Problem Set 3 for Lie Groups: Fall 2024

September 28, 2024

Problem 1. Let G be a connected Lie group and $(U, e, -1, \Omega, m)$ a sub local Lie representing the germ G at the identity. Let $W \subset U$ be an open neighborhood of e with the properties that $W^{-1} = W$ and $W \times W \subset \Omega$. Show that for any path $\omega\colon [0,1] \to G$, there is a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ with the properties asserted at the beginning of the proof of Theorem 3.1.

Problem 2. Show that for the Lie group $S^1 \times \cdots \times S^1$ the exponental mapping is the universal covering. Let N be Lie group of upper triangular 3×3 matrices with 1's down the diagonal. Show that the exponential mapping from its Lie algebra to N is a global diffeomorphism.

Problem 3. Let $\{X_1, \ldots, X_k\}$ generate the Lie algebra L. Show that any element of L is a linear combination of brackets of the form

$$[Z_1, [Z_2, [Z_3, \cdots Z_n]]] \cdots],$$

where each Z_i is one of the X_j. (Here we include brackets of length one, which is the just the X_i.)

Problem 4. Let L be a finite dimensional Lie algebra over K. Fix a basis $X_1, \ldots, X_n$ for L. Let $a^j\colon L \to \mathbb{R}$ assign to X the coefficient of X_i in the expression of X as a linear combination or the basis elements. Define constants $c^i_{j,k}$ by

$$[X_j, X_k] = \sum_i c^i_{j,k} X_i.$$

Show that

$$[\sum_j a^j X_j, \sum_k b^k X_k] = \sum_{i,j,k} a^j b^k c^i_{j,k} X_i$$

Show that any iterated bracket $Y = [X_{i_1}, [X_{i_2}, [X_{i_3}, \ldots X_{i_n}] \cdots]$ is given as $Y = \sum_n p^n X_n$ where the p^n are polynomials of degree $n-1$ in the variables $c^i_{j,k}$.

Now writing $Y_j = \sum_i a^i_j X_i$ consider the n-fold bracket

$$[Y_1, [Y_2, [Y_3, \cdots, [Y_{k-1}, Y_k] \cdots].$$

Show that this element in the Lie algebra has coefficients that are polynomial functions of the coordinate functions of the a^i_j of the Y_j with coefficients that are themselves polynomial functions of the $c^i_{j,k}$.

Problem 5. A Lie Algebra N is *nilpotent* if for some k, all brackets of k elements of N are zero. A Lie algebra A is *abelian* if all brackets of elements of A are zero. An *ideal* I in a Lie algebra A is a linear subspace of A with the property that $[A, I] \subset I$. Show that if I is an ideal of A then it is a subalgebra of A and there is an induced Lie bracket on A/I. Conversely, show that if $\varphi\colon L \to L'$ is a map of Lie algebras then (i) the image of φ is a sub Lie algebra of L', (ii) the kernel of φ is an ideal of L and (iii) φ induces a Lie algebra isomorphism between the kernel of φ and the image of φ.

The *center* of a Lie algebra A is the set of $X \in A$ such that $[X, A] = 0$. Show that the center of A is an ideal of A. A subalgebra of A is said to be *central* if it is contained in the center of A. Show that if N is a nilpotent Lie algebra, then there is a filtration $0 = N_k \subset N_{k-1} \subset \cdots N_2 \subset N_1 = N$ such that each N_i is an ideal in N and N_i/N_{i-1} is central in N/N_{i-1}

Problem 6. Let L be the linear space of strictly upper triangular 3×3 matrices. Show that L is a sub Lie algebra of $\mathfrak{gl}(3, \mathbb{R})$ and that L is a nilpotent Lie algebra. Compute explicitly the exponential map on L. What is the subgroup of $GL(3, \mathbb{R})$ whose Lie algebra is L?

Problem 7. Show that if A is a Lie algebra then $[A, A]$ is an ideal, and more generally, defining $A_1 = A$ and $A_n = [A_{n-1}, A_{n-1}]$, each A_n is an ideal. A Lie algebra is *solvable* if $A_n = 0$ for some $n \geq 0$. Show each of the following is equivalent to A being solvable.

- There is a sequence of ideals $A = A_0 \supset A_1 \supset A_2 \supset \cdots \supset A_n = 0$ such that $[A_i, A_i] \subset A_{i+1}$.
- There is a sequence of subalgebras $A = A_0 \supset A_1 \supset A_2 \supset \cdots \supset A_n = 0$ with A_{i+1} an ideal in A_i such that $\dim(A_i/A_{i+1}) = 1$ for all i.
- $[A, A]$ is nilpotent.

Problem 8. Show that if B, C are solvable ideals of a Lie algebra A so is $B \times C$. Deduce that a finite dimensional Lie algebra has a unique maximal solvable ideal, one containing all solvable ideals. This is the *radical* of A.

Problem 9. Let $PSL(2, \mathbb{C})$ be the Lie group which is the quotient of $SL(2, \mathbb{C})$ by $\{\pm \mathrm{Id}\}$. What is its Lie algebra?

Problem 10. Let G be a connected Lie group and $Z \subset G$ a discrete subset normal subgroup of G. Show that Z is an abelian subgroup and indeed is in the center of G.

Problem 11. Show that the quadratic, cubic, and quartic terms in the power series $\log(\exp(A)\exp(B))$ can be written as a sum of iterated brackets of A and B.