Lie Groups: Fall, 2024
Problem Set 4

September 28, 2024

Problems on the Frobenius Theorem.

In all these problems $\mathcal{D}$ is a distribution of dimension k on an n manifold M and $p \in M$ is a point.

F1. Show that there is a "flow box" for $\mathcal{D}$ near p. By this we mean that there are of balls $B_1^k \subset \mathbb{R}^k$ and $B_2^{n-k} \subset \mathbb{R}^{n-k}$, each centered at the origin and a diffeomorphism $B_1^k \times B_2^{n-k} \cong U \subset M$ sending $(0,0)$ to p producing coordinates $(x^1, \ldots, x^k, y^1, \ldots, y^{n-k})$ defined on U with the property that for each $u \in U$ the projection $\pi_1 \colon U \to B_1^k$ maps the k-plane $\mathcal{D}(u)$ isomorphically onto $T_{\pi_1(u)} B_1^k$.

F2. Let $B_1 \times B_2 \cong U \subset M$ be a 'flow box" as in Problem F1. Show that there unique vector fields $\xi_1, \ldots, \xi_k$ defined on U with the following properties:

- Each ξ_i is tangent to $\mathcal{D}$.
- For each $u \in U$ the image of $\xi_i(u)$ under the projection to the first factor is $(\partial/\partial x_i)|_{\pi_1(u)}$.
- For each $u \in U$, the vectors $\{\xi_1(u), \ldots, \xi_k(\mathrm{u})\}$ form a basis for $\mathcal{D}(u)$.

F3.. Keep the notation and assumptions from the previous two problems. Suppose that $\mathcal{D}$ is involutive. Show that $[\xi_i, \xi_j] = 0$ for all i and j.

4. Let X and Y be vector fields on a manifold M. We define the *Lie derivative*, $L_X(Y)$, of X on Y as follows. Fix $q \in M$. Let $\gamma_X(t)$ be the integral curve for X with $\gamma_X(0) = q$. Then

$$L_X(Y)(q) = \lim_{t \mapsto 0} \frac{\gamma_X(-t)_*(Y(\gamma_X(t))) - Y(\gamma_X(0))}{t}.$$

Show that $L_X(Y) = [X, Y]$.

F5. Suppose that X and Y are commuting vector fields. For $q \in M$, let $\gamma_X(q, s)$ be the integral curve for X with $\gamma_X(q, 0) = q$ and let $\gamma_Y(q, t)$ be the integral curve for Y with $\gamma_Y(q, 0) = q$. Fix $\epsilon > 0$ suffiiently small so that all the flow lines we write down in what follows exist. Show that for $|s|, |t| < \epsilon$

$$\gamma_Y(\gamma_X(p, s), t) = \gamma_X(\gamma_Y(p, t), s).$$

F6. With the "flow box" and vector fields $\xi_1, \ldots, \xi_k$ as in Problem F2, let $\gamma_i(u, t^i)$ be the integral curve for ξ_i with $\gamma_i(u, 0) = u$. show that for a sufficiently small ball $B_2(\epsilon) = B_2(0, \epsilon)^{n-k} \subset \mathbb{R}^{n-k}$ centered at 0, for any $q \in B_2(\epsilon)$, for any ordering of the indices $\{1, \ldots, k\}$, say $\{i_1, \ldots, i_k\}$, and for any $|t^1|, \ldots, |t^k| < \epsilon$ the flow

$$\gamma_{i_1}(\gamma_{i_2}(\cdots(\gamma_{k-1}((\gamma_{i_k}(q, t^{i_k}), t^{i_{k-1}}), \cdots t^{i_2}), t^{i_1})$$

is defined and its value depends only on the choice of variables $t_1, \ldots, t_k$ not on the ordering of the terms.

Show that letting the t^i vary over $(-\epsilon, \epsilon)$ produces an integral submanifold to $\mathcal{D}$ though q. The coordinates $y^1, \ldots, y^{n-k}$ on $B_2(\epsilon)$ extended to functions constant on these local leaves and coordinates $t^1, \ldots, t^k$ on the integral submanifolds give a flow box for the foliation produced by $\mathcal{D}$ in the sense that the integral submanifolds are given by the equations

$$\{y^1 = c_1; \ldots; \quad y^{n-k} = c_{n-k}\}.$$

This establishes the local Frobenius Theorem in the neighborhood of p and, since p was an arbitrary point, proves the result. Now we need to construct the global integral submanifolds.

Problem 7. Let X be a second countable topological space (meaning there is a countable collection of open sets $B_i \subset X$, the basis for the topology, with the property that every open set of X is a union of the some subset of the $\{B_i\}_i$). Show that for any open covering of X there is a countable sub-cover. That is to say, given any collection $\{U_a\}_{a \in A}$ of open sets covering X, There is a countable subset $A' \subset A$ such the $\{U_a\}_{a \in A'}$ is also a overing, [Hint: For each B_i that is contained in U_a for some $a \in A$, choose an index $a_i \in A$ with the property that U_{a_i} contains B_i. Show that these U_{a_i} are a countable covering of X.]

Problem F8. Let $\mathcal{D}$ be a k-dimensional involutive distribution on M. Recall that a flow box is an open subset of M identified with $U^k \times V^{n-k} \subset$

$\mathbb{R}^k \times \mathbb{R}^{n-k}$ with coordinates $(x^1, \ldots, x^k, y^1, \ldots, y^{n-k})$ with the property that the family $U \times \{v\}$ of k-manifolds are local integral submanifolds of the distribution. Cover M by flow boxes and define points p, q in a flow box to be elementarily equivalent if they lie on the same local integral submanifold in that flow box. Let this elementary equivalence relation generate an equivalence relation. An equivalence class is by definition a *global leaf*. Show that two points p and q of M are contained in the same global leaf if and only if there is a smooth path in M everywhere tangent to $\mathcal{D}$ connecting p and q. Show that the resulting foliation $\mathcal{F}$ on M is covered by countably many flow boxes $\{F_i\}_{i=1}^{\infty}$. Show that each global leaf meets any of these countably many flow boxes in a subset of the form $U \times \Delta$ where Δ is a countable subset of V. The leaf topology on a global leaf L is generated by the open sets in connected components of the intersection of the global leaf any of the F_i. Show that in this topology L is a second countable k-dimensional manifold. Show that L inherits a smooth structure from the smooth structure on M and with this structure L is smoothly one-one immersed in M.