Lie Groups: Fall, 2024
Lecture VIII:
Root Systems

October 31, 2024

For this section we fix a compact connected Lie group G, a maximal torus $T \subset G$. We have the action of the Weyl group W on T and the induced linear action of W on $\mathfrak{t}$. We define the dual action $W \times \mathfrak{t}^* \to \mathfrak{t}^*$ by $w\varphi(X) = \varphi(w^{-1}x)$.

We fix a Weyl invariant metric, denoted $\langle \cdot, \cdot \rangle$. This allows us to identify $\mathfrak{t}$ and $\mathfrak{t}^*$ by $x \mapsto \langle x, \cdot \rangle$. We then transport the metric from $\mathfrak{t}$ to $\mathfrak{t}^*$ by this identification. The identification of $\mathfrak{t}$ with $\mathfrak{t}^*$ is Weyl-equivariant. For $\varphi \in \mathfrak{t}^*$, we denote by $x_\varphi \in \mathfrak{t}$ the corresponding element. It satisfies $\langle x_\varphi, y \rangle = \varphi(y)$. With these choices, we see that $\alpha^\perp \subset \mathfrak{t}^*$ is identified with $\mathrm{Ker}(\alpha) = x_\alpha^\perp \subset \mathfrak{t}$.

Let $Z \subset T \subset G$ be the center of G. We denote by $\Lambda \subset \mathfrak{t}$ be $\exp^{-1}(Z)$. It is the *co-weight subspace*. The *weight space*, Λ^* in $\mathfrak{t}^*$, is the dual to Λ,; i.e., the set of $\lambda \in \mathfrak{t}^*$ that take integral values at every point of Λ. The adjoint action of the Weyl group W on T presevres the center. Hence, the induced action of W on $\mathfrak{t}$ preserves $\Lambda \subset \mathfrak{t}$ and the dual action of W on $\mathfrak{t}^*$ preserves Λ^*.

Claim 0.1. *Every root is a weight.*

Proof. Every root $\alpha\colon T \to S^1$ vanishes on the center and hence every $|root\colon \mathfrak{t} \to \mathbb{R}$ takes integral values on Λ. □

Claim 0.2. *The action of the reflection w_α on $\mathfrak{t}^*$ is given by*

$$\varphi \mapsto \varphi - \frac{2\langle \alpha, \varphi \rangle \alpha}{\langle \alpha, \alpha \rangle}.$$

The action of w_α on $\mathfrak{t}$ is given by

$$X \mapsto h - \frac{2\alpha(X))x_\alpha}{\langle \alpha, \alpha \rangle}.$$

Proof. Let w'_α be the map defined by the formula in the first equation of the claim. Clearly, w'_α a linear map $\mathfrak{t}^* \to \mathfrak{t}^*$ that is the identity on $\alpha^\perp$. But under the identification of $\mathfrak{t}$ with $\mathfrak{t}^*$, the subspace $\alpha^\perp$ is tidentiied with he kernel of the root $\alpha\colon \mathfrak{t} \to \mathbb{R}$. On the other hand, one checks immediately that $(w'_\alpha)^2 = \mathrm{Id}$, that w'_α interchanges the half-spaces $\langle\alpha, \cdot\rangle > 0$ and $\langle\alpha, \cdot\rangle < 0$, and that w'_α is an isometry. Thus, w'_α is the unique orthogonal reflection in the codimension-1 subspace $\alpha^\perp$. These properties also characterize w_α, so that $w_\alpha = w'_\alpha$. This proves the first statement.

The second, dual, statement is prove analogously. □

Claim 0.3. *For every root α and every weight $\lambda \in \Lambda^*$, we have*

$$-\frac{2\langle\alpha, \lambda\rangle}{\langle\alpha, \alpha\rangle} \in \mathbb{Z}.$$

Proof. (Following Adams) Fix a root α and choose $v \in L(T)$ a vector with $\alpha(v) = 1$. Then $\exp(v) \in \ker(\alpha\colon T \to S^1)$. Hence, is fixed by w_α. This means that $w_\alpha(v) - v = \frac{-2\alpha(v)\alpha}{\langle\alpha,\alpha\rangle}$ is in Λ. Hence, for any weight $\lambda \in \Lambda^*$ we have

$$\frac{-2\alpha(v)\langle\alpha, \lambda\rangle}{\langle\alpha, \alpha\rangle} \in \mathbb{Z}.$$

Since $\alpha(v) = 1$, we see that

$$-\frac{2\langle\alpha, \lambda\rangle}{\langle\alpha, \alpha\rangle} \in \mathbb{Z}..$$

□

Claim 0.4. *For roots α and β we have*

$$\beta - \frac{2\langle\alpha, \beta\rangle\alpha}{\langle\alpha, \alpha\rangle}$$

is a root, and

$$\frac{-2\langle\alpha, \beta\rangle}{\langle\alpha, \alpha\rangle} \in \mathbb{Z}$$

Proof. The second statement follows from Claim 0.3. The first statement follows from Claim 0.2 and the fact that the Weyl group action preserves the roots. □

1 Root Systems

In this section we formalize the properties that we established in the previous section and the previous lecture.

Definition 1.1. A *root system* consists of a finite dimensional real vector space V and a finite set of non-zero vectors $R = \{\alpha_1, \dots, \alpha_k\}$ of V, *the roots,* a finite group W, *the Weyl group,* acting linearly and effectively on V and a positive definite W-invariant inner product on V, satisfying the following properties:

- If α is a root then so is $-\alpha$ but no other real multiple of α is a root.
- W preserves the set of roots.
- For each $\alpha \in R$ there is an element $w_\alpha \in W$ that is the reflection in the hyperplane orthogonal to α.
- The reflections associated with the roots generate W.
- For each pair of roots α, β we have

$$-\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}.$$

Theorem 1.2. *For any compact, connected Lie group G with maximal torus T, the data* ($\mathfrak{t}^*$, roots of T, $W(T)$, W-invariant metric) *form a root system.*

Proof. The first four properties are established in Theorem 3.5, Proposition 2.16, Corollary 3.7, and Corollary 4.3, respectively, of the previous lecture. The last property is established in the previous section of this lecture. □

2 Properties of Root Systems

We fix a root system $(V, R, W, \langle\cdot,\cdot\rangle)$.

Lemma 2.1. *Let α, β be a pair of roots with $\beta \neq \pm\alpha$. Then*

$$0 \leq \left(\frac{-2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\right)\left(\frac{-2\langle\beta,\alpha\rangle}{\langle\beta,\beta\rangle}\right) < 4.$$

Proof. This is an immediate consequence of the fact that since α and β are not multiples of each other we have $\langle\alpha,\beta\rangle^2 < \langle\alpha,\alpha\rangle\langle\beta,\beta\rangle$. □

Corollary 2.2. *Let α, β be roots with $\beta \neq \pm\alpha$ and with $\langle \alpha, \beta \rangle \leq 0$. Then either $\langle \alpha, \beta \rangle = 0$ or one of*

$$\frac{-2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \quad \text{or} \quad \frac{-2\langle \beta, \alpha \rangle}{\langle \beta, \beta \rangle}$$

is equal to 1 and the other takes value in the set $\{1, 2, 3\}$.

Proposition 2.3. *Suppose $\beta \neq \pm\alpha$.*
1. If $\langle \alpha, \beta \rangle = 0$, then the angle between α and β is $\pi/2$, the reflections in the hyperplanes perpendicular to α and β commute and generate dihedral group of order 4.
2. Suppose that

$$\frac{-2\langle \beta, \alpha \rangle}{\langle \beta, \beta \rangle} = 1,$$

then the angle between α and β is $\pi/3$, $\pi/4$, or $\pi/6$ depending on whether

$$v = \frac{-2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} = 1, \quad 2, \quad \text{or } 3.$$

Also, $\frac{|\beta|}{|\alpha|} = \sqrt{v}$. In these cases the reflections in the hyperplanes perpendicular to α and β generate a dihedral group of order 6, 8, or 12.
3. Furthermore, under the hypothesis of previous statement, $\beta + k\alpha$ is a root for all

$$0 \leq k \leq \frac{-2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}.$$

Proof. The first statement is a tautology. Let us consider Statement 2. Let θ be the angle between α and β, so that $0 \leq \theta \leq \pi/2$. Set

$$v = \frac{-2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle\rangle}.$$

Then

$$\cos^2(\theta) = \frac{v}{4},$$

or

$$\cos(\theta) = \frac{\sqrt{v}}{2}.$$

Statement 2 is now clear.

If $\frac{-2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle\rangle} = 1$, then the reflection of β is the hyperplane perpendicular to α is $\beta + \alpha$.

If $\frac{-2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle\rangle} = 2$, then reflection of α in the hyperplane perpendicular to β is $\beta+\alpha$, whereas the reflection of β in the hyperplane perpendicular to α is $\beta+2\alpha$.

Finally, if $\frac{-2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle\rangle} = 3$, the reflection of α in the hyperplane perpendicular to β is $\alpha+\beta$, and reflection of $\alpha+\beta$ in the hyperplane perpendicular to α is $\beta+2\alpha$. Lastly, reflection of β in the hyperplane perpendicular to α is $\beta+3\alpha$.

Since the roots are invariant under the Weyl group action, this establishes Statement 3 in all cases. □

3 Examples

3.1 $SU(n)$

$SU(n)$ is the real sub Lie group of $GL(n,\mathbb{C})$, consisting of all matrices such that $\overline{A}^{\text{tr}} = A^{-1}$ and $\det(A) = 1$. Notice that this is not a complex Lie group since the first equation, which involves conjugation, is not holomorphic equation. Indeed $SU(2)$ is isomorphic to S^3.

The usual maximal torus for $SU(n)$ consists of the matrices in $SU(n)$ that have non-zero entries only along the diagonal.

$$\begin{pmatrix} \theta_1 & 0 & \cdots & 0 \\ 0 & \theta_2 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \theta_n \end{pmatrix}$$

where the $\theta_i \in S^1$ and $\prod_{i=1}^n \theta_i = 1$.

The roots are $\{\alpha_{i,j}\}_{1\leq i\neq j\leq n}$ where $\alpha_{i,j}(\theta_1,\ldots,\theta_n) = \theta_i\theta_j^{-1}$. The associated reflection $w_{i,j}$ interchanges θ_i and θ_j, and the Weyl group they generate is the symmetric group permuting the n coordinates.

The Lie algebra $L = \mathfrak{su}(n)$ is the subspace of $\mathbb{R}^n$ consisting of

$$\{(t_1,\ldots,t_n) | \sum_i t_i = 0\}.$$

Passing to L the differential of $\alpha_{i,j}$ is the root $t_i - t_j \colon \mathfrak{t} \to \mathbb{R}$. The co-weight lattice is $L\cap\mathbb{Z}^n$. Each root acts by permuting two of the coordinates. Thus, the Euclidean metric in the coordinates $(t_1,\ldots,t_{n+1})$ restricts to a Weyl invariant metric on $\mathfrak{t}$. In this metric, $\langle t_i - t_j, t_i - t_j\rangle = 2$ for all $i \neq j$ and

$\langle t_i - t_j, t_k - t_l \rangle$ is equal to ± 1 if the sets $\{i,j\}$ and $\{k,l\}$ have exactly one element in common and 0 if the sets $\{i,j\}$ and $\{k,l\}$ are disjoint.

The dual vector space to L is $\mathbb{R}^n/\mathbb{R}(1,1,\ldots,1)$ and the weight lattice (the dual to the lattice in $\mathfrak{t}^*$ spanned by the roots) is the quotient of $\mathbb{Z}^n/\mathbb{Z}(1,1,\ldots,1)$. The weight lattice consists of all $(a_1,\ldots,a_n) \in L$ with $na_i \in \mathbb{Z}$ for all i, with $a_i \equiv a_j \pmod{\mathbb{Z}}$ for all i,j. This contains the co-weight lattice as a sub lattice with quotient a cyclic group of order n generated by the element $(1/n, 1/n, \ldots, 1/n)$.

A Weyl chamber is

$$\{(t_1,\ldots,t_n) | \sum_i t_i = 0 \text{ and } t_1 > t_2 \cdots > t_n\}.$$

Its walls are given by the equations $\{t_i = t_{i+1}\}$, for $1 \leq i \leq n-1$, which are the kernels of the roots $(t_1 - t_2), \ldots, (t_{n-1} - t_n)$. The other Weyl chambers are given by $\{t_{\sigma(1)} > t_{\sigma(2)} > \cdots > t_{\sigma(n)}\}$ as σ ranges over the permutations of $\{1,\ldots,n\}$.

3.2 $SO(2n)$ for $n \geq 3$

$SO(2n)$ is the sub Lie group of $GL(n,\mathbb{R})$ consisting of all matrices that satisfy $A^{tr} = A^{-1}$ and $\det(A) = 1$. Its standard maximal torus consists of all 2×2 block diagonal matrices

$$\begin{pmatrix} \begin{pmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{pmatrix} & & & \\ & \begin{pmatrix} \cos(\theta_2) & -\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{pmatrix} & & \\ & & \ddots & \\ & & & \begin{pmatrix} \cos(\theta_n) & -\sin(\theta_n) \\ \sin(\theta_n) & \cos(\theta_n) \end{pmatrix} \end{pmatrix}$$

where the $\theta_i \in S^1$.

The Lie algebra of $SO(2n)$ is the space of skew symmetric matrices. The tangent space to the maximal torus described above is the space of matrices of the form

$$\begin{pmatrix} \begin{pmatrix} 0 & -t_1 \\ t_1 & 0 \end{pmatrix} & & & \\ & \begin{pmatrix} 0 & -t_2 \\ t_2 & 0 \end{pmatrix} & & \\ & & \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} & \\ & & & \begin{pmatrix} 0 & -t_n \\ t_n & 0 \end{pmatrix} \end{pmatrix}$$

The rest of the Lie algebra decomposes into four dimensional subspaces of skew symmetric matrices. Each of these subspaces consists of a 2×2 block with columns in $2s-1, 2s$ and rows $2r-1, 2r$, with $r > s$ together with the negative of the transpose below the diagonal of this block. We decompose one of these blocks as direct sum of matrices of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

and matrices of the form

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix}.$$

These subspaces are invariant under the action of the torus and these root spaces have roots $(\theta_r\theta_s^{-1})^{\pm 1}$ and $(\theta_r\theta_s)^{\pm 1}$ respectively. Thus, the roots of T are $\theta_r^{\pm 1}\theta_s^{\pm 1}$ with $1 \leq r < s \leq n$.

The Lie algebra $\mathfrak{t}$ of this maximal torus is identified with $\mathbb{R}^n$ with coordinates $t_1, \ldots, t_n$. Under passage to $\mathfrak{t}$ the roots become the linear functionals $\pm t_r \pm t_s$ for $1 \leq r < s \leq n$. Each root acts by exchanging a pair of the t_i coordinates, possibly also changing the sign of each exchanged coordinate. Thus, the Euclidean metric in the coordinates $(t_1, \ldots, t_n)$ is Weyl-invaraint. The weight lattice is

$$\{a_1, \ldots, a_n)\big| 2a_i \in \mathbb{Z} \text{ and } a_i \equiv a_j \quad (\mathbb{Z})\}.$$

The co-weight lattice is $\mathbb{Z}^n \subset \mathbb{R}^n$. Thus, the center of $SO(2n)$ is $\mathbb{Z}/2\mathbb{Z}$. As in the case of $SU(n)$ all the roots have the same length and the possible angles between two roots α and $\beta \neq \pm\alpha$ are $\pi/2$ and $\pi/3$.

A Weyl chamber in $\mathfrak{t}$ is given by $\{t_1 > t_2 > \cdots > t_{n-1} > t_n\}$ and $\{t_{n-1} + t_n > 0\}$. The walls of this chamber are the kernel of the roots

$$\{t_1 - t_2, t_2 - t_3, \ldots, t_{n-1} - t_n, t_{n-1} + t_n\}.$$

Consider the semi-direct product of $\prod_{i=1}^{n} \mathbb{Z}/2\mathbb{Z}$ with the permutation group Σ_n, where i^{th} factor of the product is reflection of the t_i coordinate, and Σ_n acts by permuting the factors. The Weyl group is $R \rtimes \Sigma_n$, where $R \subset \prod_{i=1}^{n} \mathbb{Z}/2\mathbb{Z}$ is the subgroup of index 2 that reverses an even number of coordinate directions. This is the group generated by the reflections in the walls of the given Weyl chamber.

3.3 $SO(2n+1)$ for $n \geq 2$

As before this group consists of the matrices in $GL(2n+1, \mathbb{R})$ that satisfy $A^{tr} = A^{-1}$ and $\det(A) = 1$. The Lie algebra consists of the skew-symmetric matrices in $M((2n+1) \times (2n+1), \mathbb{R})$. The standard maximal torus is the image under the natural embedding $SO(2n) \subset SO(2n+1)$ of the standard maximal torus for $SO(2n)$. Thus, all the root spaces with roots $\theta_r^{\pm 1}\theta_s^{\pm 1}$ for $1 \leq r < s \leq n$ are roots of the maximal torus for $SO(2n+1)$ and the sum of $\mathfrak{t}$ and these roots spaces form $\mathfrak{so}(2n) \subset \mathfrak{so}(2n+1)$. The additional root spaces consist of skew symmetric matrices that are zero except in their last row and column. The dimension of this space is $2n$ and hence there are n more pairs of roots for $T \subset SO(2n+1)$ that are not roots coming from $SO(2n)$. Direct computation shows that these roots are $\theta_n^{\pm 1}$ for $1 \leq r \leq n$.

The Lie algebra of the maximal torus of $\mathfrak{so}(2n+1)$ is equal to the Lie algebra of the maximal torus of $SO(2n)$. Hence, there are coordinates $(t_1, \ldots, t_n)$ for the Lie algebra of this maximal torus just as for the maximal of $SO(2n)$. The roots are $\pm t_i \pm t_j$ for $1 \leq i < j \leq n$ and $\pm t_i$ for $1 \leq i \leq n$. This is the first time we encounter roots of different lengths. The long roots act by permutation of two of the t_i, possibly with a sign change of the exchanged variables. The short roots act by sign change of one of the t_i. Thus, the Euclidean metric on the coordinates $(t_1, \ldots, t_n)$ is Weyl invariant. In this metric the roots $\pm t_r$ all have length 1 and the roots $\pm t_r \pm t_s$ have length $\sqrt{2}$. For $r \neq s$ the angle between t_r and t_s is $\pi/2$. The angle between t_r and and $\pm t_r \pm t_s$ is $\pi/4$. The angle between the roots $t_r + t_s$ and $t_r - t_s$ is $\pi/2$. A Weyl chamber is given by $t_1 > t_2 \cdots > t_n > 0$. The walls of this Weyl chamber are given by the kernels of the roots $t_1 - t_2, \ldots, t_{n-1} - t_n, t_n$. The Weyl group is $\left(\prod_{i=1}^{n} \mathbb{Z}/2\mathbb{Z}\right) \rtimes \Sigma_n$, where the i^{th} factor acts by reversal of the i^{th} coordinate.

The co-weight lattice is $\mathbb{Z}^n \subset \mathbb{R}^n$ and the root lattice is dual to the co-weight lattice, so that the weight lattice is equal to the co-weight lattice. This means that the center of $SO(2n+1)$ is the trivial group.

3.4 The compact symplectic groups

Let K be a field of characteristic 0 and let V be a finite dimensional K-vector space. A *symplectic form* on V is a non-degenerate skew symmetric pairing $V \otimes V \to K$. Such a pairing exists if and only if the dimension of V is even. In fact, for any such pairing there is a basis $\{e_1, \ldots, e_{2n}\}$ for V so that the pairing, denoted $\langle \cdot, \cdot \rangle$ is given by

$$\langle e_{2i-1}, e_{2j-1} \rangle = \langle e_{2i}, e_{2j} \rangle = 0 \quad \text{for all } 1 \leq i, j \leq n$$

and

$$\langle e_{2i-1}, e_{2j} \rangle = -\langle e_{2j}, e_{2i-1} \rangle = \delta_{i,j} \quad \text{for all } 1 \leq i, j \leq n.$$

Writing elements of V as column vectors using the basis $\{e_1 \ldots, e_{2n}\}$ the form is given by

$$\langle x, y \rangle = x^{tr} I y$$

where I is the block diagonal matrix with 2×2 blocks

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

down the diagonal. Then an element $A \in GL(2n, F)$ is in the symplectic group if and only if

$$A^{tr} I A = I.$$

This algebraic group over K is denoted $Sp(2n, K)$

We are especially interested in $Sp(2n, \mathbb{R})$, which is a real Lie group, and $Sp(2n, \mathbb{C})$, which is a complex Lie group. These groups are not compact. Indeed, $Sp(2, \mathbb{R}) = SL(2, \mathbb{R})$. The compact form of the symplectic group, denoted $Sp(n)$ is the intersection $Sp(2n, \mathbb{C}) \cap U(2n)$ in $GL(2n, \mathbb{C})$. Clearly, this is a compact real Lie group.

There is a different description of $Sp(n)$. First we take the usual positive definite hermitian inner product on $\mathbb{C}^{2n}$, and the resulting isometry group is the unitary group $U(2n)$. We choose a slightly different symplectic form. In the standard basis $\{e_1, \ldots, e_{2n}\}$ the complex symplectic form is given by

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

where I is the $n \times n$ identity complex matrix.

Next we define the structure of a quaternion (left) vector space structure on $\mathbb{C}^{2n}$ extending the given complex structure by defining $j(\zeta e_i) = \overline{\zeta} e_{i+n}$

for all $1 \leq i \leq n$ and $\zeta \in \mathbb{C}$. It follows that $j\zeta e_{n+i} = -\overline{\zeta} e_i$ for all $1 \leq i \leq n$ and $\zeta \in \mathbb{C}$. Multiplication by i, j, k are all isometries of the positive definite inner product.

One of the exercises is to show that a unitary matrix (acting from the left) commutes with the quaternion structure (i.e., commutes with left multiplication by j) if and only if it is an element of the complex symmetric group defined by I. This then tells us that $Sp(n)$ is the group of automorphisms (acting on from the left) of the left quaternion vector space structure that we have defined on $\mathbb{C}^{2n}$ that preserve the positive definite hermitian inner product.

Taking the second description, we see that the Lie Algebra of $Sp(n)$ is the sub Lie algebra of $\mathfrak{u}(2n)$ consisting of the matrices $M(2n \times 2n, \mathbb{C})$ that satisfy

$$\begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}$$

with $A \in \mathfrak{u}(n)$ and B is symmetric. The standard maximal torus of $Sp(n)$ consists of subspace matrices with diagonal entries of norm 1 such that for $1 \leq i \leq n$ the $(n+i, n+i)$ diagonal entry equal to the inverse of the (i, i) diagonal entry.

The adjoint action of the maximal torus on the sub Lie algebra given by $B = 0$ is the usual adjoint action of the maximal torus of $U(n)$ on its Lie algebra so that the roots for this action are $(\theta_i \theta_j^{-1})^{\pm 1}$ for $1 \leq i < j \leq n$. The adjoint action of the maximal torus on an element in position $(i, n+j)$ is $\theta_i \theta_{n+j}^{-1} = \theta_i \theta_j$. Thus, the roots for the action of the maximal torus on B are $(\theta_i \theta_j)^{\pm 1}$ for all $1 \leq i \leq j \leq n$.

The maximal torus is identified with $\prod_{i=1}^{n} S^1$ given by the coordinates $(\theta_1, \ldots, \theta_n)$. the differential of this map is an identification of the Lie algebra of the maximal torus with $\mathbb{R}^n$ with coordinates $(t_1, \ldots, t_n)$. On the Lie algebra of the maximal torus, there are $2n(n-1)$ roots of the form $\pm t_i \pm t_j$ for $1 \leq i < j \leq n$, the short roots, and $2n$ roots of the form $\pm 2t_i$, the long roots. Every short root acts by permutation of two of $\{t_1, \ldots, \pm t_n\}$, possibly reversing the sign of the exchanged coordinates. The long roots act by changing the sign of one of the coordinates. Hence, the usual Euclidean metric on the coordinates $(t_1, \ldots, t_n)$ is Weyl invariant. Under this inner product the first group of roots have length $\sqrt{2}$, then the second group have length 2. As in the case of $SO(2n+1)$ the angles between the roots are $\pi/2, \pi/3$, and $\pi/4$.

A Weyl chamber is given by $t_1 > t_2 > \cdots > t_n > 0$. Its walls are given by the vanishing hyperplanes of the roots $\{(t_1 - t_2), (t_2 - t_3), \ldots, (t_{n-1} - t_n), 2t_n\}$.

The Weyl group $\prod_{i=1}^{n} \mathbb{Z}/2\mathbb{Z} \rtimes \Sigma_n$ just as in the case of $SO(2n+1)$. The co-weight lattice is the usual integral lattice in the coordinate $(t_1, \ldots, t_n)$ whereas the weight lattice is given by the equations $t_i \cong t_j \pmod{\mathbb{Z}}$ and $2t_i \in \mathbb{Z}$. It follows that the center of $Sp(n)$ is $\mathbb{Z}/2\mathbb{Z}$.

4 Positive Roots and Simple roots

In each of the examples above we specified a Weyl chamber and then described things in terms of that choice. Of course, all Weyl chambers are equivalent under the action of the Weyl group, so different choices lead to an isomorphic description up to conjugation. We now formalize the important notions relative to a fixed Weyl chamber. For this section we fix a root system and use all the associated notation from above.

Definition 4.1. We fix a Weyl chamber C_0, the *fundamental* Weyl chamber. Since C_0 is disjoint from the walls defined by the roots, each root is either positive or negative on C_0. Those that are positive on C_0 are called *positive* roots, and those that are negative are called *negative* roots (relative, of course, to C_0 which we consider as fixed for this discussion).

Remark 4.2. Every root is either positive or negative and the involution -1 on V interchanges positive and negative roots. Thus, each wall is defined as the kernel of a unique positive root.

In the last lecture we established the following claim.

Claim 4.3. *Let $\alpha_1, \ldots, \alpha_k$ define the walls of C_0, chosen so that $\alpha_i > 0$ on C. Then $C_0 = \cap_{i=1}^{k} \{\alpha_i > 0\}$.*

Lemma 4.4. *The fundamental Weyl chamber C_0 is the only chamber on which all positive roots are positive.*

Proof. Let C be a chamber distinct from C_0. We claim that there is a wall separating C_0 and C. If not then all the positive roots defining walls of C_0 are positive on C and hence $C \subset C_0$ which implies $C = C_0$. If the wall W_α associated to a positive root α separates C and C_0, then α is negative on C. □

4.1 Simple Roots

Lemma 4.5. *If α and β are positive roots and $\alpha+\beta$ is a root, then $\alpha+\beta$ is a positive root. A non-trivial sum of positive roots with positive coefficients is never zero.*

Proof. The first statement is obvious from the definition. As to the second any sum of positive roots with positive coefficients takes positive value at any point of C_0. □

Definition 4.6. A positive root is a *simple* root if it cannot be written as a sum of two positive roots.

Lemma 4.7. *Every positive root is a sum of simple roots.*

Proof. Suppose that α is a positive root that cannot be written as a sum of simple roots. Then α is not simple so that it can be written as a sum $\beta_1+\beta_1'$. If each of β_1 and β_1' can be written as a sum of positive roots then so can α. Thus, renumbering if necessary, we can assume β_1 cannot be written as a sum of simple roots.

We continue this process $\beta_1 = \beta_2+\beta_2'$ with β_2 not a sum of simple roots, etc., creating a list $\{\beta_i = \beta_{i+1} + \beta_{i+1}'\}_{i=1}^{\infty}$ with each β_i not expressible as a sum of simple roots.

Claim 4.8. *In any expression for α as a positive sum of two or more positive roots the coefficient of α in the sum is* 0.

Proof. Otherwise, $\alpha = \alpha+\mu$ where μ is a positive sum of positive roots.Then $\mu = 0$ which by Lemma 4.5 means μ is the trivial sum. □

Since

$$\begin{aligned} \alpha &= \beta_1' + \cdots + \beta_{i-1}' + (\beta_i + \beta_i') \\ \beta_j &= \beta_{j+1}' + \cdots \beta_{i-1}' + (\beta_i + \beta_i') \quad \text{for} \quad j < i \end{aligned}$$

it follows from the previous claim $\alpha, \beta_1, \beta_2, \ldots$ are all distinct. Since there is a finite number of roots, this is a contradiction. □

Lemma 4.9. *If α and β are simple roots, then $\langle \alpha, \beta \rangle \leq 0$.*

Proof. If $\langle \alpha, \beta \rangle > 0$, the by Part 3 of Proposition 2.3 the element $\beta - \alpha$ is a root. Either $\beta - \alpha$ or $\alpha - \beta$ is a positive root and consequently, either $\beta = \alpha + (\beta - \alpha)$ or $\alpha = \beta + (\alpha - \beta)$ is not simple. □

Definition 4.10. We denote by S be the set of simple roots.

Proposition 4.11. *The simple roots are linearly independent.*

Proof. Suppose we have a linear relation $\sum_{\alpha \in S} \lambda_\alpha \alpha = 0$. We form two disjoint subsets of simple roots: $S_+ = \{\alpha | \lambda_\alpha > 0\}$ and $S_- = \{\alpha | \lambda_\alpha < 0\}$. Then define v by

$$v = \sum_{\alpha \in S_+} \lambda_\alpha \alpha = \sum_{\alpha \in S_-} -\lambda_\alpha \alpha$$

with both sides having positive numerical coefficients. We have

$$\langle v, v \rangle = \sum_{(\alpha,\beta) \in S_+ \times S_-} \lambda_\alpha \lambda_\beta \langle \alpha, \beta \rangle \leq 0.$$

This implies that $v = 0$, and consequently that $\sum_{\alpha \in S_+} \lambda_\alpha \alpha = 0$. It follows from Lemma 4.5 that $S_+ = \emptyset$. The same argument shows that $S_- = \emptyset$, proving the linear independence of the simple roots. □

Corollary 4.12. *The simple roots are a basis for the orthogonal space to the subspace on which the Weyl group acts trivially, and*

$$\cap_{\alpha \in S} \{\alpha > 0\}$$

is the fundamental Weyl chamber.

Proof. Since every root is either a positive linear combination or negative linear combination of the simple roots, the simple roots span the same space as all the roots. This is clearly the orthogonal complement to the maximal linear subspace on which Weyl group action is trivial. By the linear independence of the simple roots, they form a basis for this subspace.

Let $\alpha_1, \cdots, \alpha_k$ be the simple roots. Then every positive root is positive on $D = \cap_{i=1}^k \{\alpha_i > 0\}$. This means that no wall meets this sub space and hence it is contained in a Weyl chamber C. On the other hand, $\overline{D} \setminus D$ is contained in the union of the walls so that no connected open subset of V that properly contains D is contained in a chamber. Thus, D is a chamber and its walls are the walls defined by $\alpha_1, \ldots, \alpha_k$. □

4.2 The Dynkin diagram

Definition 4.13. The Dynkin diagram has nodes and connections between the nodes. The nodes are indexed by the simple roots. Two nodes have no connection if the roots that index the nodes are orthogonal. Two nodes, indexed by α, β have a single line connection between them if the angle between α and β is $\pi/3$; they have a double line connection between them if the angle between α and β is $\pi/4$, and they have a triple line connection between them if the angle between α and β is $\pi/6$. In the last two cases we add an arrow to the multiple connection that points toward the shorter root.

4.3 Examples of Dynkin Diagrams

The Dynkin diagram of a product $G \times H$ is the disjoint union of the Dynkin diagrams of G and H. The Dynkin diagram of any torus is empty.
$SU(n)$. The maximal torus T of $SU(n)$ is diagonal $n \times n$ matrices with complex numbers of norm 1 down the diagonal whose product is 1. The Lie algebra $\mathfrak{t}$ is diagonal $n \times n$ matrices with purely imaginary entries down the diagonal that sum to zero. Let $t_j\colon \mathfrak{t} \to \mathbb{R}$ be the function that assigns to a matrix in $\mathfrak{t}$ the imaginary part of its (j, j)-entry. The roots of $SU(n)$ are $\pm t_i \mp t_j$ for all $1 \leq i \neq j \leq n$. The usual fundamental Weyl chamber is the region of $\mathfrak{t}$ given by $\{t_1 > t_2 \ldots > t_n\}$. The positive roots are $t_i - t_j$ with $i < j$ and the simple roots are $(t_1 - t_2), (t_2 - t_3), \ldots, t_{n-1} - t_n$. Clearly, for $i \neq k$ we have $\langle t_i - t_{i+1}, t_k - t_{k+1}\rangle = 0$ unless $k = \{i - i, i + 1\}$, and in the exceptional case

$$\frac{-2\langle\alpha, \beta\rangle}{\langle\alpha, \alpha\rangle} = 1.$$

Thus, the nodes of the Dynkin diagram for $SU(n)$ are indexed by $\{1, \ldots, n-1\}$ and two nodes are connected (by a single line) if and only if the nodes are adjacent in the order determined by their indices.

A_n : $SU(n+1)$
$n \geq 1$

n NODES.

$SO(2n)$. The usual maximal torus for $SO(2n)$ is block diagonal matrices with two-by-two blocks down the diagonal, with the i^{th}- block being

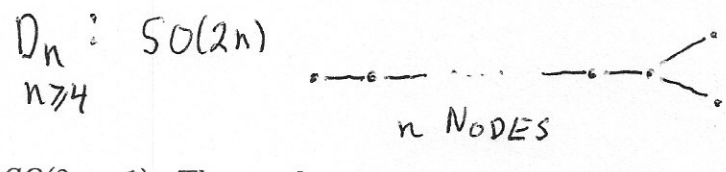
$$\begin{pmatrix} \cos(\theta_i) & -\sin(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i). \end{pmatrix}$$

Thus, the lie algebra is block diagonal matrices with two-by-two blocks down the diagonal of the form

$$\begin{pmatrix} 0 & -t_i \\ t_i & 0. \end{pmatrix}$$

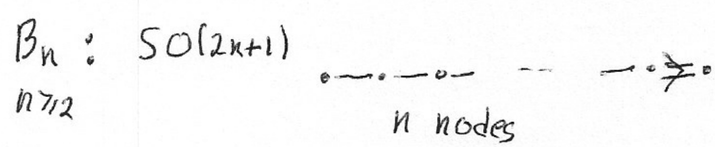
We restrict now to the case when $n \geq 4$. The roots are $\pm t_i \pm t_j$ for all $a \leq i \neq j \leq n$ and the usual fundamental Weyl chamber is $\{t_1 > t_2 > \cdots > t_n\}$ and

$\{t_{n-1} + t_n\} > 0$. The positive roots are $t_i - t_j$ and $t_i + t_j$ for $1 \le i < j \le n$ and the simple roots are $t_1 - t_2, t_2 - t_3, \ldots, t_{n-1} - t_n, t_{n-1} + t_n$. The first $n-1$ simple roots form the Dynkin diagram for $SU(n)$, namely a chain of length $(n-1)$ with simple connections. The last simple root is orthogonal to all other simple roots except $e_{n-2} - e_{n-1}$ and together with the first $(n-2)$ simple roots forms another chain with single connections of length $(n-1)$.

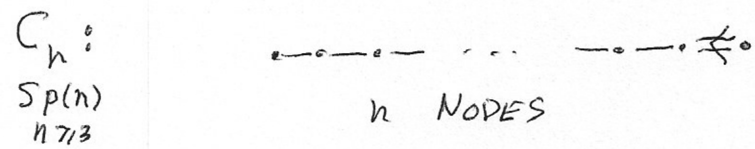


$SO(2n+1)$. The usual maximal torus for $SO(2n+1)$ is the image under the natural embedding of the usual maximal torus for $SO(2n)$. Likewise, the maximal torus for $SO(2n+1)$ is the image of the maximal torus from $SO(2n)$ and we use the same functions t_i on $\mathfrak{t}$.

We assume that $n \geq 2$. As we have seen the roots of $SO(2n+1)$ are $\pm(t_i - t_j), \pm(t_i + t_j)$ for $1 \le i \ne j \le n$ and $\pm t_i$ for $1 \le i \le n$. The usual Weyl chamber is $t_1 > t_2 \cdots > t_n > 0$ and the simple roots are $t_1 - t_2, \ldots, t_{n-1} - t_n, t_n$. The first $(n-1)$ simple roots form the Dynkin diagram $SU(n)$, a chain of length $(n-1)$ with single connections. The last root is orthogonal to all roots except the $(n-1)$ and the angle between these roots is $\pi/4$. Thus, there is a double line connection between the two nodes indexed by these roots. The last root is the shorter of the two, so the arrow points toward the n^{th} node.



$Sp(n)$. The group $Sp(n)$ is a subgroup of $U(2n)$. Its standard maximal torus is a sub torus of the standard maximal torus of $U(2n)$ consisting of diagonal matrices with diagonal entries of norm 1 with the (i,i)-entry equal to minus the $(n+i, n+i)$-entry for all $1 \le i \le n$. The maximal torus is diagonal matrices with purely imaginary entries down the diagonal with the (i,i)-entry equal to minus the $(n+i, n+i)$-entry. For $1 \le j \le n$, we set $t_j \colon \mathfrak{t} \to \mathbb{R}$ equal to the imaginary part of the (j,j)-entry. Then the roots are $\pm(t_i - t - j)$ for $1 \le i \ne j \le n$ and $2t_j$ for $1 \le j \le n$. The standard Weyl chamber is $\{t_1 > t_2 > \cdots > t_n > 0\}$ and the simple roots are $(t_1 - t_2), \ldots, (t_{n-1} - t_n), 2t_n$. As before the first $(n-1)$ simple roots form the Dynkin diagram for $SU(n)$, namely a chain of length $n-1$ with single line connections. The last simple root is orthogonal to all other simple roots except the $(n-1)^{st}$ and it makes angle $\pi/4$ with this root. Thus, the n^{th} root is connected to the $(n-1)^{st}$ by a double line connection. Since the last root is longer than the others, the arrow on the connection points away from the n^{th} node.



4.4 Classification of Connected Dynkin diagrams

There is a complete classification of connected Dynkin diagrams of root systems. In addition to the four infinite series listed above: A_n for $n \geq 1$; B_n for $n \geq 2$; C_n for $n \geq 2$; and D_n for $n \geq 4$ there are exacly 5 exceptional connected Dynkin diagrams of root systems. They are E_6, E_7, E_8, F_4, and G_2 as pictured below. The general Dynkin diagram is a finite disjoint union of diagrams of these types.