Lie Groups: Fall, 2024
Lecture VII:
Compact Lie Groups

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Fix for this lecture a non-trivial, compact, connected Lie group G with Lie algebra $\mathfrak{g}$. We begin with a basic lemma.

Lemma 0.1. *The component group of any compact Lie group is finite.*

Proof. Since a Lie group is a manifold and hence locally connected, each connected component of G is an open subset of G. Were there infinitely many connected components, this would give an infinite covering by disjoint open sets, contradicting compactness. □

1 Linear Actions of S^1 and tori

1.1 Complex Actions of S^1

Let $S^1 \times V \to V$ be a finite dimensional, complex linear action. Let $\mathbb{R} \to S^1$ be the exponential map $t \mapsto \exp(it)$. The induced map on Lie algebras sends $1 \in \mathbb{R}$ to some $A \in M(n \times n, \mathbb{C})$. According to the Jordan canonical form, we can find a basis of V in which $A = A_{ss} + A_{nil}$ with A_{ss} is diagonalizable and A_{nil} is a strictly upper trianglar matrix commuting with A_{ss}. Let $\lambda_1, \ldots, \lambda_n$ be the diagonal entries of A_{ss}. Since $\exp(2\pi A)$ is trivial, we see that each λ_j is of the form in_j for some integers n_j.

Since $\exp(itA)$ and $\exp(itA_{ss})$ are periodic of period 2π and since A_{ss} and A_{nil} commute, it follows that $\exp(itA) = \exp(itA_{ss})\exp(itA_{nil})$, so that (itA_{nil}) is also periodic of period 2π. On the other hand, since A_{nil} is strictly upper triangular, some power of A_{nil} is identically zero. Thus, $\exp(itA_{nil})$ a finite polynomial expression in itA_{nil} whose constant term is Id and the linear term is itA_{nil}. The only periodic polynomials of t are constant polynomials.. This implies that $A_{nil} = 0$ and $A = A_{ss}$ is diagonalizable with eigenvalues in_j for integers n_j.

Definition 1.1. A *character* of S^1 is a homomorphism $S^1 \to S^1$. The group of characters of S^1 is naturally identified with $\mathbb{Z}$ given by $\theta \mapsto \theta^n$.

A representation of S^1 on a complex vector space of dimension n is, up to conjugation, given by n characters. That is to say there is a basis in which the action of given by diagonal matrices, and each diagonal entry is a character of S^1.

1.2 Complex Actions of a Torus

Definition 1.2. By a *torus* we mean a compact, connected abelian Lie group T. The Lie algebra $\mathfrak{t}$ of T is an abelian Lie algebra and hence the BCH series is $H(X, Y) = X+Y$. This converges on all of $\mathfrak{t}\times\mathfrak{t}$ and defines a group structure of $\mathfrak{t}$ which is the usual addition. The exponential map is a Lie group map from $\mathfrak{t}$ with its addition to T and is a local diffeomorphism. Hence, the kernel of exp is a discrete subgroup Λ of $\mathfrak{t}$ and the exponential mapping induces an isomorphism from $\mathfrak{t}/\Lambda \to T$. Since T is compact, $\Lambda \subset \mathfrak{t}$ must be a lattice; i.e., a discrete subgroup generated by an $\mathbb{R}$-basis $\{a_1, \ldots, a_n\}$ of $\mathfrak{t}$, where $n = \dim(\mathfrak{t})$.

Remark 1.3. The circle is a one-dimensional torus. Any torus is isomorphic as a Lie group to a finite product of circles with the product Lie group structure. (This is a homework problem.)

The results about complex actions of S^1 generalize to any torus T.

Definition 1.4. A *character of a torus* T is a homomorphism $T \to S^1$. If we write $T = \mathfrak{t}/\Lambda$ then a character of T is a linear map $\mathfrak{t} \to \mathbb{R}$ that sends $\Lambda \to \mathbb{Z}$. The group of characters is the dual group $\Lambda^* = \mathrm{Hom}(\Lambda, \mathbb{Z})$, to Λ. The formula for the character $T \to S^1$ associated to the linear function $\alpha\colon \mathfrak{t} \to \mathbb{R}$ sending $\Lambda \to \mathbb{Z}$ is

$$\exp(v) \mapsto \exp(2\pi i\alpha(v)).$$

Let $T \times V \to V$ be a complex linear action. We write the torus as a product of commuting circles. Let $X_1, \ldots, X_k$ be the elements of $\mathfrak{t}$ generating these circles as before. We have seen that each X_i is diagonalizable. Since the X_i commute, they have common eigenspaces. This means that we can find a basis for V, $\{e_1, \dots, e_n\}$ so that T stabilizes each of the complex lines $\mathbb{C}e_i$. The action of T on $\mathbb{C}e_i$ is a by a character of T, and up to conjugation, an action of T on an n-dimensional vector space is the same as n characters of T.

Definition 1.5. The characters of this action are the *weights* of the representation of T on V.

1.3 Real Actions

Now let V be a finite dimension real vector space $S^1 \times V \to V$ be a real linear action. We can complexify the action and diagonalize the result:

$$V \otimes_{\mathbb{R}} \mathbb{C} = E_0 \oplus_{j \in I} E_{n_j}$$

where the action on E_0 is trivial and the action of the E_{n_j} are given by $e^{it} \cdot w = e^{in_j t} w$ for $w \in E_{n_j}$ a non-zero integer.. Since the action is real, we have $\overline{E}_{n_j} = E_{-n_j}$. In particular, E_0 is real, meaning that $E_0 = (E_0 \cap \mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C}$, and each $E_{n_j} \oplus E_{-n_j}$ is real. The action of S^1 on $(E_0 \cap \mathbb{R}^n)$ is trivial. The intersection of $E_{n_j} \oplus E_{-n_j}$ with the real subspace projects equivariantly and isomorphically onto each of E_{n_j} and E_{-n_j}. Depending on the choice of which subspace we project onto, we see that the action of S^1 on this real subspace is given by e^{it} rotates by either e^{int} or e^{-int}. (These two actions are equivalent by the isomorphism $e^{it} \mapsto e^{-it}$ of S^1.)

This generalizes to tori. Any real linear action of a torus T on V is a direct sum of a trivial action and actions on two-dimensional spaces given by a character (i.e., a homomorphism) $\alpha_j \colon T \to S^1$ followed by the standard action of S^1 on $\mathbb{R}^2$. As in the case of the circle the character is only defined up to inverse. The weights of the real action are defined to be $\{\alpha_j^{\pm 1}\}_j$. In fact, these are the weights of the complexification of the representation.)

From now on we view characters of the torus $T = \mathfrak{t}/\Lambda$ as $\mathrm{Hom}(\Lambda, \mathbb{Z})$ and write characters additively instead of multiplicatively.

2 Maximal Tori

2.1 Definition and Existence

Proposition 2.1. *G contains a positive dimensional torus.*

Proof. Since G is positive dimensional, its Lie algebra is non-trivial. For $X \neq 0$ in $\mathfrak{t}$. Then $\exp(tX)$ is a non-trivial one-parameter subgroup $A \subset$

G. The group A is connected, positive dimensional and abelian. So is its closure, which is a Lie subgroup according to Theorem 3.10 of Lecture 2. By definition, this sub group is a positive dimensional torus. □

Corollary 2.2. *There is a positive dimensional torus in G that is not properly contained in any other torus in G.*

Proof. We have seen that G contains a positive dimensional torus. Let T be a torus of maximal dimension in G. Then T is not properly contained in any other torus in G. For, if T is properly contained in a torus T', then since T' is connected, the Lie algebra of T' is strictly larger than that of T. This means that the dimension of T' is larger than the dimension of T. □

Definition 2.3. Any torus satisfying the conclusion of the previous claim is a *maximal torus.*

2.2 Generators for Abelian Lie Groups

Definition 2.4. Let A be an abelian Lie group. An element $g \in A$ is said to *generate* the A if the cyclic group generated by g is dense in A.

Corollary 2.5. *Every torus has a generator.*

Proof. Let T be a torus written as V/Λ, a vector space modulo a lattice Λ. A codimension-1 subtorus is determined by a linear surjection $\pi\colon V \to \mathbb{R}$ that induces $\pi_{\mathbb{Z}}\colon \Lambda \to \mathbb{Z}$. The subtorus is the quotient of the kernel of π modulo the lattice by a lattice $\ker(\pi_{\mathbb{Z}})$. There are only countably many such maps and subtori.

Consider the union over the countable collection of all these maps of $\pi^{-1}(\mathbb{Q}) \subset V$. This is a nowhere dense subset $\widetilde{D}$ invariant under the action of Λ. Let D be the image in T of $\widetilde{D}$. It is nowhere dense in T. For any $g \notin D$, no positive power of g is contained in a codimension-1 subtorus. Let C be the closure of $\{g^n\}_{n=1}^{\infty}$. This is an abelian sub Lie group of G. The component group of C is finite and hence some positive power of g is contained in the component of the identity C_0 of C. Being a connected, abelian Lie group C_0, is a torus. Since it contains a positive power of g, it follows from the fact that g and all its positive powers are in the complement of D that no positive power of g is contained in a proper subtorus of T. Thus, $C_0 = T$. □

Corollary 2.6. *1. Let $A \subset G$ be an abelian Lie sub group containing a torus T with finite cyclic quotient. Then A has a generator.*

2. If $A \subset G$ is the closure of an abelian group that is generated by a connected sub group of G and a single element of G, then A has a generator.

Proof. We prove the first statement. Let $a \in A$ generate the finite cyclic quotient. Let n be the order of this quotient. Then $a^n \in T$. Let $g \in T$ be such that $a^n g$ generates T. Since T is divisible, there is $h \in T$ with $h^n = g$. Then the element ha generates the finite cyclic quotient and $(ha)^n$ generates T. The first statement follows.

Suppose that $A \subset G$ is the closure of an abelian sub group of G generated by a connected sub group R and an element of $g \in G$. The component group of A is finite. Let $B \subset A$ be the union of the connected components of A that contain a power of g. Then B is a closed sub group of A that contains both R and g. This means $B = A$ and g generates the component group, implying that the component group is cyclic. Since the component of the identity of A is a compact, connected abelian Lie group, it is a torus. The result now follows from the first statement. □

2.3 The normalizer and Weyl group

Lemma 2.7. *The automorphism group of a torus is a discrete group.*

Proof. An automorphism of a torus T, lifts to a linear automorphism of its Lie algebra which stabilizes the kernel, Λ, of the exponential map. Since a linear isomorphism of a vector space that fixes the lattice Λ point-wise is the identity, we have an embedding of $\mathrm{Auto}(T) \subset \mathrm{Auto}(\Lambda)$. [It is easy to see that these automorphism groups are equal.] There are only countable many automorphisms of a lattice, and hence this is a discrete Lie group. □

Definition 2.8. Let T be a maximal torus of a connected Lie group H. The *Weyl group* $W(H,T)$ of T is defined to be the quotient of the normalizer $N_H(T)$ of T in H by T:

$$W(H,T) = N_H(T)/T.$$

When H is the fixed Lie group G we denote $W(G,T)$ by $W(T)$ by we denote by $N(T)$ the normalizer $N_G(T)$.

Proposition 2.9. *The Weyl group of G is finite and is the component group of $N(T)$.*

Proof. Let $N_0(T)$ be the component of the identity of the normalizer, $MN(T)$, of T. First of all we have a surjection $W(T) = N(T)/T \to N(T)/N_0(T)$ with kernel $N_0(T)/T$. The proposition follows once we show that $N_0(T) = T$.

We suppose that $N_0(T)$ properly contains T and deduce a contraction. Since T and $N_0(T)$ are connected, the Lie algebra $\mathfrak{t}$ of T is properly contained in the Lie algebra of $N_0(T)$. Choose a X in the Lie algebra of $N_0(T)$ that is not contained in $\mathfrak{t}$. Since the automorphism group of the torus is discrete, the Adjoint action of the component of the identity $N_0(T)$ on T is trivial, and consequently the adjoint action of the Lie algebra of $N_0(T)$, and in particular the adjoint action of X, on $\mathfrak{t}$ is trivial. Consider the subspace V of $\mathfrak{g}$ spanned by $\mathfrak{t}$ and X. As we have just argued, $[X, \mathfrak{t}] = 0$. Of course, $[X, X] = 0$. It follows that V is an abelian Lie sub-algebra properly containing $\mathfrak{t}$. The image of V under the exponential map is a connected abelian group A properly containing T. The closure of A is a torus properly containing T, which contradicts the fact that T is a maximal torus. □

Corollary 2.10. *(of the proof) If T is a maximal torus, then its Lie algebra $\mathfrak{t}$ is not properly contained in an abelian sub Lie algebra of $\mathfrak{g}$. If $g \in T$ is a generator of T, then the Lie algebra of the centralizer $Z(g)$ of g is $\mathfrak{t}$.*

Applying the discussion of the first subsection and this corollary, we have the following.

Corollary 2.11. *The action of T decomposes $\mathfrak{g}$ as*

$$\mathfrak{g} = \mathfrak{t} \oplus V_1 \oplus \cdots \oplus V_r$$

where each V_i is two-dimensional and on which T acts by a non-trivial character $\alpha_i \colon T \to S^1$ followed by a standard semi-free rotation action of the circle on V_i.

Remark 2.12. The characters $\alpha_i \colon T \to S^1$ are only defined up to sign, since reversing the orientation of V_i replaces α_i by $-\alpha_i$.

Definition 2.13. The non-zero weights of the action of T on $\mathfrak{g}$; i.e., the non-trivial characters $\{\pm\alpha_i\}$ of the action of T on $\mathfrak{g}$ are the *roots* of (G, T), or simply the roots of G if T is clear from context. The associated two-dimensional subspaces $V_i \subset \mathfrak{g}$ are the *root spaces*, with V_i being the root space for $\pm\alpha_i$.

Proposition 2.14. *We define a Weyl group action on $T^* = \mathrm{Hom}(T, S^1)$ by $w\varphi = \varphi \circ w^{-1}$ so that $\langle w\varphi, wt\rangle = \langle \varphi, t\rangle$ for every $\varphi \in T^*$ and $t \in T$. The Weyl group action on T^* preserves the set of roots.*

Proof. Fix $w \in W(G, T)$. By definition, for $\varphi \in T^*$ and $w \in W(T)$, we have $w \cdot \varphi = \varphi \circ \mathrm{Ad}(w^{-1})$. Consider the composition $\mathrm{ad} \circ \mathrm{Ad}(w^{-1}) \colon T \to$

Auto($\mathfrak{g}$) and the action ad: $T \to$ Auto($\mathfrak{g}$). Since these two actions differ by pre-composition with an automorphism of the torus, these two actions give exactly the same decomposition of $\mathfrak{g}$ as a direct summand of eigenspaces, though the eignevalues associated to the various invariant subspaces may be (and indeed are) different for the two actions. In fact, the non-trivial characters for ad $\circ$ (Ad(w^{-1}) are $\{w\alpha\}$ as α ranges over the roots of T.

On the other hand, for $g \in T$ and $X \in \mathfrak{t}$, the first representation is given by

$$\langle g, X\rangle = w^{-1}gw(X)w^{-1}g^{-1}w = \mathrm{ad}(w^{-1}) \circ \mathrm{ad}(g) \circ \mathrm{ad}(w)(X).$$

That is to say ad $\circ$ Ad(w^{-1}) is the conjugate to ad(g) by ad(w): $\mathfrak{g} \to \mathfrak{g}$. Hence, ad($w$): $\mathfrak{g} \to \mathfrak{g}$ sends an eigenspace for ad $\circ$ Ad(w^{-1}) to an eigenspace for ad with the same character. That is to say the roots of ad are identified with the non-trivial characters, $w\alpha$, of ad $\circ$ Ad(w^{-1}). Thus, we see that ad(w): $\mathfrak{g} \to \mathfrak{g}$ permutes the roots of T. □

2.4 All maximal tori are conjugate and the maximal tori cover G

Theorem 2.15. *Let $T \subset G$ be a maximal torus. Then every point $g \in G$ is contained in a conjugate of T. All maximal tori of G are conjugate.*

Proof. Let $g \in G$. Then $g \in xTx^{-1}$ if and only if $g(xT) = xT$. Said another way, $g \in G$ is in a conjugate of T if and only if, under the natural left action of $G \times G/T \to G/T$, the element g has a fixed point.

Lefschetz theory tells us that if $f\colon M \to M$ is a continuous map of compact oriented manifolds and if $L(f) = \sum_i (-1)^i \mathrm{Trace}(f_*\colon H_i(M;Q) \to H_i(M;Q))$ is non-zero, then f has a fixed point. Of course, $L(f)$ depends only on the homotopy class of f. Furthermore, if f is smooth, has only isolated fixed points, and at each fixed point the graph of f is transverse to the diagonal, then $L(f)$ is the sum over the fixed points of ± 1 that measures the local intersection number of the graph of f with the diagonal. A fixed point $x \in M$ is a transverse intersection of the graph of f and the diagonal if and only if $Df_*(x)$ does not have 1 as an eigenvalue. In this case, the local intersection number of the graph of f and the diagonal at x is sign(det($df_*(x) - \mathrm{Id}$)) as a map of $TM_x \to TM_x$.

Let us compute the $L(g_0)$ for g_0 a generating element of T. First, if $g_0 \in xTx^{-1}$, then since the cyclic group generated by g_0 is dense in T, it follows that $T = xTx^{-1}$, meaning that $x \in N(T)$. Conversely, if $x \in N(T)$ then $T = xTx^{-1}$ and $g_0 \in xTx^{-1}$. Thus, the fixed points of the action of left multiplication by g_0 on G/T are the finite set $W(T) = N(T)/T$.

Let us compute the local Lefschetz number of g_0 at $[T] \in G/T$. We have seen that the adjoint action of T on $\mathfrak{g}$ is given by $V_0 \oplus_{i=1}^r V_i$ where T acts trivially on V_0 and by a non-trivial characters, α_i, on the two dimensional spaces V_i. The tangent space of G/T at $[T]$ is identified with $\oplus_{i=1}^r V_i$. Since $g_0 \in T$, left multplication by g_0 on G/T agrees with the map induced on G/T by the action ofAd(g_0) on G. Hence, the differential of action of g_0 on $T_{eT}(G/T)$ is the restriction of the adjoint action to ad(g_0) to $\oplus_{i=1}^r V_i$ That is to say, $D(g_0 \cdot)_*$ acting on $T_{eT}(G/T) = \oplus_{i=1}^r V_i$ preserves the direct sum decomposition and acts on V_i by

$$\begin{pmatrix} \cos(\alpha_i(g_0) & -\sin(\alpha_i(g_0)) \\ \sin(\alpha_i(g_0) & \cos(\alpha_i(g_0)) \end{pmatrix}$$

Thus

$$\det(Df_*(g_0 \cdot) - \mathrm{Id}) = \prod_{i=1}^r (2 - 2\cos(\alpha_i(g_0))).$$

Since g_0 is a generator of T, each non-trivial character of T is non-trivial on g_0. Hence, the graph of the action of g_0 is transverse to the diagonal and the local intersection number is a sign, $+1$.

Now let $w \in N(T)$. Left multiplication by $w \colon G/T \to G/T$ sends $w^{-1}T$ to eT and conjugates the left multiplication of g_0 to left multiplcation of wg_0w^{-1}. Thus, the local Lefschetz number of left multiplication by g_0 at the fixed point $w^{-1}T$ is the same as the local Lefschetz number of left multiplication of wg_0w^{-1} at eT. Since wg_0w^{-1} is a generator for T, the commutation above showing that the local Lefschetz number of left multiplication by g_0 at eT applies equally well to wg_0w^{-1}, proving that the local Lefschetz number for left multiplication by wg_0w^{-1} at eT is 1. This shows that its local intersection number for left multiplication by g_0 at $w^{-1}T \in G/T$ is $+1$. This is true for every $winW(G,T)$ and consequently, the intersection number $\Gamma(g_0 \cdot \Delta) = \#W(G,T) > 0$.

Now consider an arbitrary $g \in G$. Since G is connected, g and g_0 are connected by a path. Thus, left multiplication by g and g_0 on G/T are homotopic, and hence these actions on G/T have the same Lefschetz number, which we have just seen is non-zero. It follows that $g \colon G/T \to G/T$ has a fixed point xT, meaning that $g \in xTx^{-1}$. This proves that the conjugates of T cover G. Of course, each of these conjugates is a maximal torus.

Now let T' be another maximal torus of G and let g be a generator for T'. Then g is contained in a conjugate xTx^{-1}. Since g generates T', it follows that $T' \subset xTx^{-1}$. Since T' is maximal, $T' = xTx^{-1}$. This proves that all maximal tori are conjugate □

Definition 2.16. The dimension of a maximal torus T in G is the *rank* of G.

2.5 Consequences

Corollary 2.17. *The exponential map* $\exp\colon \mathfrak{g} \to G$ *is onto.*

Proof. Let $T \subset G$ be a maximal torus. Then the Lie algebra $\mathfrak{t}$ is an abelian Lie algebra and the exponential map is surjective homomorphism from $(\mathfrak{t}, +0)$ to T. Thus, T is in the image of exp. We have just seen that every $g \in G$ is contain in a maximal torus. $\square$

Proposition 2.18. *Let $T \subset G$ be a maximal torus. Then the action of $W(T) \times T \to T$ is effective.*

Proof. Let $Z(T) \subset N(T)$ be the centralizer of T. The statement in the proposition is equivalent to the statement that $Z(T) = T$. So suppose there is an element $z \in Z(T) \setminus T$. We have already seen that $W(T)$ is the component group of $N(T)$. This implies that $A = \{z\} \cup T$ generates an abelian group containing T as a subgroup with finite cyclic quotient. According to Corollary 2.6 there is a generator a for A. The element a is contained in a maximal torus, T'. Since a generates A, $A \subset T'$, and *a fortiori* $T \subset T'$ Since $z \in T' \setminus T$, $T \neq T'$. This contradicts the fact that T is a maximal torus. $\square$

Corollary 2.19. *The center of a compact, connected Lie is contained in every maximal torus.*

Proof. Let z be a central element of G. Then z is contained in a maximal torus T. Since every maximal torus is conjugate to T, every maximal torus contains z. This shows that every element of the center is contained in every maximal torus. $\square$

Corollary 2.20. *1. Let A be an abelian subgroup of G containing a torus with cyclic quotient. Then A is contained in a maximal torus.*
2. If $A \subset G$ be an abelian group generated by a connected abelian group A_0 and a single element g, then A is contained in a maximal torus.

Proof. According to Corollary 2.6, in either case the closure of A has a generator. That generator is contained in a maximal torus and hence so is the closure of A. $\square$

Proposition 2.21. *For $g \in G$ the Lie algebra of the centralizer $Z(g)$ is the kernel of* $\mathrm{ad}(g)$*.*

Proof. Suppose that $\mathrm{ad}(g)(X) = 0$. Then g commutes with the one-parameter subgroup generated by X showing that this one-parameter subgroup is contained in $Z(g)$. Since $Z(g)$ is a topologically closed subgroup of G, by Theorem 3.10 of Lecture 2, it is a sub Lie group, This proves that the Lie algebra of $Z(g)$ contains X. The converse is clear. □

Since all maximal tori are conjugate, the roots of the maximal torus, the Weyl group $W(T)$ and its action on T are independent, up to isomorphism, of the choice of maximal torus T.

Definition 2.22. We have defined an effective action of $W(T)$ on T. Taking the differential of this action at the identity gives us a action $W(T) \times \mathfrak{t} \to \mathfrak{t}$ preserving ker(exp). It is also an effective action.

3 Subgroups of G whose derived subgroups are rank 1

3.1 Rank-1 groups

Theorem 3.1. *Let the rank of G be* 1*. Then G is isomorphic to one of the three following groups:*

- S^1
- $SO(3)$
- S^3.

In the second and third case there is one pair of roots for G and the Weyl group is a group of order 2 *acting on the maximal torus by $t \mapsto t^{-1}$.*

Proof. Let G be a rank one group and $T \subset G$ a maximal torus. Then G/T is even dimensional. We claim that this dimension is either 0 or 2. If the dimension is 0, then since G is connected $T = G$ and we have the first of the groups listed above. Suppose that the dimension of G is $n > 0$. Choose a positive definite inner product on $\mathfrak{g}$ that is invariant under the adjoint representation. Denote by $S(\mathfrak{g})$ be the unit sphere about 0 in $\mathfrak{g}$. It is diffeomorphic to S^{n-1}. Let v be a unit vector in $\mathfrak{t}$. The map $\widetilde{\rho}\colon G \to S^{n-1}$ defined by $g \mapsto \mathrm{ad}(g)(v)$ factors to define a smooth map $\rho\colon G/T \to S(\mathfrak{g})$. We claim this map is one-to-one and is a local diffeomorphism. For if $\mathrm{ad}(g_1)(v) = \mathrm{ad}(g_2)(v)$, then $\mathrm{ad}(g_1^{-1}g_2)(v) = v$ and $\mathrm{ad}(g_1^{-1}g_2)$ fixes v and hence $g_1^{-1}g_2$ commutes with T. Since T is a maximal torus his implies that $g_1g_2^{-1} \in T$. A direct computation shows that the kernel of $D\widetilde{\rho}_g$ is the linear subspace of $\mathfrak{t}$ generated by $\widetilde{\rho}(g)$ and hence $D\rho$ is an isomorphism at every point of G/T.

Since both G/T and are compact, connected manifolds of dimension $(n-1)$, it follows that ρ is a diffeomorphism. In particular there is $g \in G$ with $\mathrm{ad}(g)(v) = -v$ so that $g \in N(T)$ and has non-trivial image in $N(T)/T$. It acts on T by sending θ to θ^{-1}. Connecting g by a path $g(t)$ to e, we have a path of homomorphisms $\mathrm{ad}(g(t)) \colon S^1 \to G$ from the identity map to the inverse map. From this it follows that the map $\pi_1(T) \to \pi_1(G)$ sends twice the generator of $\pi_1(T)$ to the trivial element in $\pi_1(G)$. From the long exact sequence of a fibration

$$\pi_2(G/T) \to \pi_1(T) \to \pi_1(G)$$

we conclude that $\pi_2(G/T) \neq 0$. Since G/T is homeomorphic to S^{n-1} we conclude that $n = 3$.

The adjoint map is a homomorphism $G \to SO(\mathfrak{g}) \equiv SO(3)$ with kernel equal to the center of G. Also, the center of G is finite since G has rank 1 and is not abelian. In particular, the adjoint form of G (by definition $G/(\mathrm{center}(G))$) is three dimensional and is a subgroup of $SO(3)$. This shows that the adjoint form of G is $SO(3)$. It follows that $G \equiv SO(3)$ or S^3, the simply connected double cover of $SO(3)$.

The statement about the roots and Weyl group follow immediately. □

3.2 Reflections in $W(T)$

We begin with a tehnical lemma.

Lemma 3.2. *Let K be a compact, connected Lie group. Let $T \subset K$ be a maximal torus and let $H \subset T$ be a normal sub Lie group of K. Then:*

- *The pre-image in K of $N_{K/H}(T/H)$ is $N_K(T)$.*
- *T/H is a maximal torus of K/H.*
- *The map $K \to K/H$ induces an identification*

$$W(K,T) = W(K/H, T/H).$$

Proof. Clearly, if $w \in K$ normalizes T, then the image, $\overline{w}$, of w in K/H normalizes T/H. Conversely, suppose that $\overline{w} \in K/H$ normalizes T/H and let $w \in K$ be a lift of $\overline{w}$. Then for $t \in T$, $wtHw^{-1} = t'H$ for some $t' \in T$. Since H is normal in K, the element w normalizes H. This implies that $wtw^{-1} = t'h'$ for some $h' \in H$. Since $H \subset T$, this implies that $wtw^{-1} \in T$. Since this is true for every $t \in T$, we conclude that that $w \in N_K(T)$. This establishes Item 1.

T/H is a torus in K/H. Let $U/H \subset K/H$ be a maximal torus in K/H containing T/H. Then U/H commutes with T/H and hence normalizes T/H. From Item 1 it follows that $T \subset U \subset N_K(T)$. Since T is a maximal torus of K, $\dim(T) = \dim(N_K(T)$. It follows that $\dim(T) = \dim(U)$ and hence $\dim(T/H) = \dim(U/H)$. Since both these groups are tori, they are equal.

Item 3 is immediate from Items 1 and 2. □

Theorem 3.3. *Let k be the rank of G. Let $T \subset G$ be a maximal torus and let α be a root for G. Let $\hat{U}_\alpha = \ker(\alpha\colon T \to S^1)$ and let U_α be its component of the identity.*

1. *U_α is a codimension-1 torus in T and the component group of $\hat{U}_\alpha$ is a cyclic.*
2. *The component of the identity of the normalizer of U_α, $N_0(U_\alpha)$, has dimension $k+2$ and there are the only roots of G that vanish on U_α, namely $\pm\alpha$.*
3. *T is a maximal torus of $N_0(U_\alpha)$ and $W(N_0(U_\alpha), T) \cong \mathbb{Z}/2\mathbb{Z}$. Let w_α be the non-trivial element of $W(N_0(U_\alpha), T)$. The action of w_α on T fixes U_α and acts by inverse on the quotient T/U_α.*
4. *$\hat{U}_\alpha$ is central in $N_0(U_\alpha)$, so that $N_0(\hat{U}_\alpha) = N_0(U_\alpha)$.*
5. *$\hat{U}_\alpha$ has at most 2 components and w_α also acts on $T/\hat{U}_\alpha$ by inverse.*

Proof. 1). U_α is the component of the identity of $\ker(\alpha)$ and thus is a subtorus of T of codimension-1. The group of components of $\hat{U}_\alpha$ is cyclic group of order equal to the order of divisibilty of α in the group of characters. 2). Since the automorphism group of U_α is finite $N_0(U_\alpha)$ centralizes of U_α. Being a a torus, U_α has a generator g and $N_0(U_\alpha)$ is equal to the centralizer, $Z(g)$, of g. According to Corollary 2.10, the Lie algebra of this centralizer is equal to $\ker(\mathrm{ad}(g))$. This kernel is $\mathfrak{t} \oplus_{\{\beta|\beta(g)=1\}} V_\beta$. In particular, the dimension of $N_0(U_\alpha)$ is $k + \#\{\beta|\beta(g) = 1\}$.

We must show that $\pm\alpha$ are the only roots sending g to the identity, or equivalently the only roots vanishing on U_α. Suppose that there were a

second pair $\{\pm\beta\} \neq \{\pm\alpha\}$ vanishing on g. Then $\mathrm{ad}(g)$ vanishes on $\mathfrak{t}\oplus V_\alpha \oplus V_\beta$ the dimension of $Z(g) = N_0(U_\alpha)$ is at least $k+4$. It follows from Lemma 3.2 that T/U_α is a maximal torus of $N_0(U_\alpha))/U_\alpha$. Thus, $N_0(U_\alpha)/U_\alpha$ is of rank 1 and dimension ≥ 5, contradicting Theorem 3.1. This shows that is exactly one pair of roots that vanish on U_α; namely $\pm\alpha$.
3). Since T is a maximal torus of G contained in $N_0(U_\alpha)$, it is a maximal torus of $N_0(U_\alpha)$. Lemma 3.2 identifies the Weyl group $W(N_0(U_\alpha), T)$ with the Weyl group $W(N_0(U_\alpha)/U_\alpha, T/U_\alpha)$. But $N_0(U_\alpha)/U_\alpha$ is a group of rank one and dimension 3, and hence is isomorphic to $SO(3)$ or S^3, and the non-trivial element in the Weyl group acts by inverse on the maximal torus.
4). Under the adjoint representation $\hat{U}_\alpha$ acts trivially on the root space V_α. Since $\hat{U}_\alpha \subset T$, it commutes with T and its adjoint action on $\mathfrak{t}$ is trivial. Being an extension of finite cyclic group by a torus, $\hat{U}_\alpha$ has a generator, say v and the adjoint action of v on the Lie algebra of $N_0(U_\alpha)$ is trivial. It follows that v is contained in the center of the connected group $N_0(U_\alpha)$ as is its closure $\hat{U}_\alpha$. This shows that $N_0(U_\alpha)$ centralizers, and hence normalizes, $\hat{U}_\alpha$. Clearly any element that normalizes $\hat{U}_\alpha$ also normalizes U_α. It follows that $N_0(U_\alpha) = N_0(\hat{U}_\alpha)$.
5). Since w_α centralizes $\hat{U}_\alpha$ it acts trivially on $\hat{U}_\alpha/U_\alpha \subset T/U_\alpha$. But since the action of w_α on T/U_α is inverse, there are only two fixed points of the action. Thus, $\hat{U}_\alpha/U_\alpha$ has cardinality 1 or 2. Since w_α centralizes $\hat{U}_\alpha$ it acts on the quotient $T/\hat{U}_\alpha$ and the natural projection $T/U_\alpha \to T/\hat{U}_\alpha$ is a covering map. Since w_α acts by inverse on T/U_α, it also acts by inverse on $T/\hat{U}_\alpha$. □

One item is worth restating:

Corollary 3.4. *Let $\alpha\colon T \to S^1$ be a root of G and $w_\alpha \in W(G,T)$ be the reflection in $\alpha^\perp$. Then w_α fixes* $\ker(\alpha) = \exp(\alpha^{-1}(\mathbb{Z}))$.

Remark 3.5. Examining the proof above closely, one can see that we have given a complete description of $N_0(U_\alpha)$. We have the inclusion $i\colon U_\alpha \subset N_0(U_\alpha)$ whose image is a central subgroup. We also have the three dimensional Lie algebra generated by a basis X, Y of V_α. The element $Z = [X,Y] \in \mathfrak{t}$ and the line it generates is complementary to $L(U_\alpha)$. Since the adjoint action of $\mathfrak{t}$ on $\mathfrak{g}$ stabilizes V_α, we see that $\mathbb{R}(Z) \oplus V_\alpha$ is closed under bracket and is the Lie algebra $\mathfrak{so}(3)$. Hence, there is a map of Lie groups $\rho\colon S^3 \to N_0(U_\alpha)$ whose Lie algebra image is exactly this $\mathfrak{so}(3)$. Since $i\colon U_\alpha \subset N_0(U_\alpha)$ is central, we can form the product map

$$i \times \rho\colon U_\alpha \times S^3 \to N_0(U_\alpha).$$

This Lie group homomorphism induces an isomorphism on Lie algebras. The kernel of the homomorphism is a discrete central subgroup A. Since $i\colon U_\alpha \subset N_0(U_\alpha)$ is an injection, $A \cap U_\alpha = \{e\}$. This means that the projection onto S^3 induces an injection from $A \to S^3$ whose image obviously lies in the center of S^3. There are two possibilities for $N_0(U_\alpha)$:

- $A = \{e\}$ and $N_0(U_\alpha) \cong U_\alpha \times S^3$.
- $A \cong \mathbb{Z}/2\mathbb{Z}$ and
$$N_0(U_\alpha) \cong U_\alpha \times_A S^3$$
where the projection to S^3 induces an isomorphism to A to the center of S^3 and the projection of A to U_α is an element of order 2 or 1 of T.

In the first case $\hat{U}_\alpha$ has two components and in the second it has one.

Definition 3.6. The *derived sub-algebra* of a Lie algebra L is the sub-algebra $[L, L]$.

Corollary 3.7. *The derived sub-algebra of the Lie algebra of $N_0(U_\alpha)$ is $\mathfrak{so}(3)$ generated by a basis X, Y of $V_\alpha \subset \mathfrak{g}$. Furthermore, the line spanned by $[X, Y]$ is contained in $\mathfrak{t}$, is invariant under w_α, and the action of w_α on this line is multiplication by -1.*

Proof. Since the Lie algebra of $N_0(U_\alpha)$ is isomorphic to $L(U_\alpha) \oplus \mathfrak{so}(3)$, it is clear that its derived algebra is $\mathfrak{so}(3)$. A direct computation with the presentation of $\mathfrak{so}(3)$ shows that under the decomposition $\mathfrak{so}(3) = \mathfrak{t} \oplus V_\alpha$, the bracket of two elements X, Y forming a basis of V_α lies in $\mathfrak{t}$. The inclusion of $\mathfrak{so}(3)$ into the Lie algebra of $N_0(U_\alpha)$ integrates to give a map $S^3 \to N_0(U_\alpha, T)$ that sends the Weyl group of S^3 isomorphically onto the Weyl group of $W(N_0(_\alpha), T)$. Thus, the line spanned by $[X, Y]$ in $\mathfrak{t}$ is invariant under the Weyl group and the non-trtivial element of this groups acts by -1 on the line spanned by $[X, Y]$. $\square$

Corollary 3.8. *For each pair of roots $\pm\alpha$ of T, there is an element of order two $w_\alpha \in W(G, T)$ that fixes the kernel of $\alpha\colon \mathfrak{t} \to \mathbb{R}$ pointwise and acts by -1 on the quotient $\mathfrak{t}/\ker(\alpha)$.*

Proof. The previous theorem constructs exactly such an element in $W(N_0(U_\alpha), T)$. Of course, $W(N_0(U_\alpha), T)$ is naturally a subgroup of $W(G, T)$. $\square$

Definition 3.9. The element $w_\alpha \in W(G, T)$ associated to $\pm\alpha$ is called the *reflection in* $\ker(\alpha)$.

Corollary 3.10. *Let $T \subset G$ be a maximal torus. Any element in $g \in T$ that is not in the kernel of any root is contained in no maximal torus distinct from T. If $g \in \ker(\alpha) \subset T$ for some root α, then g is contained in at least two distinct maximal tori.*

Proof. If $g \in T$ is not contained in the kernel of any root, then g acts non-trivially on each root space V_α. Thus, the subspace of $\mathfrak{g}$ on which $\mathrm{ad}(g)$ acts by the identity is $\mathfrak{t}$. By Proposition 2.21, this means that the Lie algebra of the centralizer of g is $\mathfrak{t}$. This means that the component of the identity of the centralizer of g is T. Obviously, then g is contained in only one maximal torus, which is T.

Now suppose that $g \in \ker(\alpha)$. Then $g \in \hat{U}_\alpha$ which is contained in the center of $N_0(U_\alpha)$. Thus, g is contained in every maximal torus $N_0(U_\alpha)$. Since T is a maximal torus of $N_0(U_\alpha)$, all these tori are maximal tori of G. Since $\dim(N_0(U_\alpha) > \dim(T)$, there is more than one maximal torus of $N_0(U_\alpha)$. (In fact, there is a two-dimensional family of them.) □

4 Weyl Chambers and the Weyl Group action

For each pair of roots $\pm\alpha$ there is the torus U_α, which is the component of the identity of $\ker(\alpha\colon T \to S^1)$, and its Lie algebra $W_\alpha \subset \mathfrak{t}$. Each W_α is a codimension-1 linear subspace of $\mathfrak{g}$. The adjoint action of w_α on $\mathfrak{t}$ fixes W_α and interchanges the half-spaces that W_α divides $\mathfrak{t}$ into. The W_α are the *walls* of the Weyl group action.

Definition 4.1. Let $T \subset G$ be a maximal torus and let $R \subset \mathrm{Hom}(\mathfrak{t}, \mathbb{R})$ be the set of roots for (G, T). A *Weyl chamber* is a connected component of $\mathfrak{t} \setminus (\cup_{\alpha \in R} W_\alpha)$ where the wall W_α is the kernel of the root $\alpha\colon \mathfrak{t} \to \mathbb{R}$. Notice that $W_\alpha = W_{-\alpha}$. The hyperplane W_α is a *wall of a Weyl chamber C* iff intersection of the closure of C and W_α contains a non-empty open subset of W_α. If W_α is a wall of a chamber C, then there is a unique chamber $C' \neq C$ such that $\overline{C}' \cap W_\alpha = \overline{C} \cap W_\alpha$. Those chambers are interchanged by the element in the Weyl group that is the reflection in W_α.

4.1 Weyl chambers as intersections of half-spaces

Definition 4.2. A straight-line segment $\omega\colon [0, 1] \to \mathfrak{t}$ is *generic* if:

- each of its endpoints is contained in a chamber, and
- it is disjoint from all the codimension-2 intersections of walls, $W_\alpha \cap W_\beta$ for roots α and $\beta \neq \pm\alpha$.

Lemma 4.3. *Let $\omega\colon [0,1] \to \mathfrak{t}$ be a straight-line segment with end points in chambers $\omega(0) \in C$ and $\omega(1) \in C'$. Then there is an arbitrarily close, generic straight-line segment. It has end points in the same chambers as ω does.*

Proof. Fixing one endpoint, the condition that a straight-line segment meet a codimension-2 linear subspace is a single linear condition on the other endpoint of ω. The result follows easily. $\square$

Proposition 4.4. *Let C be a chamber and let $W_{\alpha_1} \dots W_{\alpha_k}$ be the walls of C, with the α_i chosen so that $\alpha_i|_C > 0$. Then*

$$C = \cap_{i=1}^{k}\{\alpha_i > 0\}.$$

Lemma 4.5. *Let ω is a generic path with $\omega(0) \in C$. If ω crosses a wall, then the first wall it crosses is a wall of C.*

Proof. Suppose that ω crosses a wall and let $t \in (0,1]$ be the smallest number with the property that $\omega(t)$ is in a wall, say W_β with $\beta > 0$ on C. Clearly, $\omega(s) \in C$ for $s \in [0,t)$ and $\omega(t) \notin C$, so that $\omega(t) \in \overline{C}$. Since ω is generic, $\omega(t)$ does not lie in any $W_\beta \cap W_\gamma$ for any root $\gamma \neq \pm\beta$. Thus, there is an open subset $U \subset \mathfrak{t}$ around $\omega(t)$ that meets no wall except W_β. In addition, we can suppose that U meets W_β in a connected set and meets each of $\beta > 0$ and $\beta < 0$ in a connected set. The intersection of U with the open half-space $\{\beta > 0\}$ contains $\omega(t - \epsilon)$ for all $\epsilon > 0$ sufficiently small. Thus, $U \cap \{\beta > 0\}$ is contained in C. It follows that $\overline{C} \cap W_\beta$ contains $U \cap W_\beta$, and hence that W_β is a wall of C. $\square$

Proof. (of the proposition) It is clear that $C \subset \cap_{i=1}^{k}\{\alpha_i > 0\}$.

We must prove the converse. Arguing by contradiction, suppose that C is properly contained in $\cap_{i=1}^{k}\{\alpha_i > 0\}$. Then some wall W_β must meet $\cap_{i=1}^{k}\{\alpha_i > 0\}$. For if not, then $\cap_{i=1}^{k}\{\alpha_i > 0\}$ is contained in a chamber which, since it contains C, must then be C.

Thus, there is a root $\beta \notin \{\alpha_1, \dots, \alpha_k\}$ with $\beta|_C > 0$ and with

$$W_\beta \cap_{i=1}^{k} \{\alpha_i > 0\} \neq \emptyset.$$

Let q be a point of this intersection. Then there is a generic path with one endpoint $p \in C$ and the other endpoint arbitrarily close to q with $\beta(q) < 0$. This path is contained in $\cap_{i=1}^{k}\{\alpha_i > 0\}$. As such it crosses no wall of C. But it crosses W_β. This contradicts Lemma 4.5. $\square$

4.2 The Weyl Group Acts Simply Transitively on the set of Weyl Chambers

Theorem 4.6. *The action of the Weyl group on* $\mathfrak{t}$ *sends Weyl chambers to Weyl chambers. The Weyl group acts simply transitively on the set of Weyl chambers.*

Proof. According to Proposition 2.14 the action of the Weyl group preserves the set of roots and hence stablizes the $\cup_{\alpha\in R}W_\alpha$. Consequently, it preserves the union of the Weyl chambers and hence permutes the chambers.

Fix an Weyl-invariant metric on $\mathfrak{t}$. Then the reflection element associated with U_α is an orthogonal reflection in the codimension 1 linear subspace W_α of $\mathfrak{t}$ in $W(T)$. Notice that each Weyl chamber is the intersection of a finite collection of open half spaces, the half-spaces determined by the codimension-1 linear subspaces $\{\alpha_i > 0\}$ as α_i range over the roots that are positive on the Weyl chamber and whose walls are walls of C. As such, each Weyl chamber is a convex subset of $\mathfrak{t}$.

Suppose that C is a chamber fixed by $w \in W(T)$ with $w \neq e$. Let $n > 1$ be the order of w. Let $v \in C$ and consider the average

$$\hat{v} = \frac{1}{n}\sum_{k=1}^{n} w^k v.$$

Since $w^k v \in C$ for all k and C is convex, $\hat{v} \in C$ and is invariant under w.

Let $H = \exp(t\hat{v})$ and $\widetilde{w} \in N_G(T)$ be a lift of w. Since $w \cdot \hat{v} = \hat{v}$, the element $\widetilde{w}$ commutes with H. This means that the group generated by H and $\widetilde{w}$ is an abelian group, as is its closure K. Some power of $\widetilde{w}$ is contained in the component of the identity of K. Also, H is contained in the component of the identity of K. Thus, the component group of K is cyclic. It follows from Corollary 2.20 that K is contained in a maximal torus T'. Since $w \in W(T)$ and $w \neq e$, if follows that $\widetilde{w} \notin T$, and, as a result, $T \neq T'$. Thus, H is contained in two distinct maximal tori. That implies that $H \subset \cup_\alpha \hat{U}_\alpha$. Since he intersection of H with each component of each $\hat{U}_\alpha$ is a closed subset, for some $\epsilon > 0$ and for some root α $\exp(t\hat{v})$ is contained in U_α for all $t \in [0, \epsilon)$. Hence, that $\hat{v} \in W_\alpha$. This is a contradiction, showing that no non-trivial element of $W(T)$ fixes any Weyl chamber.

For any W_α, by Corollary 3.8 there is a reflection in W_α in $W(T)$. Notice that if $\beta \neq \pm\alpha$, then by Theorem 3.3 W_β and W_α are distinct hyperplanes so that their intersection is of codimension 2. Let C and C' be Weyl chambers. There is a generic straight-line segment γ in $\mathfrak{t}$ from a point of C to a point of C'. Enumerate in order the chambers that γ meets $C = C_0, C_1, \ldots, C_k = C'$

and let $\mathcal{W}_{\beta_i}$ be the wall that γ crosses in going from C_i to C_{i+1}. Then the closures of C_i and C_{i+1} each contain a non-empty open neighborhood of $\gamma \cap W_{\beta_i}$ and thus W_{β_i} is a wall of both C_i and C_{i+1}. Let $w_{\beta_i} \in W(T)$ be the reflection in W_{β_i}. Then w_{β_i} maps C_i to C_{i+1}. We easily see by induction that there is an element of $W(T)$ that sends C_0 to C_i for every $i \leq n$. In particular, there is an element $w \in W(T)$ that sends $C = C_0$ to $C' = C_n$. This show that the action of $W(T)$ on the set of Weyl chambers is transitive. □

Corollary 4.7. *The reflections w_α in the roots α generate the Weyl group.*

Proof. Fix a Well chamber C_0. By the previous theorem the map

$$W(T) \to \text{Weyl Chambers}$$

defined by $w \mapsto wC_0$ is a bijection. On the other hand, in the proof of the theorem we constructed a product of reflections in walls that sent C_0 to any other chamber, so that the subgroup of $W(T)$ generated by the reflections in the walls also acts transitively on the chambers. Hence, this subgroup must be all of $W(T)$. □