Lie Groups: Fall, 2024
Lecture IX:
The Affine Weyl Group

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We keep our standard notation for this lecture: G is a compact Lie group, T is a maximal torus of G. The Lie algebras of these Lie groups are denoted $\mathfrak{g}$ and $\mathfrak{t}$, respectively. The roots of G are characters $\alpha\colon G \to S^1$, or their differentials (also denoted by the same name) $\alpha\colon \mathfrak{t} \to \mathbb{R}$. The set of roots is R. For each root α there is a reflection $w_\alpha\colon T \to T$ or $w_\alpha\colon \mathfrak{t} \to \mathfrak{t}$ that fixes the kernel of α and acts as a reflection. In the case of the Lie algebra the kernel of each w_α, called the wall of α is a codimension-1 linear subspace of $\mathfrak{t}$. These walls divide $\mathfrak{t}$ into the Weyl chambers. The Weyl group, which is the normalizer of T, denoted $N(T)$, modulo T is denoted W and is generated by the reflection in the kernels of the roots. It acts simply transitively on the set of Weyl chambers. We have also fixed a Weyl invariant inner product on $\mathfrak{t}$.

The *fundamental lattice*, Λ, is the kernel of the map $\mathfrak{t} \to T$.

1 The Affine Weyl Chambers and the Affine Weyl Group

1.1 The Definition and First Results

Definition 1.1. Let $\mathcal{W}_{\rm aff}$ be the set of walls in $\mathfrak{t}$ given by $\{\alpha = k\}_{(\alpha,k)\in R\times\mathbb{Z}}$. These are called the *affine walls*. (The subset $\{\alpha = 0\}_{\alpha\in R}$ are exactly the walls of the Weyl chambers.) There are infinitely many walls but only finitely many meet any compact set. The *affine Weyl group*, denoted $W_{\rm aff}$ is the group generated by reflections in all the affine walls. The *affine Weyl chambers* are the components of the complement in $\mathfrak{t}$ of the union of the affine walls.

Theorem 1.2.
- *The set of affine walls is discrete in the sense that there are only finitely many walls meeting any given compact set.*
- *The set of affine walls is stablilized by the action of the affine Weyl group.*
- *The affine Weyl group acts as a discrete group of affine linear transformations of* $\mathfrak{t}$.
- *It acts simply transitively on the affine Weyl chambers and the quotient of this action is a single chamber.*

The proof of the various statements is contained in the next subsection.

1.2 The Image of the Affine Walls and Chambers in T

Recall that for each root α, the kernel of $\alpha\colon T \to S^1$ is denoted by $\hat{U}^\alpha$. The component of the identity U_α is a sub-torus of codimension 1 and $\hat{U}^\alpha \subset T$ is either one or two componentw,

Proposition 1.3. *The affine walls in* $\mathfrak{t}$ *are the pre-image of* $\cup_{\alpha_i \in R} \hat{U}^{\alpha_i} \subset T$.

Proof. $t \in \hat{U}_\alpha$ if and only if it is covered by an element $\widetilde{t}$ in $\mathfrak{t}$ on which α takes an integral value if and only if α takes integral values on the pre-image of t in $\mathfrak{t}$. Result is immediate from this □

The lattice[1] in $\mathfrak{t}$ where all the roots take integral values is the dual lattice to the root lattice. It contains the fundamental lattice since roots take integral values on Λ.

2 Proof of Theorem 1.2

We divide the proof into the proof of five different statements.

2.1 Discreteness of the Affine Walls

Let $X \subset \mathfrak{t}$ be a compact set. Then there is $K < \infty$ such that for each root $\alpha \in R$ $\alpha(X) \subset [-K, K]$. If an affine wall $\{\alpha = k\}$ meets X, then $-K \leq k \leq K$. Since there only finitely many roots, this is a finite set of affine walls.

[1] This is a lattice in the technical sense only when the center of G is finite. In other cases, it is the product of the Lie algebra of the center and a full lattice in the orthogonal complement

Corollary 2.1. *Every point $p \in \mathfrak{t}$ has a neighborhood that meets only finite many Weyl chambers.*

Proof. Let U be a connected neighborhood of p with compact closure. Then U meets only finitely many affine walls, say r walls. The complement of these r walls in $\mathfrak{t}$ has at most 2^r connected components. [Exercise to prove this by induction.] The intersection of U with any one of these components is contained in a chamber, so U meets at most 2^r affine Weyl chambers. $\square$

2.2 The action of the affine Weyl Group is a linear action stabilizing the union of the Affine Walls

Reflections are orthogonal transformations of $\mathfrak{t}$, so the affine Weyl group is a group of linear transformations of $\mathfrak{t}$.

An easy exercise establishes the following claim.

Claim 2.2. *The formula for reflection in $\{\alpha = k\}$ is given by*

$$x \mapsto x - (k - \alpha(x)) \frac{2x_\alpha}{\langle \alpha, \alpha \rangle},$$

where x_α is perpendicular to the wall, W_α, associated to α and $\alpha(x_\alpha) = \langle \alpha, \alpha \rangle$.

Clearly, reflection in the wall $\{\alpha = k\}$ of the wall $\{\alpha = r\}$ is reflection in the wall $\{\alpha = 2k - r\}$.

Consider now the case of reflecting the wall $\{\beta = r\}$ in the wall $\{\alpha = k\}$ where α and β are linearly independent. These walls meet along an affine linear codimension-2 subspace of $\mathfrak{t}$. Choose a point x this subspace and translate by $-x$ so that the subspace passes through the origin. Denote the images after translation of $\{\alpha = k\}$ by W_a and $\{\beta = r\}$ by W_b. Then W_a and W_b are two of the walls of the Weyl chamber structure. Thus, the reflection in W_a of W_b is another wall W_c of the Weyl chamber structure, say associated to a root γ. The last thing we need to see is that $\gamma(x) \in \mathbb{Z}$, so that there is an affine wall parallel to W_c passing through x, for this is the image of $\{\beta = r\}$ under reflection in $\{\alpha = k\}$. But

$$\gamma = \beta - \frac{2\langle \alpha, \beta \rangle \alpha}{\langle \alpha, \alpha \rangle}.$$

Since α and β take integral values on x and

$$\frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z},$$

this is immediate.

Since W_{aff} stabilizes the union of the affine walls, it stabilizes the complement which is the disjoint union of the Weyl chambers. But stabilizing this union means that it stabilizes the set of its connected components which are the affine Weyl chambers.

2.3 Reflections in the walls of a fixed affine chamber C_0 generate W_{aff} which acts transitively on set of the affine Weyl Chambers

We have to fix our conventions for groups acting. We choose to have them act from the left, so that $(ab)\cdot C = a\cdot(b\cdot C)$

Fix an affine Weyl chamber C_0.

Claim 2.3. *The subgroup generated by reflections in the walls of C_0 acts transitively on the set of affine Weyl chambers.*

Proof. Given a chamber C there is a smooth path in $\mathfrak{t}$ from an interior point of C_0 to an interior point of C that misses all intersections of two distinct walls (which is a locally finite union of codimension-2 affine linear subspaces) and with each intersection point with a wall being a transverse point of intersection. Enumerate in order the chambers this path crosses $C_0, C_1 \ldots, C_k = C'$. Reflection w_{α_1} in the wall of C_0 that is also a wall of C_1 sends C_0 to C_1. The next wall-crossing point from C_1 to C_2 is in the common wall W_1 of C_1 and C_2. Being a wall of $w_{\alpha_1}\cdot C_0$ his wall is of the form $w_{\alpha_1}\cdot W_{\alpha_2}$ for some wall W_{α_2} of C_0. Thus, the product of

$$(w_{\alpha_1} w_{\alpha_2} w_{\alpha_1}^{-1}) w_{\alpha_1} = w_{\alpha_1} w_{\alpha_2}$$

acts carrying C_0 to C_2. Suppose by induction on i that there is a product γ_i of reflections in walls of C_0 carrying C_0 to C_i. Then the wall common to C_i and C_{i+1}, say W_{i+1}, is of the form $\gamma_i\cdot W_{\alpha_{i+1}}$ for $W_{\alpha_{i+1}}$ a wall of C_0. Thus, reflection in this wall is the product $\gamma_i\cdot w_{\alpha_{i+1}}\cdot\gamma_i^{-1}$ and the product $\gamma_i\cdot w_{\alpha_{i+1}}$ carries C_0 to C_{i+1}. This completes the induction. As a result, there is a product of reflections in the walls of C_0 that carries C_0 to $C_k = C'$, proving the claim. □

Corollary 2.4. *The reflections in the walls of C_0 generate W_{aff}.*

Proof. By construction, the generators of W_{aff} are the reflections in all affine walls. Any such reflection is the reflection in a wall W of some Weyl chamber, say C. We have just seen that there is a product γ of reflections in the walls of C_0 that moves C_0 to C. The wall W is then $\gamma \cdot W_\alpha$ for some wall W_α of C_0 and arguing as before, we see that reflection in W is the product $\gamma \cdot w_\alpha \cdot \gamma^{-1}$ of reflections in walls of C_0. $\square$

Here is what we have just shown. Given an element $\gamma \in W_{\mathrm{aff}}$ and given a chamber C_0 we write this element as a product $w_1 w_2 \cdots w_k$ where the w_i are reflections in walls W_i of C_0. Set $\gamma_i = w_1 \cdots w_i$. Let $C_i = \gamma_i C_0$ for all $i \leq k$. Then $\gamma_k = \gamma$ and $C_0, C_1, \ldots, C_k$ is a path of chambers with C_{i-1} and C_i sharing the wall that is the image $\gamma_{i-1} W_i$.

Corollary 2.5. *Given $\gamma \in W_{\mathrm{aff}}$ for any wall C_0, let C_i be as described immediately above. Let W_i' be the wall common to C_{i-1} and C_i.. Then there is a path $\omega \colon [0,1] \to \mathfrak{t}$ and points $0 = t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} = 1$ such that $\omega(0) \in C_0$, $\omega(1) \in C_k$ and for each $1 \leq i$, the open path $\omega((t_{i-1}, t_i)) \subset C_{i-1}$ and $\omega(t_i) \in W_i'$. Furthermore, the product $r_{W_k'} r_{W_{k-1}'} \cdots r_{W_2'} r_{W_1'} = \gamma$, where $r_{W_i'}$ denotes the reflection in the wall W_i'.*

2.4 W_{aff} action on the set of affine Weyl chambers is simply transitive

Given what has already been established, we need only show that given two products of reflections in the walls of C_0 carry C_0 to C', these products give the same element of W_{aff}. Each of these products represent a 'path' of chambers from C_0 to C'. As described in the previous corollary, we represent these two 'paths' by smooth curves in $\mathfrak{t}$ from a point of C_0 to a point of C', smooth curves ξ_1 and ξ_2 that miss all codimension-2 intersections of walls and crosses transversely in order the walls between successive elements of the path of chambers.

Consider the rectangle made sides ξ_1, ξ_2, top, the arc, A_{initial}, connecting the initial points of ξ_1 and ξ_2 by a path in C_0, and bottom, the arc, A_{final}, connecting the final points of ξ_1 and ξ_2 by a path in C'.

Since this is taking place in the vector space $\mathfrak{t}$, this map extends to a smooth map of solid rectangle $\overline{R}$ into $\mathfrak{t}$. By small deformation relative to its boundary, we can assume that the smooth map $\overline{R} \to \mathfrak{t}$ misses all codimension-3 intersections of three distinct walls, and is transverse to each wall and each the codimension-2 intersections of two distinct walls. The pre-image of the union of the walls is a one-dimensional graph Γ embedded in $\overline{R}$. The edges of Γ either end at the sides of $\overline{R}$ (the top and bottom of $\overline{R}$ are disjoint from the affine walls) or at nodes, points of intersection with the codimension-2 where $4, 6$, or 12 arcs meet.

The product of reflections in the walls of C_0 represented by ξ_i is read off from the points of intersection of ξ_i-side of $\overline{R}$ with Γ, together with the walls that contain the points of intersection.

The argument proceeds by changing the map of $\overline{R}$ into $\mathfrak{t}$ so as to simplify Γ, without changing the element of W_{aff} represented by the product of elements associated with of each side of $\overline{R}$. The goal is to simplify Γ until the only components of γ that meet the sides of $\overline{R}$ are arcs with one endpoint in each side of $\overline{R}$. At that point will we have modified the words represented by the sides until they are the same word, without having changed the elements in the affine Weyl group the sides represent. This will prove that the original products represent the same elements in the affine Weyl group. Since this is true for arbitrary products of reflections in walls of C_0 that carry C_0 to C', this will show that there is a unique element in W_{aff} carrying C_0 to C'.

Removing Simple Closed Curve Components of Γ.

If there is a simple closed curve component S of Γ. Then S bounds a disk D in the rectangle. (It is possible that D contains other components of Γ.) The image of S lies in a wall W and is disjoint from codimension-2 intersections of walls. The intersection of a small neighborhood of S with $\overline{R} \setminus D$ maps into a chamber, say C_- with W as a wall. Let S' be embedded curve that is a small deformation of S in this neighborhood. Its image in C_-. The simple closed curve S' bounds a disk D" containing D in R. Since C_- is convex, there is a map $D' \to C_-$ that agrees with the given map of $S' \to C_-$. We can choose this map so that together with the original map $R \setminus \text{int}D'$) it forms a new smooth map $\overline{R} \to \mathfrak{t}$.

This operation has removed from Γ the simple closed curve S together with any components of Γ that lie in D. Since this map has not changed ξ_1 nor ξ_2, it has not changed the words in the reflections in the walls of C_0 that these paths represent.

In this way we remove all simple closed curve components from Γ. If in future moves we create simple closed curve components, we immediately remove them in the same way.

Removing certain arc components with boundary points in the same side.

Let $\rho\colon \overline{R} \to \mathfrak{t}$ be map producing a graph Γ. Suppose that A is an arc component of Γ with both endpoints in the same side of $\overline{R}$, say for definiteness, the ξ_1 side. This arc is the frontier of a disk $D \subset \overline{R}$ that meets only The ξ_1 side of $\overline{R}$. Denote by I the intersection of D with this side of $\overline{R}$. We suppose that there are no endpoints of Γ in the interior of I, though they can well be components of Γ in D. Let $A' \subset \overline{R}$ be a small deformation of A away from D, a deformation that keeps the endpoints in the side of $\overline{R}$ and that is small enough that the track of the homotopy is disjoint from Γ except for its initial position A. Let A' be the result of this deformation. It is disjoint from Γ. Let D' be the disk cut off by A' that contains D We remove the relative interior of D' from $\overline{R}$ producing a new rectangle $\overline{R}'$. Let $\rho'\colon \overline{R}' \to \mathfrak{t}$ be the restriction of ρ. There is a diffeomorphism from $\overline{R} \to \overline{R}'$ that sends I' to the relative frontier of D' in $\overline{R}$ and is the identity on the rest of $\partial\overline{R}$. The composition of this diffeomorphism followed by ρ' is a new map of $\overline{R}$ whose graph is obtained from Γ by removing the arc A and all components of Γ contained in the relative interior of D.

The image of A lies in a wall W. Let C_- be the affine Weyl chamber that contains the image of A'. The new map $\overline{R} \to \mathfrak{t}$ restricted to the ξ_1-side we have replaced an arc that goes from C_- crosses W to the neighboring chamber C_+ and then returns by crossing W to C_- by an arc that remains in C_-. In the associated word we have removed successive reflections in the wall W.. Of course, while this changes the word, it does not change the element of $W_{\rm aff}$ represented by the square of reflection is W is trivial in $W_{\rm aff}$.

The path ξ_2 and the word it represents are unchanged by this operation.

Removing certain points of intersection with codimension-2 intersections of walls.

Again suppose that $\rho\colon \overline{R} \to \mathfrak{t}$ produces a graph Γ. Let $p \in \overline{R}$ be a node of Γ; i.e.; a point in the pre-image of a codimension-2 intersection of walls. Suppose that we can connect p to a side, say ξ_1 for definiteness, by an arc A that meets Γ only in p. Let D be a small relative regular neighborhood of A. Then D meets the ξ_1-side in an arc I that maps to the interior of a chamber. We cut out the relative interior of D from $\overline{R}$ producing a subspace $\overline{R}'$ of $\overline{R}$ diffeomorphic to $\overline{R}$ by a diffeomorphism which is the identity on $\partial\overline{R} \setminus I$ and sends I to the relative frontier J of D in $\overline{R}$. The new map of $\overline{R} \to \mathfrak{t}$ is the composition of this diffeomorphism followed by the restriction of ρ to $\overline{R}'$ Thus, we have produced a new map of $\overline{R} \to \mathfrak{t}$ whose associated graph is identified with the subgraph Γ' of Γ with a small neighborhood of the node removed. On the ξ_1-side we have replaced an arc that has no points of intersection with Γ by an arc the multiple intersections with the truncated subgraph Γ'. This inserts a product of reflections in walls of C_0 into the word represented by this side.

But this product is the product, in order, of reflections in the successive walls around the codimension-2 intersection. This product represents the trivial element of $W_{\rm aff}$. So, while we have changed the word represented by the ξ_1-side, we have not changed the element in $W_{\rm aff}$ it determines.

The ξ_2-side and the word it represents are unchanged by this operation.

Arranging that the only components of Γ that meet the sides are arcs from one side to the other.

For any graph $\Gamma \subset \overline{R}$ that is the pre-image of the union of the affine walls under a map, generic as above, define $P(\Gamma) = (d_1, d_2)$ where d_1 is the number of nodes of Γ; i.e.; pre-images of codimension-2 intersections of walls and d_2 is the number of components of Γ that are arcs with both endpoints in the same side. We order lexicographically the $P(\Gamma)$ by $(d_1, d_2) < (d_1', d_2')$ if either $d_1 < d_1'$ or $d_1 = d_1'$ and $d_2 < d_2'$.

Claim 2.6. *Let $\rho\colon \overline{R} \to \mathfrak{t}$ be a map as above producing a graph $\Gamma \subset \overline{R}$. If Γ either has an edge that connects a node to the side of $\overline{R}$ or if Γ has a component that is an arc with both endpoints in the same side, then we can perform one of the two operations above replacing ρ by a map $\rho'\colon \overline{R} \to \mathfrak{t}$ giving a graph Γ' with $P(\Gamma') < P(\Gamma)$ while keeping the elements of $W_{\rm aff}$ represented by the sides of the maps the same.*

Proof. If there is a node connected by an arc of Γ to a side, then there is arc in $\overline{R}$ connecting this node to the same side that meets Γ at only the node. Using the second move above we produce a new map with graph Γ' which has fewer nodes than Γ leaving the elements in $W_{\rm aff}$ represented by the sides unchanged. Since Γ' has fewer nodes than Γ, we have $P(\Gamma') < P(\Gamma)$.

Suppose that there are no such nodes, but there is an arc component A of Γ that has both endpoints in the same side. Let I be the interval in this side between the endpoints of A and let D be the disk with frontier A that contains I. If there are no endpoints of Γ in the interior of I, then we can use the first move above to change the map so as to remove this arc and all components of Γ inside the disk cut off by A. This reduces $P(\Gamma)$, while leaving the elements in $W_{\rm aff}$ represented by the two sides the same.

If there are endpoints in the interior of Γ, then, by our assumption that no node of Γ is connected by an edge of Γ to either side of $\overline{R}$, any such endpoints must be from arc components of Γ, and those arcs are contained in D. We can replace the arc A by an arc component A_1 of Γ contained in the relative interior of D and repeat the argument. Since there are only finitely many components of Γ contained in D, we can repeat this argument only finitely many times until we arrive at an arc A_k bounding a disk D_k for which there are no endpoints of Γ in the interval $D' \cap \partial\overline{R}$, and then argue as in the previous paragraph using A_k . $\square$

Completion of the Proof that the Action is Simply Transitive

Any strictly monotone decreasing string for the given lexicographic ordering on pairs (d_1, d_2) of non-negative numbers must be finite. Thus, starting a graph Γ associated with a map $\overline{R} \to \mathfrak{t}$ as above, after a finite number of steps with the process of simplification described in the previous lemma must terminate. But the process only terminates if the graph produced by the last step has no node connected to the boundary no arc component of Γ with both ends in the same side. This means that every component of the graph that meets the sides of $\overline{R}$ is an arc with one endpoint in each side.

It follows that in this case the words in the reflections in walls of C_0 represented by the two sides of the solid rectangle are the same. Since our moves leave invariant the elements in the Weyl group represented by the words coming from the sides, this means that the original words given by ξ_1 and ξ_2 represent the same element in W_{aff}.

2.5 The W_{aff} action is discrete with quotient equal to the closure of a single affine Weyl chamber

The first statement is immediate from the fact that the W_{aff} action is simply transitive on the set of affine Weyl chambers and the set of Weyl chambers is locally finite in the sense that every point has a neighborhood meeting only finite many chambers.

Now fix an affine Weyl chamber C_0. Define a map from any affine Weyl chamber C to C_0. It is the map given by the unique element in W_{aff} that sends C to C_0. In this way we define isomorphisms of each closed Weyl chamber onto the closure of C_0. We claim that this gives a well-defined function from $\mathfrak{t}$ to the closure of C_0. To prove this we need only see that if C and C' share a wall then the maps we have given $C \to C_0$ and $C' \to C_0$ agree on the intersection of their closures. Let C and C' share an affine wall W and suppose that γ_C and $\gamma_{C'}$ there the unique elements of the affine Wel group carrying C and C', respectively, to C_0. Then by the uniqueness of these elements we see that the Weyl element $\gamma_C = \gamma_{C'} w$, where w is the reflection in W. Thus, γ_C and $\gamma_{C'}$ agree on W and hence on the intersection of the closures of C and C'. Thus, we have a continuous map $\mathfrak{t} \to \overline{C}_0$

The same argument, using the uniqueness of the affine Weyl elements, shows that this map is a W_{aff}-inavariant on $\mathfrak{t} \to \overline{C}_0$ whose fibers are exactly the orbits of the W_{aff} action.

3 The Structure of W_{aff} and its Translations Subgroup

3.1 The Structure of W_{aff}

Associating to an affine linear transformation its differential defines a homomorphism

$$\rho\colon W_{\text{aff}} \to W.$$

Proposition 3.1. • *The affine Weyl group sits in an exact sequence*

$$\{1\} \to \Lambda_0 \to W_{\text{aff}} \xrightarrow{\rho} W \to \{1\},$$

where Λ_0 is a lattice of translations in $\mathfrak{t}$.

- *The natural inclusion of W into W_{aff} spits this sequence so that W_{aff} is a semi-direct product $\Lambda_0 \rtimes W$ with the action of W on Λ_0 induced from the defining action of W on $\mathfrak{t}$.*

Proof. Since W_{aff} is generated by reflections in walls parallel to walls of the Weyl chamber, the differential of any element of W_{aff} is an element of W. x The kernel of the map from linear transformations to $GL(n,\mathbb{R})$ given by taking the differential is the subgroup of translations. Thus, the kernel of ρ, Λ_0, is a group of translations of $\mathfrak{t}$. A translation of $\mathfrak{t}$ is identified with an element of $\mathfrak{t}$. The group Λ_0 is naturally an additive subgroup of $\mathfrak{t}$. (We will see a generating set later.) Since W_{aff} is a discrete group of affine transformations, Λ_0, is a lattice in $\mathfrak{t}$.

The natural inclusion $W \subset W_{\text{aff}}$ as the subgroup of reflections in walls passing through 0 gives a splitting of ρ so that $W_{\text{aff}} = \Lambda_0 \rtimes W$, meaning that Λ_0 is a Weyl invariant lattice in $\mathfrak{t}$. □

3.2 The Subgroup Λ_0 of translations

Here is the first result.

Proposition 3.2. *For each root α let λ_α be the image of 0 under reflection in $\{\alpha = 1\}$. For every $\alpha \in R$, $\lambda_\alpha \in \Lambda_0$ and $\{\lambda_\alpha\}_{\alpha\in R}$ generates Λ_0. Also, Λ_0 is a sub lattice of the co-weight lattice Λ. That is to say every the arc in $\mathfrak{t}$ connecting 0 to λ_α projects to a loop in T. This loop bounds a disk in G.*

Proof. First, notice that we can rewrite the formula in Claim 2.2 for reflection in $\{\alpha = k\}$ acting on x as

$$x \mapsto x + (k - \alpha(x))\lambda_\alpha.$$

The composition of reflection in $\{\alpha = 0\}$ followed by reflection in $\{\alpha = 1\}$ is translation by λ_α. This shows each $\lambda_\alpha \in \Lambda_0$.

For a root α, there is a 3-dimensional Lie group $H \subset G$ whose Lie algebra $\mathfrak{t}_H \oplus V_r$ maps isomorphically onto $\mathbb{R}(\lambda_\alpha) \oplus V_\alpha$ preserving the direct sum decomposition. This map sends the element $\lambda_r \in \mathfrak{t}_H$ associated to the root r to the element $\lambda_\alpha \in \mathfrak{t}$.

There are two possibilities for H: either $SO(3)$ and S^3. In the first case the generator x of the co-weight lattice, Λ_x, satisfies $r(x) = \pm 1$, so that the co-weight lattice is the dual to the root lattice and λ_r is $2x$. In the second case the generator y of the co-weight lattice satisfies $r(y) = \pm 2$, so that the co-weight lattice is twice the dual of the of the root lattice. and $\lambda_r = y$. Since, in both cases, $r(\lambda_r) = 2$, we have $\lambda_r \in \Lambda_H$. In the first case the arc connecting 0 to λ_r projects to the square of the generator of $\pi_1(T_H)$ and in the second it projects to the generator of $\pi_1(T_H)$. In both cases this element in trivial in $\pi_1(H)$

By naturality both statements pass from H to G: the element $\lambda_\alpha \in \Lambda$ and the arc in $\mathfrak{t}$ connecting 0 to λ_α projects to a loop in T that represents the trivial element in $\pi_1(G)$.

Now let us show by induction on the length of products of reflections that for any $\gamma \in \mathcal{W}_{\text{aff}}$ we have $\gamma(0) = \sum_\alpha n_\alpha \lambda_\alpha$ for integers n_α. Clearly, the result holds for any single reflection: Reflection of 0 in $\{\alpha = k\}$ is $k\lambda_\alpha$. Suppose $\xi = w\xi'$ where ξ' has shorter length than ξ and w is a reflection. Then by induction $\xi'(0) = \sum_\alpha n_\alpha \lambda_\alpha$. Suppose that w is reflection in $\{\beta = k\}$. Then this reflection sends $\sum_\alpha n_\alpha \lambda_\alpha$ to

$$\sum_\alpha n_\alpha \lambda_\alpha + (k - \sum_\alpha n_\alpha \beta(\lambda_\alpha))\lambda_\beta.$$

Since the $\lambda_\alpha \in \Lambda$, $\beta(\lambda_\alpha) \in \mathbb{Z}$ and hence this expression is an integral linear combination of the λ_α.

Applying this to the subgroup Λ_0 of translations, we see that every element in Λ_0 is a translation by an integral linear combination of the λ_α.

This proves that the λ_α generate Λ_0. Since each $\lambda_\alpha \in \Lambda$, it follows that $\Lambda_0 \subset \Lambda$. $\square$

Actually, we have proved more:

Corollary 3.3. *The lattice $\Lambda_0 \subset \mathfrak{t}$ is contained in $\Lambda = \pi_1(T)$ and is contained in the kernel of the map $\pi_1(T) \to \pi_1(G)$.*

Proof. In the above proof we established that each λ_α represents a loop in T that bounds in G. Since these elements generate Λ_0, the corollary follows. $\square$

Remark 3.4. For each root α, the element $\lambda_\alpha = 2x_\alpha/\langle\alpha, \alpha\rangle$ as defined before. In the literature λ_α is denoted $\alpha^\vee$ and is called the *coroot dual to* α.