Problem Set 5 for Lie Groups: Fall 2024

September 30, 2024

For the first 3 problems we use the notation from the lecture: S is a finite set, $F(S)$ is the free associative algebra generated by S, $FL(S)$ is the free Lie algebra generated by S and $\hat{\psi}\colon U(F(S)) \to T(S)$ is the isomorphism between the universal enveloping algebra for $FL(S)$ to the tensor algebra of S. All these algebras are graded algebras induced from the grading $S_\infty = \coprod_{n=1}^{\infty} S_n$

Problem 1. Let $P\colon \mathcal{F}^1(T(S)) \to \mathcal{F}^1(T(S))$ be the map defined by the linear extension of

$$P(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = \mathrm{ad}(x_1) \circ \mathrm{ad}(x_2) \circ \cdots \circ \mathrm{ad}(x_{n-1})(x_n),$$

for $x_1, \ldots, x_n \in S$. Show that P preserves the gradings, has image contained in $FL(S)$, and is the identity on $S \subset T(S)$.

Problem 2. We keep the notation of P for the map defined in Problem 1. Define a map $\theta\colon T(S) \to \mathrm{End}(FL(S))$ by setting $\theta(x) = \mathrm{ad}(x)$ for $x \in S$ and extending to all of $T(S)$ using the fact that $T(S)$ is the free associative algebra generated by S.
(a) Show that the restriction of θ to $FL(S)$ is a Lie algebra map from $FL(S)$ to $\mathrm{End}(FL(S)$ that sends $u \in FL(S)$ to the endomorphism $\mathrm{ad}(u)$.
(b) Show that for all $u \in FL(S))$ and all $v \in \mathcal{F}^1(T(S))$ we have $P(uv) = \theta(u)P(v)$ for P the map in Problem 1.
(c) For $u, v \in FL(S)$ show that $P[u, v] = [P(u), v] + [u, P(v)]$.
(d) Show that for every k, the restriction of P to $FL^k(S) = T^k(S) \cap FL(S)$ is multiplication by k, so that the function $\overline{P}$ defined by

$$\pi = \oplus_{k\geq 1} \frac{1}{k} P|_T^k(S)\colon T^k(S) \to FL^k(S)$$

is a projection $\pi\colon T(S) \to FL(S)$ (meaning that $\pi|_{FL(S)} = \mathrm{Id}$).

Problem 3. Fix positive numbers m, r, s. Define $\alpha(m, r, s)$ to be the set of pairs of sequences $\mathbf{r} = (r_1, \ldots, r_m)$ and $\mathbf{s} = (s_1, \ldots, s_m)$ with the properties that (1) for each i we have $r_i, s_i \geq 0$ and $r_i + s_i \geq 1$ and (2) $|\mathbf{r}| := \sum_{i=1}^m r_i = r$ and $|\mathbf{s}| := \sum_{i=1}^m s_i = s$. Let $\alpha(m) = \sum_{r+s} \alpha(m, r, s)$. We denote sequences in $\alpha(m)$ by $(\mathbf{r}, \mathbf{s})$. We denote by $|\mathbf{r}| = \sum_{i=1}^m r_i$ and by $|\mathbf{s}| = \sum_{i=1}^m s_i$. Show that

$$\log(\exp(X)\exp(Y)) = \sum_m \frac{(-1)^{m-1}}{m} \left(\sum_{r,s \geq 0} \frac{X^r Y^s}{r!s!} \right)^m$$
$$= \sum_m \frac{(-1)^{m-1}}{m} \sum_{(\mathbf{r},\mathbf{s}) \in \alpha(m)} \frac{X^{r_1} Y^{s_1}}{r_1! s_1!} \frac{X^{r_2} Y^{s_2}}{r_2! s_2!} \cdots \frac{X^{r_m} Y^{s_m}}{r_m! s_m!}.$$

Problem 4. For $(\mathbf{r}, \mathbf{s}) \in \alpha(m)$: if $s_m \geq 1$ set

$$H(\mathbf{r}, \mathbf{s})(X, Y) = \frac{1}{|\mathbf{r}| + |\mathbf{s}|} \frac{\mathrm{ad}(X)^{r_i} \mathrm{ad}(Y)^{s_i}}{r_i! s_i!} \cdots \frac{\mathrm{ad}(X)^{r_m} \mathrm{ad}(Y)^{s_m - 1}}{r_m! s_m!}(Y),$$

and if $s_m = 0$ set

$$H(\mathbf{r}, \mathbf{s})(X, Y) = \frac{1}{|\mathbf{r}| + |\mathbf{s}|} \frac{\mathrm{ad}(X)^{r_i} \mathrm{ad}(Y)^{s_i}}{r_i! s_i!} \cdots \frac{\mathrm{ad}(X)^{r_m - 1}}{r_m!}(X),$$

(a) For all r, s set

$$H(r, s)(X, Y) = \sum_m (-1)^{m-1} \frac{1}{m} \sum_{(\mathbf{r},\mathbf{s}) \in \alpha(m,r,s)} H(\mathbf{r}, \mathbf{s})(X, Y).$$

Show that $H(r, s)$ is a finite sum of elements in $FL(S)$, each being a bracket of exactly r copies of X and s copies of Y. Show that we have an equality of formal power series.

$$\log(\exp(X)\exp(Y)) = \sum_{r,s} H(r, s)(X, Y).$$

[Hint: Use the fact that $\log(\exp(X)\exp(Y)) \in FL(S)$.]

Problem 5. Consider the function $f(u, v) = -\log(2 - \exp(u + v))$. The power series centered at $(u, v) = (0, 0)$ for f is

$$\sum_{r,s} \eta_{r,s} u^r v^s$$

where

$$\eta_{r,s} = \sum_m \frac{1}{m} \sum_{\alpha(m,r,s)} \frac{1}{r_1! s_1! \cdots r_m! s_m!}.$$

Show that this series is absolutely convergent for $u, v > 0$ and $u + v < 2$.

Problem 6. Let L be finite dimensional real Lie algebra. Fix a positive definite inner product on L with associated norm denoted $|\cdot|$. Show that there is a $1 \leq M < \infty$ such that $|[U, V]| \leq M|U||V|$ for all $U, V \in L$. Show that for any fixed r, s and for elements $U, V \in L$

$$|H(r, s)(U, V)| \leq M^{r+s-1}|U|^r|V|^s\eta_{r,s}.$$

Problem 7. Show that with L a finite dimensional real Lie algebra and M as in the previous problem, the Hausdorff series $\sum_{r,s} H(r, s)(U, V)$ is absolutely convergent for U, V with $|U|, |V| < \frac{1}{M}$.