2024 Lie Groups Fall 2024; Problems VI

September 30, 2024

Problem 1. Let G be the group of all upper triangular (3×3)-matrices with 1's down the diagonal. What is the Lie algebra $\mathfrak{g}$ of G. Write an explicit formula for the exponential mapping. Show that the Baker-Campbell-Hausdorff formula converges for all $X, Y \in \mathfrak{g}$, so that in fact it defines a Lie group structure on $\mathfrak{g}$. Show that the exponential mapping is an isomorphism of Lie groups between this group structure on $\mathfrak{g}$ and G.

Problem 2. Recall that a nilpotent connected Lie group is one whose center is positive dimensional and whose quotient by the center is also nilpotent. This gives an inductive definition of a connected nilpotent Lie group. Generalize Problem 1 to any connected nilpotent Lie group in the following sense. Show (if you have not already in an earlier problem) that if N is a connected nilpotent group then its Lie algebra L is nilpotent in the sense that there is $k \geq 2$ such that all brackets of length $k+1$ vanish. [Proof by induction on the length of nilpotence of the group.] Show that the BCH for L is a polynomial of degree k and hence converges on all of L and hence defines a group structure on L which maps onto N by the exponential map with discrete kernel.

Problem 3. Let G be a connected Lie group whose universal covering is not diffeomorphic to $\mathbb{R}^n$ for any n. Show that the the BCH formula cannot converge absolutely and uniformly on all of $\mathfrak{g}$.

Problem 4. Suppose that L is a finite dimensional Lie algebra over the rationals $\mathbb{Q}$ that is nilpotent. Show that the BCH formula defines a group structure on L and that this group is nilpotent. This group is called the *Malcev completion* of L.