Lie Groups: Fall, 2024
Lecture VI: Real Analyticity for Real Lie Groups

September 30, 2024

1 BCH Formula and Local Lie Groups

Let $FL(X,Y)$ be the free Lie algebra generated by X and Y. In the last lecture we showed that there is a formal series, the BCH series $\sum_{n\geq 1} H_n(X,Y)$ where, for $n \geq 2$ the term $H_n(X,Y)$ is a linear combination of terms of the form

$$[Z_1,[Z_2,\cdots[Z_{n-1},Z_n]]\cdots]$$

where each $Z \in \{X,Y\}$ and $H_1(X,Y) = X+Y$.

Now consider $\mathfrak{g}l(n,\mathbb{R})$. Define a positive definite inner product on this Lie algebra by setting

$$\langle A,B\rangle = \sum_{i,j=1}^{n} A_{i,j}B_{i,j}.$$

The resulting norm is $|A| = \sqrt{\sum_{i,j=1}^{n} A_{i,j}^2}$. Your homework problems were to show that there is $\epsilon > 0$ with the property for $|X|,|Y| < \epsilon$ the series $\sum_n H_n(X,Y)$ converges uniformly and absolutely in $\mathfrak{g}l(n,\mathbb{R})$.

Let $U \subset \mathfrak{g}l(n,\mathbb{R})$ be the ball of radius ϵ centered at 0. Then there is a real analytic function

$$\mathcal{H}\colon U\times U \to L$$

whose value at (X,Y) is the limit of the convergent series $H(X,Y)$. The identity $H(0,X) = H(X,0) = X$ of formal power series imply that $\mathcal{H}(0,X) = \mathcal{H}(X,0) = X$. The identity $H(X,-X) = H(-X,X) = 0$ implies $\mathcal{H}(X,-X) = \mathcal{H}(-X,X) = 0$. The identity $H(H(X,Y),Z) = H(X,H(Y,Z))$ implies that if $(X,Y),(Y,Z),(\mathcal{H}(X,Y),Z),(X,\mathcal{H}(Y,Z))$ are all contained in Ω then $\mathcal{H}(\mathcal{H}(X,Y,Z) = \mathcal{H}(X,\mathcal{H}(Y,Z))$. That is to say there is a local Lie group

$$(U,0,-1,\Omega,\mathcal{H})$$

where $\Omega = \mathcal{H}^{-1}(U) \subset U \times U$.

We also have the identity of formal series $\exp(H(X,Y)) = \exp(X)\exp(Y)$. This immediately implies that for all $(X,Y) \in \Omega$ we have

$$\exp(\mathcal{H}(X,Y)) = \exp(X)\exp(Y)$$

as elements of $GL(n,\mathbb{R})$ Of course $\exp(0) = e$ and $\exp(-X) = \exp(X)^{-1}$. We have now established the following:

Theorem 1.1. *$(U, 0, -1, \Omega, m)$ is a local Lie group. The map*

$$\exp\colon (U, 0, -1, \Omega, m) \to GL(n,\mathbb{R})$$

is a map of local Lie groups and maps U diffeomorphically onto an open neighborhood $\exp(U) \subset G$ of the identity. In particular, $\exp$ gives an isomorphism between $(U, 0, -1, \Omega, m)$ and the local sub Lie group of G determined by $\exp(U) \subset G$.

This is not only true for $\mathfrak{g}l(n,\mathbb{R})$ but for any Lie subalgbra of $\mathfrak{g}l(n,\mathbb{R})$.

Theorem 1.2. *Let $L \subset \mathfrak{g}l(n,\mathbb{R})$ be a Lie subalgebra. With U as above, let $U_L = U \cap L$. Denote the restriction of $\mathcal{H}$ to $U_L \times U_L$ by $\mathcal{H}_L$. The image of $\mathcal{H}_L$ is contained in L. Setting $\Omega_L = \mathcal{H}_L^{-1}(U_L \times U_L)$ produces a local Lie group*

$$(U_L, 0, -1, \Omega_L, \mathcal{H}_L). \tag{$*$}$$

Let $H \to GL(n,\mathbb{R})$ be a map of Lie groups that is a one-one immersion with the image of $\mathfrak{h}$ being L. The exponential map from induces an isomorphism from this local Lie group given in (*) to the local Lie subgroup of H determined by $\exp(U_L)$.

Proof. This follows directly from the result for $GL(n,\mathbb{R})$ by restricting once we observe that $\mathcal{H}\colon U_L \times U_L \to L$ since L is a Lie subalgebra, so that the terms $H_n(X,Y) \in L$ if $X, Y \in L$. □

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Corollary 1.3. *Suppose that G is a Lie group with Lie algebra $\mathfrak{g}$. Choose a positive definite symmetric inner product on $\mathfrak{g}$ with resulting norm $|\cdot|$. Then there is $\epsilon > 0$ such that setting $U \subset \mathfrak{g}$ equal to the ball of radius ϵ, the series $H(X,Y)$ converse uniformly and absolutely for X, Y in U. As before, we form a local Lie group*

$$(U, 0, -1, \Omega, m).$$

The germ of this local Lie group is isomorphic to the germ of G by $\exp$.

Proof. By Ado's theorem there is an embedding $\mathfrak{g} \to \mathfrak{g}l(n, \mathbb{R})$ for some n. Let $G' \to GL(n, \mathbb{R})$ be the map of Lie groups that is a one-one immersion so that the induced map on Lie algebras, $\mathfrak{g}' \to \mathfrak{g}l(n, \mathbb{R})$, maps $\mathfrak{g}'$ isomorphically onto $\mathfrak{g}$. Then the previous result tells us that the corollary as stated holds for G' replacing G.

But G' and G have the same Lie algebra and thus are isogenous. It follows immediately that the germ of G is identified with the germ of G' by an isomorphism that induces the given identification of their Lie algebras. Hence, the corollary holds for G as well as for G'. $\square$

2 Real Analyticity of Real Lie groups

Theorem 2.1. *Every Lie group inherits a natural real analytic structure that makes it a real analytic Lie group. Every Lie group homomorphism is real analytic with respect to these structures.*

Proof. For any finite dimensional real Lie algebra L there is a neighborhood U of 0 invariant under $X \mapsto -X$ on which the series H converges to give an analytic function $U \times U \to L$, leading to an analytic map $\mathcal{H}_L \colon \Omega_L \to U_L$.

Now suppose that $L = \mathfrak{g}$ for some Lie group G. Possibly after replacing U by a smaller open set the exponential map from $L \to G$ identifies this local Lie group with a local sub Lie group of G generated by $\exp(U)$. Let W be a neighborhood of the identity such that $W^2 \times W^2 \subset \Omega$. The usual real analytic structure on L (determined by its real linear structure) restricts to U so that the multiplication on $W^2 \times W^2 \to U$ is a real analytic map as the map $W^2 \to W^2$ given by $w \mapsto w^{-1}$. We transported the real analytic structure on W by the exponential map to a real analytic structure on $\overline{W} = \exp(W) \subset G$. On $\overline{W}$, the multiplication and inverse mappings of G are analytic maps in this transported structure.

Now we define a real analytic structure in a neighborhood $g\overline{W}$ of $g \in G$ by transporting the real analytic structure just defined on $\overline{W}$ via left multiplication by g. This gives analytic charts covering G.

We check that on the overlap of two charts the analytic structures agree. Suppose that $g\overline{W} \cap h\overline{W} \neq \emptyset$, say $gv_0 = hw_0$ with $v_0, w_0 \in \overline{W}$. Then $h^{-1}g = w_0v_0^{-1}$ and the overlap function is given by multiplication by $w_0v_0^{-1}$. Set $\overline{V} = \overline{(}W \cap (v_0w_0^{-1}\overline{W})$. Then $v_0 \in \overline{V}$ so that $\overline{V}$ is an open neighborhood of v_0. Since the element $w_0v_0^{-1} \in \overline{W}^2$ and $w_0v_0^{-1}\overline{V} \subset \overline{W}$, multiplication by $w_0v_0^{-1}$ on $\overline{V}$ is real analytic. This shows that the two real analytic charts give the same real analytic structure on the overlaps and hence define a global real analytic structure on G. We call this the *real analytic structure generated by the BCH formula.*

Let us consider multiplication near $(g, h) \in G \times G$. The analytic structure on $g\overline{W} \times h\overline{W} \subset G \times G$ is given by (gw, hv) with the analytic structure on w and v as defined previously. The product is given by $gwhv = gh(h^{-1}wh)v$. Of course $Ad(h^{-1})$ is the transport to $\overline{W}$ by exp of a linear map on L (which, of course, is real analytic on L). For w sufficiently close to $e \in W$, the image $L(w) \in W$. Hence. the restriction of $Ad(h^{-1})$ to a neighborhood $\overline{W}'$ of $e \in \overline{W}$ is analytic function $\overline{W}' \to \overline{W}$. Since we know the product in G is analytic on $\overline{W} \times \overline{W}$, it follows that the product $(h^{-1}wh)v$ is analytic in v and w near for $w \in \overline{W}'$ and $v \in \overline{W}$. This proves that the product structure on G is real analytic in the real analytic structure generated by the BCH formula.

Lastly, let us consider the inverse map. Near g it sends gw to $w^{-1}g^{-1} = g^{-1}(gw^{-1}g^{-1})$. For $w \in \overline{W}$ the map $w \mapsto w^{-1}$ is analytic. Also, for w sufficiently close to the identity $gw^{-1}g^{-1}$ is in $\overline{W}$. As before, since $Ad(g)$ is the transport of a linear map and therefore analytic near the identity, the inverse map, which near g is the composition of the inverse on $\overline{W}$ and $\mathrm{Ad}(g)$, is analytic near g.

This proves that G with the analytic structure generated by the Baker-Campbell-Hausdorff formula is a real analytic Lie group.

Now suppose that $\psi\colon G \to H$ is a Lie group homomorphism. It induces a linear map $d_e\psi\colon \mathfrak{g} \to \mathfrak{h}$ which is obviously real analytic. Transporting by the exponential mapping, $d_e\psi$ induces a map on the germs of the local Lie groups in $\mathfrak{g}$ and $\mathfrak{h}$. We have a commutative diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "d_e\psi"] \arrow[d, "\exp"'] & \mathfrak{h} \arrow[d, "\exp"] \\ G \arrow[r, "\psi"] & H \end{tikzcd}$$

Since the real analytic structure near $e \in G$ and $e \in H$ are transported from the usual real analytic structure on $\mathfrak{g}$ and $\mathfrak{h}$ and since $d_e\psi$ is linear and hence real analytic, it follows that, near the identity in G, the map ψ is real analytic.

Now let us consider ψ in a neighborhood of $g \in G$. In a neighborhood of g, the map is given by $\psi(gw) = \psi(g)\psi(w)$ for w near the identity in G. Since multiplication by g in G and multiplication by $\psi(g)$ in H are real analytic isomorphisms, and $w \mapsto \psi(w)$ is a real analytic map by what we just observed, it follows that ψ is real analytic near g, an consequently, is a real analytic map.

This shows that the category of real analytic Lie groups is equivalent to the category of smooth Lie groups. (Assuming, of course, Ado's theorem.)

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