*

- *G_1 and G_2 are isogenous.*
- *The Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic.*
- *The germs of G_1 and G_2 are isomorphic.*

- *The universal covering groups of G_1 and G_2 are isomorphic as Lie groups.*

Proof. If G is a covering group of both G_1 and G_2, the the universal covering group of G is also the universal covering group of G_1 and G_2. This shows the first item implies the fourth. The fourth obviously implies the first, second, and third. The third implies the second since germs have Lie algebras. The second implies the fourth by Theorem 2.1. $\square$

6 Ado's Theorem

To complete the picture of the general theory of Lie groups we need a non-trivial result from the theory of Lie algebras.

Theorem 6.1. *(Ado's Theorem) Every finite dimensional real Lie algebra has a faithful finite dimension linear representation*

The proof of this theorem requires a detour through some of the more detailed parts of general Lie algebra theory. I will not prove it in this course. Nevertheless, I will use the following consequence.

Theorem 6.2. *Let G be a connected Lie group. Then there is an isogenous Lie group G' that admits a faithful finite dimensional representation; i.e., for some n there is a Lie group homomorphism $G' \to GL(n, \mathbb{R})$ that is a one-one immersion.*

Proof. (Assumping Ado's Theorem) By Ado's theorem, there is $n > 0$ and an embedding $\iota\colon \mathfrak{g} \subset \mathfrak{g}l(n, \mathbb{R})$ of Lie algebras. By Theorem 2.1 there is a group G' and a Lie group homomorphism $\psi\colon G' \to GL(n, \mathbb{R})$ that is a one-one immersion and whose differential $d_e\psi\colon \mathfrak{g}' \to \mathfrak{g}l(n, \mathbb{R})$ maps $\mathfrak{g}'$ isomorphically onto $\iota(\mathfrak{g})$. Since G and G' have the isomorphic Lie algebras by Corollary 5.7 they are isogenous. $\square$

Remark 6.3. It is not true that every Lie group has a faithful finite dimensional representation. In fact $\pi_1(SL(2, \mathbb{R})) \cong \mathbb{Z}$ and the universal covering group of $\widetilde{SL(2, \mathbb{R})}$ does not have a faithful finite dimensional representation.

Theorem 6.4. *Every finite dimensional Lie algebra is (up to isomorphism) the Lie algebra of a group, indeed of a simply connected group.*

Proof. (Assuming Ado's Theorem) Let L be a finite dimensional real Lie algebra. Then according to Ado's Theorem, there is an embedding $L \subset$

$\mathfrak{g}l(n,\mathbb{R})$ for some n. Applying Theorem 1.1 there is a Lie group H and a one-one immersion $H \to GL(n,\mathbb{R})$ so that L is the Lie algebra of H. The universal covering group of H is a simply connected Lie group with Lie algebra L. $\square$